Towards global mirror symmetry

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Contents

Introduction

Mirror symmetry is a phenomenon which inspired fundamental progress in a wide range of disciplines in mathematics and physics in the last twenty years; we will review here a number of results going from the enumerative geometry of curves to homological algebra. These advances justify the introduction of new techniques, which are interesting in their own right. Among them, Gromov–Witten theory and its variants allow us to provide a refined statement of mirror symmetry. Of course this leads to further open questions (despite much effort and progress, Gromov–Witten theory remains unknown in high genus for the quintic threefold). In this course, we will illustrate the natural problem of moving beyond the local mirror symmetry statement and completing a framework of global mirror symmetry which is gradually taking shape. We will show how the missing piece in this picture comes unexpectedly from a classical subject in algebraic geometry: the theory of curves with level structures.

Plan of the course. These notes cover the material of the course of the summer school and more; they also contain a detailed discussion of several crucial points and many examples. Lecture 1 will present the problem of stating mirror symmetry beyond the local setup. Level structures are introduced as the geometric object completing the picture; they will be preliminarily approached via examples. Lecture 2 will present some of the material covered in Chapter I of the notes: the compactification of moduli of curves with level structures, the enumerative geometry, the Grothendieck–Riemann–Roch formula (this is interesting in its own right and is related to the first week course by Gavril Farkas). Lecture 3 will fit the theory of level curves into the mirror symmetry framework; this is the so-called Landau–Ginzburg model set up in Chapter II. Finally, Lecture 4 will provide a more general treatment of this global mirror symmetry framework, beyond the case of Calabi–Yau hypersurfaces, moving from a construction due to Berglund, Hübsch and Krawitz (Chapters III and IV).

A first example. The phenomenon inspiring mirror symmetry consists of a pair of three-dimensional varieties X and $X^\vee$ of Calabi–Yau type (CY), satisfying the relations

$$h^{1,1}(X) = h^{2,1}(X^\vee) \qquad \text{and} \qquad h^{2,1}(X) = h^{1,1}(X^\vee). \tag{1}$$

The most elementary example is that of a complex smooth hypersurface X of dimension three defined by the Fermat polynomial

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0$$

in the projective space $\mathbb{P}^4(\mathbb{C})$. The canonical bundle $\omega_X = \det(T_X^\vee)$ is trivial. In fact X is a Calabi–Yau (CY) variety, which — in this introduction — will simply mean $\omega_X \cong \mathcal{O}_X$ or equivalently $\deg(X) = \dim(\mathbb{P}^4(\mathbb{C})) + 1$. Its Hodge numbers $h^{p,q} = \dim H^{p,q}(X)$ are given by

$$\begin{array}{ccccccc} & & & h^{0,0}=1 & & & \\ & & h^{1,0}=0 & & h^{0,1}=0 & & \\ & h^{2,0}=0 & & h^{1,1}=1 & & h^{0,2}=0 & \\ h^{3,0}=1 & & h^{2,1}=101 & & h^{1,2}=101 & & h^{0,3}=1 \\ & h^{1,3}=0 & & h^{2,2}=1 & & h^{3,1}=0 & \\ & & h^{2,3}=0 & & h^{3,2}=0 & & \\ & & & h^{3,3}=1 & & & \end{array} \tag{2}$$

The mirror Calabi–Yau variety $X^\vee$ satisfying (1) can be regarded as X modulo the action of the group $(\mathbb{Z}/5)^3$ spanned (for instance) by the diagonal matrices $\mathrm{Diag}(\xi_5, 1, 1, 1, \xi_5^4)$, $\mathrm{Diag}(1, \xi_5, 1, 1, \xi_5^4)$, $\mathrm{Diag}(1, 1, \xi_5, 1, \xi_5^4)$ acting on $\mathbb{P}^4(\mathbb{C})$. The quotient is singular, but has a minimal desingularization, which we will denote by $X^\vee$. This is again a of CY type, and the Hodge numbers are equal to

$$\begin{array}{ccccccc} & & & h^{0,0}=1 & & & \\ & & h^{1,0}=0 & & h^{0,1}=0 & & \\ & h^{2,0}=0 & & h^{1,1}=101 & & h^{0,2}=0 & \\ h^{3,0}=1 & & h^{2,1}=1 & & h^{1,2}=1 & & h^{0,3}=1 \\ & h^{1,3}=0 & & h^{2,2}=101 & & h^{3,1}=0 & \\ & & h^{2,3}=0 & & h^{3,2}=0 & & \\ & & & h^{3,3}=1 & & & \end{array} \qquad (3)$$

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A and B models. At a less elementary level, mirror symmetry predicts an isomorphisms involving two types of invariants attached to each CY variety X: $\mathcal{A}_X$ and $\mathcal{B}_X$. These are both spaces equipped with a vector bundle with flat connection, we usually refer to them as the A model and the B model. Here, we illustrate how they are constructed in this case. The numerical equalities of (1) are simply identities between dimensions deriving from $\mathcal{A}_X \cong \mathcal{B}_{X^\vee}$ and $\mathcal{A}_{X^\vee} \cong \mathcal{B}_X$.

On can define $\mathcal{B}_{X^\vee}$ using the space of deformations of $X^\vee$. This is may be regarded as the space of deformations of X stable with respect to the $(\mathbb{Z}/5)^3$-action used above. We get in this way the Dwork family

$$\left\{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + 5\psi \prod_{i=1}^{5} x_i = 0\right\}$$

where ψ is a parameter of an open subset Ω of the projective line

$$\begin{array}{ccc} \mathcal{F}_\psi & \longrightarrow & \mathcal{F} \\ \downarrow & \square & \downarrow \\ \psi & \longrightarrow & \Omega \subset \mathbb{P}^1. \end{array}$$

The group $(\mathbb{Z}/5)^3$ preserves the fibers. The quotient followed by the desingularization $\mathcal{F}$ yields, as before, a family of varieties $X_\psi^\vee$ for ψ varying in $\Omega \subset \mathbb{P}^1$. We recall that the odd cohomology of this family is locally trivial and yields a complex vector bundle V of rank four, naturally equipped with the so-called Gauss–Manin connections [1]. Indeed, if we write π for the total space of the family over the base scheme Ω and we consider the local system $R^3\pi_*\mathbb{Z}$, then the vector bundle $V = \mathcal{O}_\Omega \otimes R^3\pi_*\mathbb{Z}$ is equipped with a flat connection corresponding to the local system. We define in this way a complex projective line equipped, on a Zariski open subset Ω, with a rank-four vector bundle with flat connection. In the literature, the B model is given by this structure restricted to a disc, neighborhood of $\psi = \infty$ in the complex projective line. We point out that $\psi = \infty$ does not belong to Ω and that the rank-four vector bundle with connection is supported on the punctured disk. The above construction suggests the notation

$$\mathcal{B}^\infty_{X^\vee}$$

highlighting the restriction to a neighborhood of infinity.

As we shall illustrate in Chapter IV one can define another rank-four vector bundle with flat connection over a punctured disc. We consider a one parameter complex disc

$$\mathcal{A}^+_X,$$

neighborhood of the origin in $H^{1,1}(X;\mathbb{C}) \cong \mathbb{C}$, equipped with a trivial rank-four vector bundle $H^{\mathrm{even}}(X;\mathbb{C}) \otimes \mathcal{O}$. Gromov–Witten theory allows us to define Dubrovin connection over the punctured disc: it encodes the enumerative geometry of rational curves of X (genus-zero Gromov–Witten invariants, see (IV.9)).

The mirror symmetry statement 1 can be refined by requiring an isomorphism between $\mathcal{A}^+_X$ and $\mathcal{B}^\infty_{X^\vee}$ identifying the respective vector bundles with connection. In this way, (1) is just a preliminary of mirror symmetry; its proof was given in the case of the quintic three-fold by Givental [56] and Lian–Liu–Yau [92]. More precisely, there exist an isomorphism

$$\begin{array}{c} \mathcal{B}^\infty_{X^\vee} \\ \updownarrow \cong \\ \mathcal{A}^+_X \end{array} \qquad (4)$$

compatible with the two local systems.

[1] In intuitive terms one can describe easily the flat sections of this connection: since the family is locally trivial on Ω, the cohomology classes defined on a fiber of $V \to \Omega$ can be constantly continued as soon as we trivialize $(V, R^3\pi_*\mathbb{Z})$ over an open contractible set. These continuations are the flat section with respect to Gauss–Manin connection.

Beyond the local statement. It is natural to observe that, on the B side we only retained the information at a neighborhood of the point at infinity of the local system $(V, R^3\pi_*\mathbb{Z})$ naturally defined over the Zariski open set Ω of $\mathbb{P}^1$. Indeed it is natural to define $\mathcal{B}_{X^\vee}$ as the above structure without any restriction. More precisely $\mathcal{B}_{X^\vee}$ is the entire projective line equipped, over the open subspace Ω, with the local system $(V, R^3\pi_*\mathbb{Z})$. We ask the natural question of extending mirror symmetry by defining a mathematical object $\mathcal{A}_X$ matching $\mathcal{B}_{X^\vee}$. The first step in this direction is the identification of an A model invariant attached to the quintic three-fold X mirror to the the restriction of $\mathcal{B}_{X^\vee}$ to a chart near 0 in the projective line. In [31] we identify this object as the enumerative geometry of curves equipped with a level structure. In complete analogy with Gromov–Witten theory we can define a local system on a disc, which we denote $\mathcal{A}_X^-$. In Theorem IV/2.2.1 of [31] we provide a mirror map

$$\begin{array}{c} \mathcal{B}^0_{X^\vee} \\ \big\updownarrow \cong \\ \mathcal{A}^-_X \end{array} \tag{5}$$

compatible with the local systems. We extend in this way the local mirror symmetry statement.

$$\begin{array}{ccccc} \mathcal{B}^0_{X^\vee} & \longrightarrow & \mathbb{P}^1 & \longleftarrow & \mathcal{B}^\infty_{X^\vee} \\ \big\updownarrow \cong & & & & \big\updownarrow \cong \\ \mathcal{A}^-_X & & & & \mathcal{A}^+_X \end{array} . \tag{6}$$

The idea of completing a mirror image is a crucial motivation of a number of important papers in mathematics and physics. This completion is often called *Landau–Ginzburg model.* As discussed above, $\mathcal{A}^+_X, \mathcal{B}^\infty_{X^\vee}$ and $\mathcal{B}^0_{X^\vee}$, are clearly identified: they arise from the CY geometry of X, the geometry of $X^\vee_\psi$ for $\psi \to \infty$, and that of $X^\vee_\psi$ for $\psi \to 0$. The missing piece is the Landau–Ginzburg model, which allows us to define the bundle with connection on $\mathcal{A}^-_X$. This text provides a detailed presentation of this object. Up to this point our main goal has been to explain as clearly as possible this natural effort to extend a mirror image. This seems a good spot to reproduce a quotation inspired by the title of Morrison's paper "*Through the looking glass*" [102].

> *"I'll tell you all my ideas about Looking-glass House. First, there's the room you can see through the glass — that's just the same as our drawing room, only the things go the other way. I can see all of it when I get upon a chair — all but the bit behind the fireplace. Oh! I do so wish I could see that bit! I want so much to know whether they've a fire in the winter: you never can tell, you know, unless our fire smokes, and then smoke comes up in that room too? but that may be only pretence, just to make it look as if they had a fire."* Through the Looking Glass, by Lewis Carroll.

Higher genus

The correspondence between Landau–Ginzburg (LG) model and the enumerative geometry of Calabi–Yau (CY) varieties follows from the global mirror symmetry described above. Via mirror symmetry, it reflects the correspondence between $\mathcal{B}^\infty_{X^\vee}$ and $\mathcal{B}^0_{X^\vee}$, two restriction of the same geometrical object. It deserves special attention, independently of mirror symmetry, because it provides us with a new approach to the computation of Gromov–Witten of CY varieties.

LG-CY as an approach to Gromov–Witten theory. We recall that the enumeration of curves of any genus traced on CY varieties — *i.e.* the Gromov–Witten theory of CY varieties — has been a central problem in mathematics and physics for the last twenty years. Many techniques have been developed, but the actual computation still eludes both mathematicians and physicists. Consider the above example of the smooth CY quintic three-fold X. Here, the most advanced effort is Huang, Klemm, and Quackenbush's speculation [69], via a physical argument, on Gromov–Witten invariants in high genus; it is striking however that, even with these far-reaching techniques, there is no prediction beyond $g = 52$. We should also mention that in mathematics several general methods have been recently found [93, 97] and can in principle determine Gromov–Witten invariants in a wide range of cases. These methods, however, are hard to put into practice both when calculating a single invariant and when one needs to effectively compute the full higher genus Gromov–Witten theory. For the above quintic three-fold, after the famous genus-zero computations [59, 92], the genus-one theory has been determined after a great deal of hard work by Zinger [126]. Computations in higher genera remain out of mathematicians' reach for the moment.

In this current unsatisfactory state of affairs, a natural idea is to push through the LG-CY correspondence as a precise mathematical statement in terms of enumerative geometry of curves and use the computational power of the LG model as an effective method for determining the higher genus Gromov–Witten invariants of the CY manifold. Providing a rigorous definition of the LG counterpart to Gromov–Witten theory of CY manifolds is a first step towards establishing a geometric LG-CY correspondence and is likely to be interesting in its own right. For instance, in a different context, the LG-CY correspondence led to identify matrix factorization as the LG counterpart of the derived category of complexes of coherent sheaves — Orlov's equivalence [105] (see also [67]).

Witten's insight into the Landau–Ginzburg model. In [125] Witten provides a purely mathematical introduction to the Landau–Ginzburg model. Its construction fits in the formalism of geometric invariant theory and suggests an interpretation of the Landau–Ginzburg model as a quantum theory of the singularity at the origin of the function $\mathbb{C}^5 \to \mathbb{C}$ defined by $x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$. (The argument of Witten is reviewed in Section IV/3.1.) The first instance of such a quantum singularity theory is Witten's quantum theory [123] of the simplest singularity; namely, the ramification point of $x \mapsto x^2$ (A_1-singularity). This led to an enumerative theory of curves paired with square roots of the canonical line bundle — *theta characteristics*. It was proven by Kontsevich [81] that the quantitative theory set up by Witten is governed by the Korteweg–de Vries hierarchy (the integrable hierarchy attached to the simple singularity of type A_1). This was generalized to all type-A singularities by Faber, Shadrin, and Zvonkine [50] using the theorem on universal relations of Y. P. Lee [90] and the connection between KdV hierarchies and type-A singularities of Givental [58]. This result relies on a great deal of work to set up an analogue of the A_1 theory: we should quote the early papers of Witten himself [123, 125] as well as [4, 22, 28, 25, 26, 49, 72, 75, 73, 74, 76, 89, 90, 87, 88, 99, 109, 108, 116].

In [51, 52, 53], Fan, Jarvis and Ruan constructed an enumerative theory of curves, which is expected to provide a complete counterpart for Gromov–Witten theory in the context of singularity theory. The formulation of this general Landau–Ginzburg setup presents some analogies with Gromov–Witten theory, but it involves a radical change of perspective. Instead of enumerating curves equipped with maps to a target, in the Landau–Ginzburg model, we study curves C equipped with line bundles L (level structures): $L^{\otimes l} \cong \mathcal{O}$, or more generally

$$L^{\otimes l} \cong \omega^{\otimes s},$$

for integers $l \geq 1$ and $s \geq 0$.

In [26] and [28] we provide a systematic approach to the enumerative geometry of these structures. This leads to Theorem I/3.2.1 generalizing Mumford's theorem on stable curves to the case of curves with level structures. Jointly with Yongbin Ruan, in genus zero and in the case of the quintic polynomial W, we use this result to compute the the Landau–Ginzburg enumerative theory of curves defined in [51, 52, 53]. Furthermore, we show that it is equivalent to that of Gromov–Witten invariants of the quintic three-fold, see Theorem IV/2.2.1 and Corollary IV/2.3.1. We obtain in this way a conjectural formula relating the two theories in higher genera.

Structure of this text

In Chapter I we present the theory of level structures: their moduli and the compactified moduli space. In Chapter II we use some of the results on level structures to set up the Landau–Ginzburg model. In Chapter III we present the mirror symmetry framework at the level of cohomology. In Chapter IV we present the global mirror symmetry beyond the cohomology level.

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Chapter I

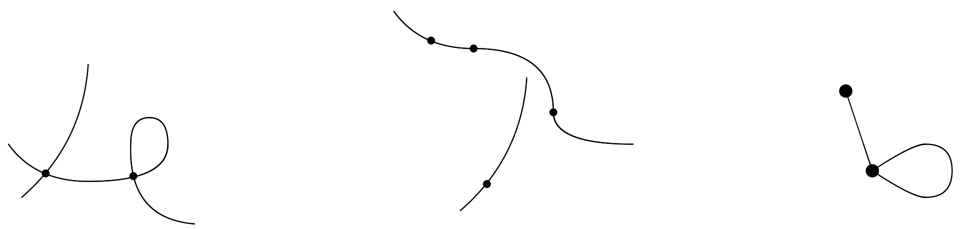
Level structures

1 Prerequisites on moduli of curves

We recall the main definitions of the theory of moduli of curves. We refer to [8, 9] for more details.

We work with algebraic curves of genus g. More precisely, we only work with schemes over the complex numbers, and we write nodal curve of genus g for a projective, one-dimensional scheme, reduced and geometrically connected, having only ordinary double points as singularities. We will usually refer to these singularities as *nodes*. A curve C has genus g when $h^0(\omega)$ equals g.

The notion of normalization allows us to make the geometry of C more explicit. We recall that the normalization of a curve at a node $p \in C$ is a finite morphism $\nu_p \colon C^\nu_p \to C$ (invertible over $C \setminus \{p\}$ and with two distinct point in C^ν mapping to p). The curve C^ν_p is either a nodal connected curve, or the disjoint union of two curves: in the second case we will write that p is a *separating* node. The genus of C can be expressed explicitly by considering the normalization of C; *i.e.* the morphism $\nu \colon C^\nu \to C$, from a smooth curve C^ν to C, given by normalizing C at all nodes. Let V be the set of connected components $C_1, \dots, C_v$ of C^ν, let E be the set of



curve C normalization $C^\nu = \bigsqcup C_j$ dual graph Γ

Figure I.1: the curves C, its normalization C^ν, its dual graph Γ.

nodes of C; then, the two branches of each node indentify a non ordered pair of elements of V. Giving (V, E) is the same as giving a graph Γ, which we call the *dual graph* of C. We have a genus formula

$$g(C) = \sum_{i=1}^{v} g(C_i) + b_1(\Gamma),$$

where b_1 is the first Betti number of the graph Γ.

The category of nodal curves. A family of nodal curves of genus g over a base scheme B is given by a morphism $C \to B$, which is flat and proper, and has as fiber on each geometric point $b \in B$ a nodal curve C_b of genus g. A morphism from the family $C \to B$ to the family $C' \to B'$ is a morphism of schemes $m \colon C \to C'$ fitting in the fiber diagram

$$\begin{array}{ccc} C & \xrightarrow{m} & C' \\ \downarrow & \square & \downarrow \\ B & \longrightarrow & B'. \end{array}$$

In this way the families of nodal curves form a category, which is not represented by an algebraic space. Indeed, due to the presence of curves with nontrivial automorphisms, the notion of algebraic space is not well suited to represent moduli of nodal curves and, more generally, categories classifying families of varieties. Algebraic stacks are more general than algebraic spaces, and allow geometric points with nontrivial automorphism groups. Hence, they provide us with the suitable framework for representing moduli of curves. In particular, the category of nodal curves is represented by an algebraic stack.

Algebraic stacks. There is a wide spectrum of types of algebraic stacks. Here we will limit ourself to algebraic stacks which are locally isomorphic to quotient stacks $[U/G]$, where U is an affine space and G is a finite group. There are only two exceptions: the case of $U = \operatorname{Spec}(\mathbb{C}[x,y]/(xy))$ modulo $G = \mathbb{Z}/l$ (operating by multiplication on each coordinate) and the case of $U = \mathbb{C}^n$ modulo $G = \mathbb{C}^\times$ (operating with weights $w_1, \dots, w_n$). We recall that the stack $[U/G]$ is the category of G-torsors T over a base scheme B equipped with a morphism $T \to U$ compatible with the G-action, see [121]. We write $\mathsf{B}G$ in the special case where $U = \operatorname{Spec}\mathbb{C}$; *i.e.* for the category of G-torsors.

Stability. The stack of nodal curve does not fit in the above local description. Furthermore, it is not separated, which makes it very hard to study its enumerative geometry. In order to reduce it to a separated stack, Deligne and Mumford have defined the notion of stable curve: a nodal curve is *stable* if its canonical bundle is ample. Let us notice, to begin with, that there is no stable curve of genus $g \le 1$. On the other hand, as soon as g is larger than 1, then this notion of stability identifies a subcategory containing the category of smooth curves, and represented by a separated algebraic stack. This stack, which we will denote by $\overline{\mathsf{M}}_g$, is in fact smooth, proper, and fits in the local description by quotient stacks $[U/G]$ given above.

Deligne–Mumford stack. In fact, the algebraic stack $\overline{\mathsf{M}}_g$ satisfies further geometric properties which have been first stated and proven in [44]. Among these properties, the finiteness of the *stabilizer* group $\operatorname{Aut}(\mathsf{p})$ for each point p. More precisely $\overline{\mathsf{M}}_g$ is a category equipped with a functor to the category of schemes, a geometric point p is an object over $\operatorname{Spec}\mathbb{C}$, and we write $\operatorname{Aut}(\mathsf{p})$ for the automorphism group of p as an object of the fibred category over $\operatorname{Spec}\mathbb{C}$. We refer to [5, Defn. 2.1.(1-3)] for a concise list of the properties of [44]; in the recent literature they are usually summarized by the words "the stack is Deligne–Mumford". The notion of Deligne–Mumford stack can be regarded as as the algebraic analogue of the notion of orbifold. We recall here a fundamental property of Deligne–Mumford stacks.

Existence of the reification. If X is a Deligne–Mumford stack, we write X for the universal object with respect to morphisms from X to algebraic spaces. By construction, this object is equipped with a morphism $\epsilon_{\mathsf{X}} \colon \mathsf{X} \to X$. Explicitly, for all algebraic spaces Y and all $h \colon \mathsf{X} \to Y$, there exists $X \to Y$ commuting with $\epsilon_{\mathsf{X}} \colon \mathsf{X} \to X$ and h. By [82], X is represented by an algebraic space. As a consequence, if f is a morphism of Deligne–Mumford stacks, we get a corresponding morphism of algebraic spaces, which we note by f. We write X and f the *reifications* of X and f. (We point out that X is usually referred to as the coarse space in the literature; here we prefer the term *reification*, which applies to f as well as to X.) In this way the stack $\overline{\mathsf{M}}_g$ admits a reification $\overline{M}_g$ represented by an algebraic space; Deligne and Mumford show that the algebraic space $\overline{M}_g$ is in fact a projective scheme of dimension $3g-3$.

Marked points. There is a natural generalization of the notion of nodal curve of genus g: the notion of nodal curve equipped with n distinct smooth points $s_1, \dots, s_n$. The genus is still given by $h^0(\omega)$. A curve of this type is stable if $\omega(\sum_i [s_i])$ is ample; clearly there is no genus-g n-pointed curve for $2g-2+n \le 0$. We assume $2g-2+n > 0$. We use the notation $\overline{\mathsf{M}}_{g,n}$ for the stack classifying n-pointed genus-g stable curves. It is a Deligne–Mumford stack, smooth, proper, of dimension $3g-3+n$. Its substack $\mathsf{M}_{g,n}$ is the subcategory n-pointed genus-g smooth curves.

Boundary morphisms. We recall the natural morphisms

$$\overline{\mathsf{M}}_{g_1,n_1+1} \times \overline{\mathsf{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathsf{M}}_{g_1+g_2,n_1+n_2}$$

and

$$\overline{\mathsf{M}}_{g-1,n+2} \longrightarrow \overline{\mathsf{M}}_{g,n}$$

naturally defined by identifying two points. These morphisms, which we will revisit in the context of level structures, allow to study the boundary $\partial\overline{\mathsf{M}}_{g,n} = \overline{\mathsf{M}}_{g,n} \setminus \mathsf{M}_{g,n}$.

2 Moduli of level structures

In this section we review the definition of the moduli space of level structures R_g^l and its compactification. It is natural to introduce R_g^l as the reification of a Deligne–Mumford stack R_g^l. We refer here to the terminology introduced in the previous section.

Let us fix $g \geq 2$ and $l \geq 1$; then, R_g^l is the category of smooth genus-g curves C equipped with a line bundle L and an isomorphism $\phi\colon L^{\otimes l} \to \mathcal{O}_C$. We regard these data as triples (C, L, ϕ) and we refer to them as *level-l curves*. A family of level-l genus-g curves is a flat family $C \to B$ of smooth curves equipped with L and ϕ on C as above. There is a natural notion of morphism from a family $(C \to B, L, \phi)$ to another family $(C' \to B', L', \phi')$; this is given by a pair (s, ρ) where s fits into a fibre diagram

$$\begin{array}{ccc} C & \xrightarrow{s} & C' \\ \downarrow & \square & \downarrow \\ B & \longrightarrow & B' \end{array}$$

and ρ is an isomorphism of line bundles $s^*L'' \to L'$ satisfying $\phi' \circ \rho^{\otimes l} = s^*\phi''$. The category R_g^l is a Deligne–Mumford stack. In particular, its points have finite stabilizers and there exists a reification R_g^l and a morphism

$$\epsilon\colon \mathsf{R}_g^l \to R_g^l.$$

Notice that the forgetful morphism $\mathsf{f}\colon \mathsf{R}_g^l \to \mathsf{M}_g$ is a finite morphism with constant fiber. The fiber (pullback of f via a geometric point) is formed by l^{2g} points; as many as the elements of $\mathrm{Pic}(C)[l]$ for any smooth curve C. It is important to notice that each of these points is isomorphic to the stack $\mathsf{B}\mathbb{Z}/l$. This happens because each point has *quasi-trivial* automorphisms acting on C as the identity (*i.e.*, α equals id_C), and scaling the fibers of L by an lth root of unity (*i.e.*, $\rho\colon z \mapsto \xi_l z$). These automorphisms are forgotten in the reification $f\colon R_g^l \to M_g$. The morphism f is still finite, but it may well be ramified.

2.1 The compactification: the problem.

The stack R_g^l is not compact (it maps onto M_g which is not compact). As a first approach one may consider allowing triples $(C_{\mathrm{st}}, L, \phi)$ where C_{st} is a stable genus-g curve in the sense of Deligne–Mumford (a nodal curve whose canonical bundle is ample). This does not work. Indeed, the proper forgetful morphism $\mathsf{f}\colon \mathsf{R}_g^l \to \mathsf{M}_g$ extends to a morphism from a category of triples $(C_{\mathrm{st}}, L, \phi)$ to the Deligne–Mumford moduli stack of stable curves $\overline{\mathsf{M}}_g$; it is easy to see that this extended morphism is not proper.

Remark 2.1.1. The forgetful morphism $\mathsf{f}\colon \mathsf{R}_g^l \to \mathsf{M}_g$ as well as the extended morphism to $\overline{\mathsf{M}}_g$ mentioned above are étale[1]. Then properness holds as soon as the fiber is constant. Consider a one-parameter smoothing of a one-noded irreducible curve. The following sequence shows

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{Pic}^0(C) \xrightarrow{\nu^*} \mathrm{Pic}^0(C^\nu) \longrightarrow 1, \tag{I.1}$$

that for the smooth fibers there are l^{2g} distinct lth roots of $\mathcal{O}$, whereas for the singular fibre the number of roots[2] of $\mathcal{O}$ is l^{2g-1}. Hence f is not proper.

2.2 Orbifold curves

The solution to this problem is remarkably simple, once we find the suitable notion of curve. This is the notion of orbifold (or twisted) curve developed by Abramovich and Vistoli [5] in the context of stable maps to a stack. Here by *orbifold curve* we mean a one-dimensional stack C of Deligne–Mumford type whose generic stabilizer is trivial, whose singularities are ordinary double points and whose reification C is a stable curve. We also add an extra condition to insure that a twisted curve can always be smoothed. Explicitly, locally at a node, consider a stabilizer $G \cong \mathbb{Z}/k$ and the local picture of the curve $\{xy = 0\} \subset \mathbb{C}^2$. In order for C to be smoothable, we always require that $k \in \mathbb{Z}/l$ acts as $k \cdot (x, y) = (\xi_l^k x, \xi_l^{-k} y)$.

Remark 2.2.1 (unmarked orbifold curves)**.** Up to Section 2.5 we do not consider marked curves and we always assume that all smooth points have trivial stabilizers.

[1]This happens because the relative cotangent complex of $\mathsf{B}\mathbb{C}^* \to \mathsf{B}\mathbb{C}^*$; $\lambda \mapsto \lambda^l$ is trivial.

[2]It is easy to generalize the above example and show that the number of points in the fiber of the morphism forgetting the l-level structure drops by a factor l^m every time we have a specialization where the Betti number of the dual graph increases by m.

All basic notions used for stable curves extend word for word to twisted curves: the genus $h^0(\omega_{\mathsf{C}})$, the normalization, and the dual graph (see Figure I.1). The vertices are in one-to-one correspondence with the connected components of the normalization; the edges are in one-to-one correspondence with the nodes. We have the classical genus formula $g = b_1(\Gamma) + \sum_j g(\mathsf{C}_j)$.

Remark 2.2.2. Orbifold curves form a non-separated stack [106] (just as usual nodal curve do). Consider a family of orbifold curves $\mathsf{C} \to \Delta$, where Δ may be thought as a one-dimensional scheme with a special point (*e.g.* the spectrum of a discrete valuation ring R). On the generic point (the spectrum of the fraction field K) we have a smooth curve $C^\times \to \Delta^\times = \operatorname{Spec} K$. On the special point we assume that C is an orbifold curve with nontrivial stabilizers. The reification C of C yields a family $C \to \Delta$ which differs from $\mathsf{C} \to \Delta$; by definition it is still an orbifold curve (but a representable one). In other words there exists two distinct "models" of $C^\times \to \Delta^\times$: the orbifold curves $\mathsf{C}' = \mathsf{C} \to \Delta$ and $C \to \Delta$. Hence, the stack of orbifold curve is non-separated.

As it already occurred for nodal curves, we need a notion of stability yielding a nicely behaved substack. In [26] we describe all the stability conditions yielding a separated stack and we show that they all yield a proper and even smooth compactification of M_g (these compactifications differ from the moduli space of Deligne–Mumford stable curves only because of the stabilizers; the stack is an root construction over the normal crossings boundary divisor, see discussion in [26, Thm. 4.1.6]). Throughout this text we will constantly use a canonical choice: we simply impose that all stabilizers at all nodes have order l. In this way the stack of *l-stable orbifold curves* is a proper Deligne–Mumford stack $\overline{\mathsf{M}}_g(l\text{-st})$ providing a new compactification of M_g; an enriched version of $\overline{\mathsf{M}}_g$ with extra $\mathbb{Z}/l$-stabilizers on the boundary locus $\overline{\mathsf{M}}_g \setminus \mathsf{M}_g$. We conclude this digression by stating precisely the notion of unmarked l-stable curve.

Definition 2.2.3. An l-stable orbifold curve is a proper and connected Deligne–Mumford stack of dimension one with singularities of type nodes, trivial stabilizers on smooth points and stabilizers of order l at the nodes.

2.3 Compactified moduli of level-l curves

Now the same naive idea which failed with stable curves works: we can define *level-l curves* as triples $(\mathsf{C}, \mathsf{L}, \phi)$ where C is an l-stable curve, L is line bundle on it, and ϕ is an isomorphism of line bundles $\mathsf{L}^{\otimes l} \to \mathcal{O}$.

Definition 2.3.1. A stable level-l curve of genus g on a base scheme B consists of a triple $(\mathsf{C} \to B, \mathsf{L}, \phi)$, where $\mathsf{C} \to B$ is a family of genus-g l-stable curves, L is a line bundle, and $\phi\colon \mathsf{L}^{\otimes l} \to \mathcal{O}_{\mathsf{C}}$ is an isomorphism of line bundles.

We denote by $\overline{\mathsf{R}}_g$ the category of stable l-level curves of genus g. It is a proper stack of dimension $3g-3$. Furthermore it is étale over $\overline{\mathsf{M}}_g^{l\text{-st}}$ (and fibered in finite groups over $\overline{\mathsf{M}}_g^{l\text{-st}}$, see [26]). The reification $\overline{R}_g$ of $\overline{\mathsf{R}}_g$ is isomorphic to the normalization of $\overline{M}_g$ in the function field of R_g.

Remark 2.3.2 (multiplicity of a level-l structures at a node)**.** Consider a level-l curve $(\mathsf{C}, \mathsf{L}, \phi)$. Let us describe the local picture of C and of L at a node p; the description depends on the choice of a branch of the node, which plays a privileged role. The stabilizer G of p is a cyclic group of order l; we can choose a generator g acting on the node $\{xy = 0\}_{\mathbb{C}^2} = \operatorname{Spec}(\mathbb{C}[x,y]/(xy))$ as $g \cdot (x,y) = (\xi_l x, \xi_l^{-1} y)$ where x is the local parameter on the privileged branch. Then, let us regard L as a 2-dimensional stack (the total space of the bundle) and the local picture at the node as $\{xy = 0\}_{\mathbb{C}^3} =\subset \operatorname{Spec}(\mathbb{C}[x,y,\lambda]/(xy))$. The generator g of G operates on this space as

$$g \cdot (x,y,\lambda) = (\xi_l x, \xi_l^{-1} y, \xi_l^M \lambda) \qquad \text{(we write } \tfrac{1}{l}(1, l-1, M)\text{)}.$$

Let us refer to $M \in \{0, \dots, l-1\}$ as the multiplicity index at the node of the l-level curve (with oriented branches). In Section 4, we point out that the multiplicities M define a cycle in the dual graph, see Proposition 4.1.1.

Remark 2.3.3 (the reification of l-level curves)**.** Consider a smooth component Z of the l-level curve $(\mathsf{C}, \mathsf{L}, \phi)$. The points $\mathsf{p}_1, \dots, \mathsf{p}_k$ where Z meets the rest of the curve are nodes of C with $\mathbb{Z}/l$-stabilizer and multiplicities $M_1, \dots, M_k$ (the privileged branch here is the one lying in Z). The restrictions (N, ν) of (L, ϕ) yields a root of $\mathcal{O}$ on Z. A natural question is to describe the direct image of (N, ν) via ϵ_*, where ϵ is the natural map from Z to the reification Z

$$\epsilon\colon \mathsf{Z} \to Z.$$

For line bundles on smooth orbifold curves, we introduce the notation

$$\lfloor \mathsf{N} \rfloor = \epsilon_* \mathsf{N}. \tag{I.2}$$

In fact, $\lfloor \mathsf{N} \rfloor$ is a line bundle because ϵ is flat. Via $\epsilon_*\nu$, we can regard it as an lth root. Explicitly, we have

$$\epsilon_*\nu\colon \lfloor \mathsf{N} \rfloor^{\otimes l} \longrightarrow \mathcal{O}_Z\left(-\textstyle\sum_{i=1}^k M_i[p_i]\right), \tag{I.3}$$

where $p_1, \dots, p_k$ are the reifications of $\mathsf{p}_1, \dots, \mathsf{p}_k$.

We refer to [26] for a systematic discussion. The idea is the following: we can express N as a line bundle associated to a divisor with rational coefficients

$$D \in \sum_{\mathsf{p} \in \mathsf{Z}} \frac{[\mathsf{p}]}{\#\operatorname{Aut}(\mathsf{p})} \mathbb{Z} = \operatorname{Div}(\mathsf{Z}), \tag{I.4}$$

where the summation runs on the geometric points of Z (which is is just the same as saying "the points of Z"). Then, D can be written as

$$D = \lfloor D \rfloor + \frac{M_1}{l}\mathsf{p}_1 + \dots + \frac{M_k}{l}\mathsf{p}_k,$$

where $\lfloor D \rfloor$ is the round-down divisor (in the sense of [80, 0.4.(14)]). Divisors with integer coefficients on Z can be expressed as pullbacks of a divisor on Z (uniquely determined up to linear equivalence); let us identify in this way $\lfloor D \rfloor$ as a divisor on Z. The line bundle $\lfloor \mathsf{N} \rfloor$ can be written as $\mathcal{O}_Z(\lfloor D \rfloor)$. The claim (I.3) follows easily.

In particular (I.3) means that $\sum_i M_i$ is a multiple of l and we have

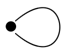
$$\#\{i \mid M_i > 0\} \neq 1. \tag{I.5}$$

If Z is not smooth, the analogous statement holds after normalization of D (in this case a node of the component Z lifts to two points in the normalization, with multiplicities in $\{0, \dots, l-1\}$ adding up to a multiple of l).

2.4 Examples of one-noded level curves

This section is devoted to examples, it does not provide any notation that will occur in the rest of the text, but it allows us to get a feeling for nodal level curves and for the structure of the boundary divisor of the moduli stack. We classify all possible *one-noded* level-2 curves; the reader may find a systematic discussion in [29]. Notice that, in this process, one determines the components of the normal crossings divisor forming the boundary locus $\overline{R}^l_g \setminus R^l_g$, see 3.1.

First recall that Deligne and Mumford's moduli of stable curves represent two types of one-noded curves: those of irreducible type and those of compact type. In other words their dual graphs are a loop ($b_1 = 1$) or a one-edged tree ($b_1 = 0$):

 , .

Irreducible one-noded curves (loop case) are parametrized by a connected locus in $\overline{\mathsf{M}}_g$; its closure forms a normal crossings divisor L^{st}. The divisor L^{st} may be described as the category of curves with *at least* one nonseparating node (a node n for which the normalization of the curve at n is connected).

The locus classifying one-noded curves of compact type (tree case) in $\overline{\mathsf{M}}_g$ consists of $\lfloor g/2 \rfloor$ connected components. Its closure T^{st} is the category of curves with at least one separating node (a node n for which the normalization of the curve at n has exactly two connected components).

In order to see this, and in view of later discussions, it is convenient to describe a two-folded étale cover of T^{st}. Consider the stack whose points represent curves with a choice of a node separating the curve into two components and a privileged branch for such node. This cover is the disjoint union

$$\mathbb{T}^{\mathrm{st}} = \bigsqcup_{0 \le i \le g} \mathbb{T}^{\mathrm{st}}_i$$

of connected loci labeled by the genus i of the subcurve containing the chosen branch (by Deligne–Mumford stability condition requiring ampleness of ω the loci $\mathbb{T}^{\mathrm{st}}_0$ and $\mathbb{T}^{\mathrm{st}}_g$ are empty).

The morphism forgetting the choice of the branch and of the node maps $\mathbb{T}^{\mathrm{st}}$ to the divisor T^{st} in $\overline{R}^l_g$; it sends $\mathbb{T}^{\mathrm{st}}_i$ to T^{st}_i so that the labellings i and $g - i$ occur for the same divisor. The case of $T^{\mathrm{st}}_{g/2}$ is special, the divisor is given by multiplying by 2 the image of $\mathbb{T}^{\mathrm{st}}_{g/2}$. In this way we have

$$T^{\mathrm{st}} = \frac{1}{2} \sum_{0 < i < g} T^{\mathrm{st}}_i.$$

Remark 2.4.1. All these facts generalize immediately to l-stable curves: the analogue loci $L^{l\text{-st}}$ and $T^{l\text{-st}}$ are normal crossings divisors. They are equipped with the same kind of two-folded étale cover classifying the branches:

$$\mathbb{L}^{l\text{-st}} \to L^{l\text{-st}}, \text{ and } \mathbb{T}^{l\text{-st}} \to T^{l\text{-st}}.$$

The stacks $\mathbb{L}^{l\text{-st}}$ and $L^{l\text{-st}}$ are connected. For $\mathbb{T}^{l\text{-st}}$ we have the same decomposition and $T^{l\text{-st}} = \frac{1}{2}\sum_{i=1}^{g-1} T_i^{l\text{-st}}$.

Example 2.4.2 (level-2 curves of compact type)**.** Consider a one-nodal curve of compact type; *i.e.* let $\mathsf{C} = \mathsf{C}_1 \cup \mathsf{C}_2$ be a union of two smooth orbifold curves C_1 and C_2 of genus i and $g-i$ meeting transversally at a orbifold point $B\mathbb{Z}/2$. We use Remark 2.3.3, (I.5) to note that the multiplicity M at the node is 0. Since C_1 identifies one branch, we may write the local structure as $\frac{1}{2}(1,1,0)$.

The line bundle L on C is determined by the choice of two line bundles L_1 and L_2 satisfying $L_1^{\otimes 2} \cong \mathcal{O}_{C_1}$ and $L_2^{\otimes 2} \cong \mathcal{O}_{C_1}$ where C_1 and C_2 are the reifications of C_1 and C_2 (see (I.3)). In this way, L_1 or L_2 may be of order 2 or of order 1 (trivial). In this way the stack $\mathbb{T}$ classifying 2-level curves with a choice of a separating node and a the choice of a branch decomposes into 4 open and closed substacks

$$\mathbb{T} = \bigsqcup_{a,b=1 \ or\ 2} \mathbb{T}^{[a],[b]}.$$

This holds over each divisor $T_i^{2\text{-st}}$; we have exactly four connected components lying over $T_i^{2\text{-st}}$

$$T_i^{[1],[1]} \sqcup T_i^{[2],[1]} \sqcup T_i^{[1],[2]} \sqcup T_i^{[2],[2]} = \mathsf{f}^* T_i^{2\text{-st}},$$

where f is the unramified forgetful morphism $\overline{\mathsf{R}}_g^2 \to \overline{\mathsf{M}}_g^{2\text{-st}}$.

Example 2.4.3 (irreducible one-nodal curves)**.** If C is irreducible and has one node, then the node is of nonseparating type: when we normalize at that point, the curve remains connected. As mentioned above, we may regard these curves as loops, because the dual graph is a loop. We can describe the reification of C as $C_{p_1,p_2} := N/p_1 \sim p_2$, where $[N, p_1, p_2]$ if a smooth genus-$(g-1)$ 2-pointed curve.

The study of C allows us to provide a decomposition into irreducible components of the divisor L of singular 2-level curves with at least one nonseparating node. It lies over $L^{2\text{-st}} \subset \overline{\mathsf{M}}_g^{2\text{-st}}$ and we show that it consists of 4 connected components. Again, it is convenient to look first at $\mathbb{L}$ classifying l-level curves, with a choice of a node of nonseparating type *and* a privileged branch for such node. In this way the multiplicity at the nonseparating node is well defined. The natural forgetful map $\mathbb{L} \to L$ turns out to preserve the four connected components, which can be described as follows

1. Assume that the multiplicity M equals 0. The level structure is a pullback of a 2-torsion bundle on C_{p_1,p_2}. Let $\nu\colon N \to C_{p_1,p_2}$ be the normalization map, then there is the exact sequence (I.1). Thus L is determined by a line bundle $L_N := \nu^*(L) \in \mathrm{Pic}^0(C)$ together with an identification of the fibers $L_N(p_1)$ and $L_N(p_2)$. If L_N is trivial, there is a canonical identification induced by $L_N \cong \mathcal{O}$. We get a trivial level structure.

2. The remaining nontrivial identification yields a second connected locus of level curves within $\mathbb{L}$. This is the locus of level curves corresponding to Wirtinger covers, see also [54, Exa. 1.4].

3. Assume $L_N \ncong \mathcal{O}_N$. For each such choice of $L_N \in \mathrm{Pic}^0(N)[l]$ there are 2 ways to glue $L_N(p_1)$ and $L_N(p_2)$. This yields another connected component in $\mathbb{L}$.

4. Finally we consider the case $M = 1$. The local picture at the node is given by $1/2(1,1,1)$ (the privileged branch is the one identified by p_1). The line bundle L is determined by a line bundle $\mathsf{L}_\mathsf{N} = \nu^*(\mathsf{L})$ on the normalization $\nu\colon \mathsf{N} \to \mathsf{C}$ together with an identification of the fibers over the node. By [2, §7], the automorphism group of the curve C is a $\mathbb{Z}/2$-extension of $\mathrm{Aut}(C_{p_1,p_2})$ and — due to these extra involutions — any two identifications of the fibers over the node can be identified via a nontrivial involution of C (see [26] and [29]).

By projecting on $\overline{\mathsf{R}}_g^2$, we have the following decomposition, with labellings referring to the above list of cases

$$L^{\text{trivial}} \sqcup L^{\text{Wirtinger}} \sqcup L^{M=0,\text{nontriv.}/N} \sqcup L^{M\neq 0} = \mathsf{f}^* L^{2\text{-st}}.$$

2.5 Variants

One of the main advantages of the above approach is that it applies systematically to lth roots of any line bundle defined on the moduli space of curves. In order to introduce all possible variants in full generality for n-pointed genus-g curves, we recall known facts about marked curves.

Assume that g and n are nonnegative integers satisfying $2g-2+n>0$. Consider a nodal curve with n ordered, distinct, and smooth markings $(C;\sigma_1,\dots,\sigma_n)$. We recall that $(C;\sigma_1,\dots,\sigma_n)$ is stable if $\omega_{\log}=\omega(\sum_{i=1}^n[\sigma_i])$ is ample.

Remark 2.5.1. So far we have only considered orbifold curves with trivial stabilizers on the smooth locus and $\mathbb{Z}/l$-stabilizers at the nodes. It is worth noticing that there is a unique way to add a $\mathbb{Z}/l$-stabilizer at a marking $D\in \mathsf{C}$. This has been described by Cadman [20] and Vistoli [1] as the category classifying the l-roots of the normal bundle $\mathcal{O}_{\mathsf{C}}(D)$.

Giving an l-stable $(\mathsf{C};s_1,\dots,s_n)$, with $\mathbb{Z}/l$-stabilizers only at the nodes, is equivalent to giving a one-dimensional stack $(\mathsf{C}';\mathsf{s}_1,\dots,\mathsf{s}_n)$ equipped of a morphism

$$\mathsf{p}\colon \mathsf{C}'\to\mathsf{C}$$

invertible on $\mathsf{C}\setminus\{s_1,\dots,s_n\}$ mapping marked points $\mathsf{s}_1,\dots,\mathsf{s}_n$, each of them with $\mathbb{Z}/l$-stabilizer, on the points $s_1,\dots,s_n$.

Remark 2.5.2. We can compare the canonical bundles $\omega_{\mathsf{C}}=\mathcal{O}(K_{\mathsf{C}})$ and $\omega_{\mathsf{C}'}=\mathcal{O}(K_{\mathsf{C}'})$. In the notation (I.4), we have a Riemann–Hurwitz formula

$$K_{\mathsf{C}'}=\mathsf{p}^*K_{\mathsf{C}}+\sum_{i=1}^n\frac{l-1}{l}[\mathsf{s}_i].$$

Hence $\mathsf{p}^*\omega_{\mathsf{C}}$ differs from $\omega_{\mathsf{C}'}$. On the other handle, for all i, we have

$$\mathcal{O}_{\mathsf{C}'}\left(\frac{[\mathsf{s}_i]}{\#\operatorname{Aut}(\mathsf{s}_i)}\right)^{\otimes l}=\mathsf{p}^*\mathcal{O}_{\mathsf{C}}\left(\frac{[s_i]}{\#\operatorname{Aut}(s_i)}\right)$$

where $\#\operatorname{Aut}(s_i)=1$ and $\#\operatorname{Aut}(\mathsf{s}_i)=l$. The two equations above imply

$$\omega_{\mathsf{C}'}\left(\sum_{i=1}^n\frac{[\mathsf{s}_i]}{\#\operatorname{Aut}(\mathsf{s}_i)}\right)=\mathsf{p}^*\omega_{\mathsf{C}}\left(\sum_{i=1}^n\frac{[s_i]}{\#\operatorname{Aut}(s_i)}\right),$$

which we can write as

$$\omega_{\log,\mathsf{C}'}=\mathsf{p}^*\omega_{\log,\mathsf{C}}$$

once we identify $\omega_{\log,\mathsf{C}'}$ with the left hand side of the previous equation. This shows why it is better to consider $\omega_{\log}$ in the context of orbifold marked curves.

Example 2.5.3. In fact, the above procedure applies also to families of curves (voir [20, Thm. 4.1]). One should take into account, however, the following subtlety. Over a base scheme B we assume that $C\to B$ is a family of scheme-theoretic curves, smooth and with only one marking. Then, clearly, it is equivalent to specify a subscheme D in C on B determined by the marking in each fiber, or to specify a section $B\to C$ sending each point of the base to the marking.

Conversely, let us suppose that $\mathsf{C}\to B$ is a family of smooth orbifold curves on a base scheme B, and with only one marking with stabilizer $\mathbb{Z}/l$ (the curve without the marking will be represented by a scheme). In this case it is not equivalent to specify the substack D specifying the marking in each fibre, or giving a section $B\to\mathsf{C}$ specifying for every $b\in B$ the marking in C_b. Indeed D projects onto B via a morphisms locally given by $\mathsf{B}\mathbb{Z}/l\to\operatorname{Spec}\mathbb{C}$. We remark that D it is not necessarily trivial; *i.e.* of the form $B\times\mathsf{B}\mathbb{Z}/l$. Giving a section $B\to\mathsf{C}$ specifying for all b the marking in C_b is equivalent to giving $\mathsf{D}\subset\mathsf{C}$ over B alongside with a trivialisation $\mathsf{D}\cong B\times\mathsf{B}\mathbb{Z}/l$.

The above example motivated the usual definition of family of marked orbifold curves.

Definition 2.5.4 (orbifold markings)**.** A family of orbifold curves $\mathsf{C}\to B$ is marked in n smooth points as soon as we specify n disjoint substacks

$$\mathsf{D}_i\hookrightarrow\mathsf{C},\qquad i=1,\dots,n, \tag{I.6}$$

mapping to the base B via a morphism locally isomorphic to $\mathsf{B}\mathbb{Z}/l\longrightarrow \mathrm{pt}$ (a $\mathbb{Z}/l$-gerbe).

Remark 2.5.5. Consider the stack $\overline{\mathsf{M}}^{l\text{-st}}_{g,n}$ of orbifold curves with a stable reification, n smooth and distinct markings, and $\mathbb{Z}/l$ stabilizers only at the nodes. As we showed, this stack can be also regarded as the stack classifying nodal orbifold curves with stable reification and $\mathbb{Z}/l$ stabilizers at the nodes *and at the markings.* We will always think of $\overline{\mathsf{M}}^{l\text{-st}}_{g,n}$ equipped with a universal curve

$$\mathsf{p}\colon \mathsf{C}^{l\text{-st}} \longrightarrow \overline{\mathsf{M}}^{l\text{-st}}_{g,n} \tag{I.7}$$

with $\mathbb{Z}/l$ stabilizers at the markings.

Via reification, above the moduli stack $\overline{\mathsf{M}}^{l\text{-st}}_{g,n}$, we also have a universal stable curve C^{st} whose fibers are ordinary orbifold curves, stable in the sense of Deligne–Mumford. As noted above, the curve is equipped with sections

$$\begin{array}{ccc} D_i & \longrightarrow & C^{\mathrm{st}} \\ & \nwarrow_{=} & \downarrow \\ & & \overline{\mathsf{M}}^{l\text{-st}}_{g,n} \end{array} \tag{I.8}$$

specifying the markings. For all i, there is a natural fiber diagram

$$\begin{array}{ccc} \mathsf{D}_i & \longrightarrow & \mathsf{C}^{l\text{-st}} \\ \downarrow & \square & \downarrow \epsilon \\ D_i & \longrightarrow & C^{\mathrm{st}}, \end{array}$$

where the horizontal arrows denote injective morphisms. We have the relation discussed above

$$\omega_{\log,\mathsf{C}^{l\text{-st}}} = \varepsilon^*\omega_{\log,C^{\mathrm{st}}}.$$

Definition 2.5.6 (psi classes)**.** We define

$$\psi_i = c_1(\omega_{\mathrm{rel}}|_{D_i}) \in H^2(\overline{\mathsf{M}}^{l\text{-st}}_{g,n};\mathbb{Q}) \qquad \forall i = 1,\dots,n$$

via the identity $D_i = \overline{\mathsf{M}}^{l\text{-st}}_{g,n}$ of equation (I.8).

We could define psi classes in a different way, as Chern classes of the restrictions of ω_{rel} on $\mathsf{C}^{l\text{-st}}$ to all each substack D_i for $i = 1,\dots,n$. One easily checks that this alternative definition boils down to dividing by l the above classes ψ_i. We follow the established practice that privileges, on this issue, the scheme-theoretic curve, see [3].

We can finally introduce a variant of level structures. For $l = 2$, $s = 1$, and $\boldsymbol{m} = \mathbf{1}$ we obtain the classical notion of theta characteristic or spin structure. Witten's notion of r-spin curve corresponds to $l = r$, $s = 1$, and $\boldsymbol{m} = \mathbf{1}$.

Definition 2.5.7. An n-pointed genus-g stable level-l curve *with respect to the line bundle* $\omega_{\log}^{\otimes s}$ on a base scheme S consists of a triple $(\mathsf{C} \to B, \mathsf{L}, \phi)$, where $\mathsf{C} \to B$ is a family of n-pointed genus-g l-stable (orbifold) curves (in $\overline{\mathsf{M}}^{l\text{-st}}_{g,n}$), L is a line bundle, and $\phi\colon \mathsf{L}^{\otimes l} \to \omega_{\log}^{\otimes s}$ is an isomorphism of line bundles.

For each fibre $(\mathsf{C}_s, \mathsf{L}_s, \phi_s)$, the local picture at the markings is the product of a complex line along the curve $\mathsf{C} = \operatorname{Spec}\mathbb{C}[x]$ and a complex line $\mathbb{C} = \operatorname{Spec}\mathbb{C}[\lambda]$ along L with $g \in \mathbb{Z}/l$ acting as

$$g(x,\lambda) = (\xi_l x, \xi_l^m \lambda) \qquad \left(\text{we write } \frac{1}{l}(1,m)\right)$$

for some $m \in \{0,\dots,l-1\}$. We refer to m as the type (or multiplicity) of the level structure at the marking.

We write $\overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m})$ for the category of level-l curves with respect to $\omega_{\log}^{\otimes s}$ of type $m_1,\dots,m_n$. The union over all types $\boldsymbol{m}$

$$\bigsqcup_{0\le m_1,\dots,m_n<l} \overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m}) \tag{I.9}$$

classifies level-l curves of any type with respect to $\omega_{\log}^{\otimes s}$. Clearly, in the above union, the terms $\overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m})$ are nonepty only if we have

$$(2g-2)s - \sum_{i=1}^{n}(m_i - s) \in l\mathbb{Z}$$

(see Remark 2.5.8).

We drop the indices s and $\boldsymbol{m}$ when s and $\boldsymbol{m}$ vanish (moduli of ordinary level-l curves). This is again a smooth $(3g-3+n)$-dimensional stack (it is étale over $\overline{\mathsf{M}}^{l\text{-st}}_{g,n}$ and equipped with a torsor structure with respect to the previous group stack $\overline{\mathsf{R}}^{l}_{g,n}$, see [26]).

Remark 2.5.8. Unravelling the above definition and using Remark 2.3.3 we point out that $\overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m})$ classifies lth roots of $\omega_{\log}^{\otimes s}(-\sum_{i=1}^{n} m_i D_i)$ on $\mathsf{C}^{l\text{-st}}$ with zero-multiplicities at the markings. Here, by a slight abuse of notation, $D_i \subset C^{\mathrm{st}}$ is regarded as a divisor on $\mathsf{C}^{l\text{-st}}$.

3 Intersection theory

We work in the rational cohomology ring of the stacks $\overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m})$. We already introduced the most important classes: the psi classes of (2.5.6), first Chern classes of the line bundles formed by cotangent lines at the markings. We provide a systematic treatment of the boundary $\overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m}) \setminus \mathsf{R}^{l,s}_{g,n}(\boldsymbol{m})$. Then, via boundary morphisms and psi classes, we compute the K theory pushforard of the level structure.

3.1 Boundary morphisms

Let S be the the singular locus in the universal curve $C^{\mathrm{st}} \to \overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m})$. Since the singularities are, fibre by fibre, ordinary double points (nodes), we can define $i : \mathbb{S} \to S$ as the double étale covering whose geometric points are nodes alongside with a choice of a branch. Section 2.4, can be regarded as an early example of this procedure.

The stack $\mathbb{S}$ is naturally equipped with two line bundles whose fibres are the cotangent lines to the chosen branch of C^{st} and to the other branch. We write

$$\psi, \psi' \in H^2(\mathbb{S}; \mathbb{Q}) \tag{I.10}$$

for their respective first Chern classes in $\mathbb{S}$. Note that, following established notation, we are systematically privileging the scheme-theoretic curve C^{st}: this happens because in this way the classes ψ and ψ' are more easily related to the classes ψ_i introduced in (2.5.6) (see Remark 3.1.2).

While in the case of moduli spaces of stable curves, the space $\mathbb{S}$ turns out to be a disjoint union of several smaller moduli spaces, the picture here is more complicated: the space $\mathbb{S}$ is not isomorphic, but can be projected to a disjoint union of smaller moduli spaces.

We can decompose $\mathbb{S}$ according to the topological type of the node. In this way we get the disjoint union

$$\mathbb{S} = \bigsqcup_{\substack{0\le i\le g\\ I\subseteq[n]}} \mathbb{S}_{i,I} \sqcup \mathbb{S}_{\mathrm{irr}},$$

where the substacks $\mathbb{S}_{i,I}$ and $\mathbb{S}_{\mathrm{irr}}$ are determined as follows:
(1) a point of $\mathbb{S}_{i,I}$ corresponds to a separating node and a branch lying on the connected component of genus i of the desingularisation carrying the markings $(s_j)_{j\in I}$,
(2) $\mathbb{S}_{\mathrm{irr}}$ corresponds to a nonseparating node, i.e. a node whose desingularization is connected.

Another natural decomposition is the disjoint union

$$\mathbb{S} = \bigsqcup_{0\le M\le l-1} \mathbb{S}^M,$$

where a point of $\mathbb{S}^M$ corresponds to a node and a branch on which L has multiplicity M. The following remark illustrates how the two decompositions above are related to each other.

Remark 3.1.1. All points of $\mathbb{S}_{i,I}$ have a common multiplicity index $M \in \{0,\dots,l-1\}$ satisfying

$$s(2i-2) - \textstyle\sum_I (m_i - s) \equiv M \mod l. \tag{I.11}$$

We denote this index by $M(i,I)$. On the other hand, on $\mathbb{S}_{\mathrm{irr}}$ the values of the index M range over the entire set $\{0,\dots,l-1\}$. Therefore, intersecting the above decompositions we get

$$\mathbb{S} = \bigsqcup_{\substack{0\le l\le g\\ I\subseteq[n]}} \mathbb{S}_{i,I} \sqcup \bigsqcup_{0\le M<l} \mathbb{S}^M_{\mathrm{irr}}.$$

The restrictions of $j = \pi \circ i$ to these components yield the morphisms

$$j_{i,I}\colon \mathbb{S}_{i,I} \longrightarrow \overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m}) \qquad \text{and} \qquad j^q_{\mathrm{irr}}\colon \mathbb{S}^M_{\mathrm{irr}} \longrightarrow \overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m})$$

as well as

$$j_M\colon \mathbb{S}^M = \left(\bigsqcup\nolimits_{M(i,I)=M} \mathbb{S}_{i,I}\right) \sqcup \mathbb{S}^M_{\mathrm{irr}} \longrightarrow \overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m}) \tag{I.12}$$

There exist natural morphisms from $\mathbb{S}_{i,I}$ and $\mathbb{S}^q_{\mathrm{irr}}$ to moduli stacks of level structures of smaller dimension.

For any $i \in \{0, \dots, g\}$ and for any $I \subseteq [n]$ write M for the integer $M(i,I)$ defined above. We have a morphism

$$\mathbb{S}_{i,I} \xrightarrow{\mu_{i,I}} \overline{\mathsf{R}}^{l,s}_{g,|I|+1}(\boldsymbol{m}_I, M) \times \overline{\mathsf{R}}^{l,s}_{g,|I'|+1}(\boldsymbol{m}_{I'}, M'),$$

where $I' = [n] \setminus I$ and $M' \in \{0, \dots, l-1\}$ is opposite to $M \mod l$. First, recall that an object $T \to S_{i,I}$ specifies an l-stable curve $\mathsf{L}_T \to \mathsf{C}_T$ over T with a T-node; *i.e.* a section n_T from T to the singular locus of the reification C_T of C_T. Furthermore, giving an object $T \to \mathbb{S}_{i,I}$ lifting $T \to S_{i,I}$ amounts to equip the above data with a lifting of the T-node to the desingularisation of C_T at n_T.

The functor $\mu_{i,I}$ assigns to each object $T \to \mathbb{S}_{i,I}$ as above the pair of level curves

$$\mathsf{L}_{T,1} \to \mathsf{C}_{T,1} \quad \text{and} \quad \mathsf{L}_{T,2} \to \mathsf{C}_{T,2}$$

induced by

1. the desingularisation $\mathsf{C}_{T,1} \sqcup \mathsf{C}_{T,2} \to \mathsf{C}_T$ of the l-stable curve C_T at the node in C_T overlying the T-node in C_T,
2. the pullback of L_T via the above normalization.

The above definition of μ^q_{irr} applies *mutatis mutandis* to $\mathbb{S}^q_{\mathrm{irr}}$. For any $M \in \{0, \dots, l-1\}$, we have

$$\mathbb{S}^M_{\mathrm{irr}} \xrightarrow{\mu^M_{\mathrm{irr}}} \overline{\mathsf{R}}^{l,s}_{g-1,n+2}(\boldsymbol{m}, M, M').$$

Remark 3.1.2. Clearly, the psi classes on $\overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m})$ of (2.5.6) and the psi classes on $\mathbb{S}$ of (I.10) are compatible; we have

$$(\mu^M_{\mathrm{irr}})^*(\psi_{n+1}) = \psi, \qquad (\mu^M_{\mathrm{irr}})^*(\psi_{n+2}) = \psi',$$

and

$$\mu^*_{i,I}(\psi_{|I|+1} \otimes \mathbf{1}) = \psi, \qquad \mu^*_{i,I}(\mathbf{1} \otimes \psi_{|I'|+1}) = \psi'.$$

for all i, I, M.

We refer to [34] for an explicit description of the generic fibre of the morphisms introduced above.

3.2 The index K class

In this section, in Theorem 3.2.1, we provide a formula playing a crucial role in the intersection theory of the moduli space of level curves.

On $\overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m})$, we have universal objects

$$\begin{array}{ccc} \mathsf{L} & \longrightarrow & \mathsf{C}^{l\text{-st}} \\ & & \big\downarrow \mathsf{p} \\ & & \overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m}) \end{array}$$

and a natural K class

$$R\mathsf{p}_*\mathsf{L} = \sum_i (-1)^i [R^i \mathsf{p}_* \mathsf{L}] \in K_0\left(\overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m})\right).$$

We can compute the Chern character of $R\pi_*\mathsf{L}$ in terms of psi classes and Bernoulli polynomials $B_h(x)$. We recall that Bernoulli polynomials are defined by the following generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{h=0}^{\infty} B_h(x) \frac{t^n}{n!}.$$

Theorem 3.2.1 ([28]). *Consider the moduli stack* $\overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m})$ *of n-pointed genus-g level-l curves of type* $\boldsymbol{m}$ *with respect to* $\omega_{\log}^{\otimes s}$*. Let* L *be the univeral level structure and let* p *denote the universal curve. Then, the degree-2h term of the Chern character of* $R\mathsf{p}_*\mathsf{L}$ *in* $H^*(\overline{\mathsf{R}}^{l,s}_{g,n}(\boldsymbol{m});\mathbb{Q})$ *equals*

$$\frac{B_{h+1}\left(\frac{s}{l}\right)}{(h+1)!}\kappa_h - \sum_{i=1}^{n}\frac{B_{h+1}\left(\frac{m_i}{l}\right)}{(h+1)!}\psi_i^h + \frac{l}{2}\sum_{0\le M<l}\frac{B_{h+1}\left(\frac{M}{l}\right)}{(h+1)!}(j_M)_*\sum_{a+a'=h-1}\psi^a(-\psi')^{a'},$$

where j_M *is the morphism from* (I.12) *and* κ_h *is the standard kappa class* $\mathsf{p}_*\left(c_1(\omega_{\log})^{h+1}\right)$. □

Remark 3.2.2. The above formula generalizes Mumford formula for the Chern character of the Hodge bundle [103].

Remark 3.2.3. It is well known that the kappa class may be regarded as a pushforward of the the $(h+1)$st power of the class ψ_{n+1} from the universal stable curve; see for instance [7]. Therefore, the above formula provides an expression of the Chern character exclusively in terms of psi classes (see also Remark 3.1.2).

3.3 Examples of index bundles

We study briefly by means of examples the properties of the index K class. This is not only motivated by pedagogical reasons, but it is actually useful to appreciate the definition of the quantum theories in the next section.

The case $s = 0$ is related to orbifold Gromov–Witten theory. This happens because moduli of level-l structures with respect to $\mathcal{O}$ are stable maps to $B\mathbb{Z}/l$ (see [10, 37, 119] and see [11, Thm.] for an explicit treatment).

Here, we focus on the case $s = 1$, the case of roots of $\omega_{\log}$, because it plays an important role in the Landau–Ginzburg model.

Each geometric point $\mathrm{pt} \in \overline{\mathsf{R}}^{l,1}_{g,n}(\boldsymbol{m})$ determines a pair $(\mathsf{C}_{\mathrm{pt}}, \mathsf{L}_{\mathrm{pt}})$ and we can consider the vector spaces $H^0(\mathsf{C}_{\mathrm{pt}}, \mathsf{L}_{\mathrm{pt}})$ and $H^1(\mathsf{C}_{\mathrm{pt}}, \mathsf{L}_{\mathrm{pt}})$, whose ranks define upper-semicontinuous functions on $\overline{\mathsf{R}}^{l,1}_{g,n}(\boldsymbol{m}) \to \mathbb{Z}/\geq 0$. In some special cases, we actually get continuous (locally constant) functions.

Example 3.3.1. Assume $g = 0$ and $m_1, \dots, m_n > 0$. Then $H^0(\mathsf{C}_{\mathrm{pt}}, \mathsf{L}_{\mathrm{pt}})$ vanishes at every point and $\overline{\mathsf{R}}^{l,1}_{g,n}(\boldsymbol{m})$ is equipped with a locally free sheaf $R^1\mathsf{p}_*\mathsf{L}$. In these special cases $\mathrm{ch}(R\mathsf{p}_*\mathsf{L})$ equals (up to a sign) the Chern character of the bundle $R^1\mathsf{p}_*\mathsf{L}$.

This can be easily seen for a smooth curve genus-zero C. Indeed

$$H^i(\mathsf{C},\mathsf{L}) = H^i(C, \lfloor \mathsf{L} \rfloor) \qquad \forall i.$$

Then the condition $H^0(\mathsf{C},\mathsf{L}) = 0$ is an immediate consequence of the fact that $\deg(\lfloor \mathsf{L} \rfloor)$ is negative when $\mathsf{L}^{\otimes l} \cong \omega_{\log}$:

$$\deg(\omega_{\log}) - \sum_{i=1}^{n}\frac{m_i}{l} = -2 - \sum_{i=1}^{n}\frac{m_i - 1}{l} < 0.$$

If the genus-zero curve is nodal the same argument holds by reasoning by induction on the components (see [31]).

Remark 3.3.2. It is immediate to point out that the claim above holds (in genus zero) even when all multiplicities *but one* are positive. This plays a special role in the study of the normalization of nodes, see Remark 3.3.6.

In general we have

$$h^1(\mathsf{C}_{\mathrm{pt}}, \mathsf{L}_{\mathrm{pt}}) - h^0(\mathsf{C}_{\mathrm{pt}}, \mathsf{L}_{\mathrm{pt}}) = (g-1)\left(1 - \frac{2}{l}\right) + \sum_{i=1}^{N}\frac{m_i - 1}{l}. \tag{I.13}$$

Let us discuss the above formula in the case of smooth curves, and in the case of one-noded curves with all possible multiplicities. These cases allow us to introduce three main tools: *orbifold Riemann–Roch*, *narrow and broad nodes*. The examples below amount essentially to a proof of the formula, but this can be found in higher generality in [2], [79], and [118].

Example 3.3.3 (Riemann–Roch for orbifold curves)**.** We assume that the curve with level structure is smooth. We consider a smooth one-dimensional proper stack C whose stabilizers are nontrivial only on a finite number of points. We write the line bundle L as $\mathcal{O}(D)$ for $D \in \mathrm{Div}(\mathsf{C})$. Then we have

$$h^0(\mathsf{C},\mathsf{L}) - h^1(\mathsf{C},\mathsf{L}) = h^0(C, \lfloor \mathsf{L} \rfloor) - h^1(C, \lfloor \mathsf{L} \rfloor),$$

where C is the reification of C and $\lfloor \mathsf{L} \rfloor$ is $\mathcal{O}(\lfloor D \rfloor)$ as in I.2. (In fact, the above relation may regarded as $R\mathsf{p}_*(\mathsf{L}) = Rp_*(\epsilon_*\mathsf{L})$ where $\mathsf{p}\colon \mathsf{C} \to \operatorname{Spec}\mathbb{C}$ is the composite of $\epsilon\mathsf{C} \to C$ and $p\colon C \to \operatorname{Spec}\mathbb{C}$).

Now we have

$$h^0(C, \lfloor \mathsf{L} \rfloor) - h^1(C, \lfloor \mathsf{L} \rfloor) = \deg(\lfloor D \rfloor) + 1 - g(\mathsf{C}).$$

Since the multiplicity at the markings is $m_1, \dots, m_n$, we get $\deg(\lfloor D \rfloor) = \deg(D) - \sum_{i=1}^n \frac{m_i}{l}$ and ultimately the orbifold Riemann–Roch formula for curves

$$h^0(\mathsf{C},\mathsf{L}) - h^1(\mathsf{C},\mathsf{L}) = \deg(\mathsf{L}) - \sum_{i=1}^{n} \frac{m_i}{l} + 1 - g \tag{I.14}$$

matching (I.13) via $\deg(\mathsf{L}) = \deg(\omega_{\log})/l$.

Example 3.3.4 (narrow nodes and broad nodes)**.** We assume that the curve has a single separating node $B\mathbb{Z}/l$ (the normalization is disconnected). We write the orbifold curve as $\mathsf{C} = \mathsf{C}_1 \cup \mathsf{C}_2$ and the normalization $\mathsf{C}^\nu = \mathsf{C}_1 \sqcup \mathsf{C}_2$; clearly $g(\mathsf{C}) = g(\mathsf{C}_1) + g(\mathsf{C}_2)$. The line bundle L is determined by a line bundle L_1 on C_1 and a line bundle L_2 on C_2. The multiplicity is defined with respect to the branch lying in C_1 and equals M. Consider the normalization $\mathsf{n}\colon \mathsf{C}^\nu \to \mathsf{C}$, the exact sequence

$$0 \to \mathsf{L} \to \mathsf{n}_*\mathsf{n}^*\mathsf{L} \to \mathsf{L}|_{B\mathbb{Z}/l} \to 0,$$

and the long exact sequence

$$\begin{aligned} 0 \to & H^0(\mathsf{C},\mathsf{L}) \to H^0(\mathsf{C}^\nu, \mathsf{n}^*\mathsf{L}) \to H^0(B\mathbb{Z}/l, \mathsf{L}|_{B\mathbb{Z}/l}) \to \\ \to & H^1(\mathsf{C},\mathsf{L}) \to H^1(\mathsf{C}^\nu, \mathsf{n}^*\mathsf{L}) \to 0. \end{aligned}$$

Now $h^0(\mathsf{C},\mathsf{L}) - h^1(\mathsf{C},\mathsf{L})$ is the sum of three terms

1. $h^0(\mathsf{C}_1,\mathsf{L}_1) - h^1(\mathsf{C}_1,\mathsf{L}_1)$,
2. $h^0(\mathsf{C}_2,\mathsf{L}_1) - h^1(\mathsf{C}_2,\mathsf{L}_1)$,
3. $-h^0(B\mathbb{Z}/l, \mathsf{L}|_{B\mathbb{Z}/l})$.

The first two terms yield $\deg(\lfloor \mathsf{L}_1 \rfloor) + 1 - g(C_1)$ and $\deg(\lfloor \mathsf{L}_2 \rfloor) + 1 - g(C_2)$. The last term is -1 or 0 depending on the multiplicity M. We have two possibilities.

Broad nodes. This is the case where $M = 0$; zero multiplicity implies two facts: first $\deg(\lfloor \mathsf{L} \rfloor) = \deg(\lfloor \mathsf{L}_1 \rfloor) + \deg(\lfloor \mathsf{L}_2 \rfloor)$, second $h^0(B\mathbb{Z}/l, \mathsf{L}|_{B\mathbb{Z}/l}) = 1$. The sum of the three terms yields $\deg(\lfloor \mathsf{L} \rfloor) + 1 - g(\mathsf{C})$ and ultimately (I.13).

Narrow nodes. This is the case $M \neq 0$; nonzero multiplicity implies two facts: first we have $\deg\lfloor \mathsf{L} \rfloor = \deg(\lfloor \mathsf{L}_1 \rfloor) + \deg(\lfloor \mathsf{L}_2 \rfloor) + \frac{M}{l} + \frac{l-M}{l}$, second the term $h^0(B\mathbb{Z}/l, \mathsf{L}|_{B\mathbb{Z}/l})$ vanishes. The sum of the three terms still yields $\deg(\lfloor \mathsf{L} \rfloor) + 1 - g(\mathsf{C})$ and (I.13).

Remark 3.3.5. The two cases appearing in the above example differ because L_1 and L_2 are tied by a broad node ($M = 0$) and a narrow node ($M \neq 0$). When the node is narrow the two sides are decoupled, the correcting term $h^0(B\mathbb{Z}/l, \mathsf{L}|_{B\mathbb{Z}/l})$ vanishes and the Riemann–Roch computations on the two sides are independent and their sum yields the correct result. If the node is broad, then when we rescale a section on C_1 we are automatically rescaling it on C_2. The two Riemann–Roch computation should be corrected by $h^0(B\mathbb{Z}/l, \mathsf{L}|_{B\mathbb{Z}/l}) = 1$.

The terminology narrow nodes/broad nodes replaces the previous terminology Neveu–Schwarz nodes/Ramond nodes. Ramond and Neveu-Schwarz sectors have a precise meaning in conformal field theory which seems to be only vaguely related to the meaning commonly used to distinguish between these two type of nodes. (This change of notation is consistent with recent versions of [51, 52, 53].)

Remark 3.3.6 (factorization of the index bundle)**.** Let us consider a family of curves over a base scheme B. Again $s = 1$ and $m_1, \dots, m_n > 0$. We assume that all fibres have genus 0 and exactly one node. Their level structure L yields an index bundle

$$I = R^1\mathsf{p}_*\mathsf{L}.$$

The normalization

$$\mathsf{n}\colon \mathsf{C}_1 \sqcup \mathsf{C}_2 \to \mathsf{C}$$

of the node yields two level-l structures $(\mathsf{p}_1\colon \mathsf{C}_1 \to B, \mathsf{L}_1)$ and $(\mathsf{p}_2\colon \mathsf{C}_2 \to B, \mathsf{L}_2)$ and two index bundles

$$I_1 = R^1(\mathsf{p}_1)_*\mathsf{L}_1 \qquad \text{and} \qquad I_2 = R^1(\mathsf{p}_2)_*\mathsf{L}_2.$$

(using Remark 3.3.2). Finding a relation expressing I in terms of I_1 and I_2 is not only a natural question, it is a crucial issue for the definition of quantum theories.

There are two possibilities.

Narrow nodes. We have

$$I = I_1 \oplus I_2.$$

Indeed let S be the singular locus of C. Since all fibres have exactly one node S projects to B via a morphism locally isomorphic to $B\mathbb{Z}/l \to \mathrm{pt}$. We have a long exact sequence

$$0 \to \mathsf{L} \to \mathsf{n}_*\mathsf{n}^*\mathsf{L} \to \mathsf{L}|_{\mathsf{S}} \to 0.$$

and a long exact sequence

$$\begin{array}{cccccccccc} 0 & \to & \mathsf{p}_*\mathsf{L} & \to & (\mathsf{p}_1)_*\mathsf{L}_1 \oplus (\mathsf{p}_2)_*\mathsf{L}_2 & \to & \mathsf{p}_*\mathsf{L}|_{\mathsf{S}} & \to \\ & \to & R^1\mathsf{p}_*\mathsf{L} & \to & R^1(\mathsf{p}_1)_*\mathsf{L}_1 \oplus R^1(\mathsf{p}_2)_*\mathsf{L}_2 & \to & 0. & \end{array}$$

Since the node is narrow and $\mathsf{p}_*\mathsf{L} = (\mathsf{p}_1)_*\mathsf{L}_1 = (\mathsf{p}_2)_*\mathsf{L}_2 = 0$, we get $I = I_1 \oplus I_2$.

Broad nodes. We have an exact sequence of the form

$$0 \to T \to I \to I_1 \oplus I_2 \to 0.$$

Set $T = \mathsf{p}_*\mathsf{L}|_{\mathsf{S}}$; then this follows by the previous argument. Notice that $T^{\otimes l}$ is trivial because $\omega_{\log}|_{\mathsf{S}}$ is trivial (residue map); hence the Euler class of I vanishes.

We may rephrase the remark by saying

$$c(I) = c(I_1)c(I_2),$$

where c is the total Chern class. Furthermore, the Euler class (or top Chern class) $c_{\mathrm{top}}(I)$, either vanishes (broad nodes), or is the product of $c_{\mathrm{top}}(I_1)$ and $c_{\mathrm{top}}(I_2)$. This is the key fact for the definition of the Landau–Ginzburg quantum theory in Chapter II.

4 The singularities of the moduli space

In this last section we address natural geometric question on moduli of level curves and on their reifications. The material presented in this section is not used in the rest of the text. The reader may skip to Chapter II.

4.1 Dual graph and multiplicity

Consider a dual graph Γ of an orbifold curve C (*i.e.*, the ordinary dual graph of the reification). It is a finite graph with vertex set V and edge set E. Let $C^0 = C^0(\Gamma, \mathbb{Z}/l)$ be the set of $\mathbb{Z}/l$-valued functions on V and let $C^1 = C^1(\Gamma, \mathbb{Z}/l)$ be the set of $\mathbb{Z}/l$-valued functions on the set E. In analogy with our study of the boundary locus via the degree-2 map $\mathbb{S} \to S$, we regard $C^1(\Gamma, \mathbb{Z}/l)$ as a set of functions g on the set of *oriented* edges $\mathbb{E}$ (for each edge there ate two orientations; so, the cardinality of $\mathbb{E}$ is twice that of E). We only consider functions $g\colon \mathbb{E} \to \mathbb{Z}/l$ satisfying $g(\overline{e}) = -g(e)$, where $\overline{e}$ and e are oriented edges with opposite orientations (see also Serre [115, §2, n°1]). The space of 0-cochains and 1-cochains C^0 and C^1 are equipped with bilinear $\mathbb{Z}/l$-valued forms

$$\langle f_1, f_2 \rangle = \sum\nolimits_{v \in V} f_1(v)f_2(v) \qquad \langle g_1, g_2 \rangle = \frac{1}{2}\sum\nolimits_{e \in \mathbb{E}} g_1(e)g_2(e)$$

with $f_1, f_2 \in C^0$ and $g_1, g_2 \in C^1$. One defines the exterior differential

$$\delta\colon C^0 \to C^1,$$
$$f \mapsto \delta f(e) = f(e_+) - f(e_-),$$

where e_+ and e_- denote the head and the tail of the oriented edge e. The adjoint operator with respect to $\langle\ ,\ \rangle$ is given by

$$\partial\colon C^1 \to C^0,$$
$$g \mapsto \partial g(v) = \sum_{\substack{e\in\mathbb{E}\\ e_+=v}} g(e).$$

Since the edges of Γ are in one-to-one correspondence with nodes of C, oriented edges of Γ are in one-to-one correspondence with branches of nodes of C. In this way, given an l-level curve, to each oriented edge e we can attach the multiplicity M_e of $(\mathsf{C}, \mathsf{L}, \phi)$ at the node (with its prescribed branch). The function $M\colon e \mapsto M_e$ clearly satisfies $M(\overline{e}) = -M(e)$ for all $e \in \mathbb{E}$; in this way M is a 1-chain in $C^1(\Gamma, \mathbb{Z}/l)$.

On the other hand, each choice of s and $\boldsymbol{m}$ determines a line bundle $T = \omega_{\log}^{\otimes s}(-\sum_{i=1}^n m_i[D_i])$. The degrees of this line bundle T on each irreducible component amount to a function $D\colon v \mapsto \deg(T|_{\mathsf{C}_v})$, where C_v is the component corresponding to the vertex v. In this way D is a 0-cochain in $C^0(\Gamma, \mathbb{Z}/l)$.

Proposition 4.1.1 ([26])**.** *We have*

$$\partial M = D.$$

In particular, when $s = 0$ and $\boldsymbol{m} = 0$, the multiplicity chain M is closed with respect to the differential ∂. □

The image of δ is the orthogonal complement of $\ker\partial$ with respect to $\langle\ ,\ \rangle$:

$$\operatorname{im}\delta = (\ker\partial)^\perp. \tag{I.15}$$

This provides us with a simple criterion to decide when a function $g\colon E \to \mathbb{Z}/l$ in C^1 belongs to $\operatorname{im}\delta$. Recall that $\ker\partial$ is generated by the circuits of Γ. A *circuit* within a graph is a sequence of n distinct oriented edges $e_0, \dots, e_{n-1}$ such that the head of e_i is also the tail of e_{i+1} for $0 \le i < n-1$ and the head of e_{n-1} is the tail of e_0. In this way a circuit identifies n vertices $v_i = (e_i)_-$ for $0 \le i < n$; we require that all these vertices are distinct. A circuit formed by a single edge will be called one-edged or, simply, loop. Via the natural map sending an edge to its characteristic function

$$e \mapsto \chi_e$$

an edge may be regarded as an element of $C^1(\Gamma, \mathbb{Z}/l)$. In this way, (I.15) may be regarded as saying: $g \in C^1$ is in $\operatorname{im}\delta$ if and only if $\langle g, C\rangle = 0$ for all circuits of Γ.

4.2 Ghost automorphisms of level-l curves

An *automorphism* of an l-level curve $(\mathsf{C}, \mathsf{L}, \phi)$ is given by (s, ρ) where s is an isomorphism of C, and ρ is an isomorphism of sheaves $\mathsf{s}^*\mathsf{L} \to \mathsf{L}$ satisfying $\phi \circ \rho^{\otimes l} = \mathsf{s}^*\rho$. Set

$$\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) = \{\mathsf{s} \in \mathrm{Aut}(\mathsf{C}) \mid \mathsf{s}^*\mathsf{L} \cong \mathsf{L}\}.$$

It is easy to see that for each element of $\underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi)$ there exists a pair $(\mathsf{s}, \rho) \in \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi)$. Two pairs of this form differ by a power of a quasi-trivial automorphism of the form $(\mathrm{id}_{\mathsf{C}}, \xi_l)$ operating by scaling the fibres. We have the following exact sequence

$$1 \to \mathbb{Z}/l \to \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi) \to \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) \to 1.$$

Quasi-trivial isomorphisms act trivially on $\mathrm{Def}(\mathsf{C}, \mathsf{L}, \phi)$, the deformation space of $(\mathsf{C}, \mathsf{L}, \phi)$.

The reification $\mathsf{s} \mapsto s$ induces group homomorphisms $r\colon \mathrm{Aut}(\mathsf{C}, \mathsf{L}, \phi) \to \mathrm{Aut}(C)$ and $\underline{r}\colon \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) \to \mathrm{Aut}(C)$. Consider

$$1 \to \ker\underline{r} \to \underline{\mathrm{Aut}}(\mathsf{C}, \mathsf{L}, \phi) \xrightarrow{\underline{r}} \operatorname{im}\underline{r} \to 1;$$

the kernel and the image of $\underline{r}$ are natural geometric object of independent interest.

Ghosts automorphisms. The kernel of $\underline{\mathrm{Aut}}(\mathsf{C},\mathsf{L},\phi)$ is the group

$$\ker \underline{r} = \underline{\mathrm{Aut}}_C(\mathsf{C},\mathsf{L},\phi)$$

of automorphisms s of C fixing at the same time the underlying curve C and the isomorphism class of the overlying line bundle L. Indeed, an automorphism of a stack M may well be nontrivial and operate as the identity on M. Consider the quotient stack $\mathsf{U} = [\{xy = 0\}/\mathbb{Z}/l]$ where $\mathbb{Z}/l$ acts as $\xi_l(x,y) = (\xi_l x, \xi_l^{-1} y)$ and all automorphisms $(x,y) \mapsto (\xi_l^b x, \xi_l^a y)$ induce the identity on the quotient space. The autormorphisms fixing the reification U up to natural transformations (2-isomorphisms) are $\mathrm{Aut}_U(\mathsf{U}) \cong \mathbb{Z}/l$ and are generated by $(x,y) \mapsto (\xi_l x, y)$. In this way, the automorphisms of a k-noded twisted curve C which fix C are freely generated by automorphisms operating as $(x,y) \mapsto (\xi_{r_i} x, y)$.

Automorphisms of C lifting to $(\mathsf{C},\mathsf{L},\phi)$. The image $\mathrm{im}\, \underline{r}$ is the group of automorphisms s of C, that can be lifted to $(\mathsf{C},\mathsf{L},\phi)$. This means that there exists a morphism s of C whose reification is s and such that $\mathsf{s}^*\mathsf{L} \cong \mathsf{L}$.

4.3 Singularities

For the rest of this chapter we only consider level structures with respect to $\mathcal{O}$. The point representing $(\mathsf{C},\mathsf{L},\phi)$ is smooth if and only if each element of $\mathrm{Aut}(\mathsf{C},\mathsf{L},\phi)$ operated on $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)$ as a quasireflection (an automorphism whose fixed space is of codimension one). This holds if and only if all elements of $\mathrm{im}\, \underline{r}$ and $\ker \underline{r}$ operate as quasireflections.

In [29] we prove the following theorem extending the level-2 statement of [96]. We need the following combinatorial tool. Given a level structure $(\mathsf{C},\mathsf{L},\phi)$ and its corresponding multiplicity M, for every k dividing l, let us consider the graph Γ_k obtained by contracting an edge of Γ if and only if k divides M_e. If two divisors k' and k'' of l satisfy $k' \mid k''$ then $\Gamma_{k'}$ can be obtained by contracting some edges of $\Gamma_{k''}$; in this case we will say that if $k' \mid k''$, then $\Gamma_{k'}$ is a contraction of $\Gamma_{k''}$. In particular Γ_1 has a single vertex and no edges and Γ_l is given by contracting the edges e of Γ with vanishing multiplicity M_e. We recall that a graph is *tree-like* if it is a tree once we eliminate all the loops (the edges connecting a vertex to itself).

Theorem 4.3.1 ([29]). *The point of $\overline{R}_{g,l}$ representing $(\mathsf{C},\mathsf{L},\phi)$ is smooth if and only if*

1. *the group of automorphisms of C lifting to $(\mathsf{C},\mathsf{L},\phi)$ (*i.e. *the group $\mathrm{im}\, \underline{r}$) is generated by involutions of elliptic tails; and*
2. *for all divisors of l of the form p^d (with p prime and $d \in \mathbb{Z}/\geq 1$) the graph Γ_{p^d} is tree-like.*

Sketch of the proof. Consider the homomorphism

$$\underline{M}\colon C^1(\Gamma,\mathbb{Z}/l) \to C^1(\Gamma,\mathbb{Z}/l)$$
$$\sum_e h_e\chi_e \mapsto \sum_e M_e h_e \chi_e,$$

where χ_e is the characteristic function of e. Then, the group of ghost automorphisms $\ker \underline{r}$ is $\underline{M}^{-1}(\mathrm{im}\,\delta)$.

Indeed, the group of automorphisms $\mathrm{Aut}_C(\mathsf{C})$ of C, which fix C is given by $C^1(\Gamma,\mathbb{Z}/l)$. By [26], given an automorphism $\alpha \in \mathrm{Aut}_C(\mathsf{C})$ and a level structure L we have

$$\alpha^*\mathsf{L} \cong \mathsf{L} \otimes \mathsf{T}_M,$$

where T_M is a line bundle on C defined by applying to $M \in C^1(\Gamma,\mathbb{Z}/l)$ the natural map $C^1(\Gamma,\mathbb{Z}/l) \to \mathrm{Pic}(\mathsf{L})[l]$. Then, The exact sequence

$$1 \to C^0(\Gamma,\mathbb{Z}/l) \xrightarrow{\delta} C^1(\Gamma,\mathbb{Z}/l) \to \mathrm{Pic}(\mathsf{C})[l] \to \mathrm{Pic}(\mathsf{C}^\nu)[l]$$

yields the desired identification between $\ker \underline{r}$ and $\underline{M}^{-1}(\mathrm{im}\,\delta)$.

Once this combinatorics is set up, the claim in [29] follows from a direct study of the action of $C^1(\Gamma,\mathbb{Z}/l)$ on $\mathrm{Def}(\mathsf{C},\mathsf{L},\phi)$. □

4.4 Global geometry of moduli of level-l structures.

We illustrate results from [29], where we focus on ordinary level-l curves without markings: *i.e.* the initial moduli space $\overline{R}^l_g$. They are proven in the above framework.

The moduli space $\overline{R}_g^l$ is a normal variety with finite quotient singularities; using the above characterization of level-l curves without ghosts we describe the singular locus, Theorem 4.3.1. Then, in order to determine its Kodaira dimension we consider a smooth model $\widehat{R}_g^l$ of $\overline{R}_g^l$ and then analyze the growth of the dimension of the spaces

$$H^0\big(\widehat{R}_g^l, K_{\widehat{R}_g^l}^{\otimes q}\big)$$

of pluricanonical forms for all $q \geq 0$. The following theorem allows us to relate the pluricanonical forms on $\widehat{R}_g^l$ to those on $\overline{R}_g^l$.

Theorem 4.4.1 ([29])**.** *We fix $g \geq 4$ and $l \leq 4$ and let $\widehat{R}_g^l \to \overline{R}_g^l$ be any desingularization. Then every pluricanonical form defined on the smooth locus $\overline{R}_g^{l,\mathrm{reg}}$ of $\overline{R}_g^l$ extends holomorphically to $\widehat{R}_g^l$; that is, for all integers $q \geq 0$ we have isomorphisms*

$$H^0\big(\overline{R}_g^{l,\mathrm{reg}}, K_{\overline{R}_g^l}^{\otimes q}\big) \cong H^0\big(\widehat{R}_g^l, K_{\widehat{R}_g^l}^{\otimes q}\big).$$

The above statement embodies Harris and Mumford's result for $l = 1$ [63, Theorem 1], Farkas and Ludwig's result for $l = 2$ [54]. See also Ludwig [96] for 2-level structures over ω (2-spin curves).

The following statement shows an application of Theorem 4.4.1.

Theorem 4.4.2. *The moduli space of level-3 curves $\overline{R}_g^3$ is of general type for $g > 10$.*

Chapter II

The Landau–Ginzburg model

In this section, we present the quantum theory introduced by Fan, Jarvis, and Ruan in [51, 52, 53] building upon work of Witten. Following [51], we refer to it as Fan–Jarvis–Ruan–Witten (FJRW) theory. The theory combines the moduli spaces of level structures and classical singularity theory.

1 The state space

The state space is an orbifold version of the space of Lefschetz thimbles. Let us review the setup.

Let us fix the notation for quasihomogeneous (or weighted homogeneous) polynomials. Write

$$W = \sum_{i=1}^{s} \gamma_i \prod x_j^{m_{i,j}}$$

with $m_{i,j} \in \mathbb{Z}/\geq 0$ and $\gamma_i \neq 0$. Then W is quasihomogeneous of charges $q_1, \dots, q_N$ if $\sum_{j=1}^{N} m_{i,j} q_j = 1$. Equivalently, with a slight abuse of notation, we write

$$W(\lambda^{q_1} x_1, \dots, \lambda^{q_N} x_N) = \lambda W(x_1, \dots, x_N)$$

and we refer to $q_1, \dots, q_N$ as the *charges* of W. In mathematical literature $q_1, \dots, q_N$ are commonly expressed in the form $w_1/d, \dots, w_N/d$ under a common minimal denominator; then $w_1, \dots, w_N$ are the *weights* of W and d is the degree of W. The physics terminology is more convenient in many formulae, and we will stick to it.

The polynomial W is *nondegenerate* if:

1. W defines a unique singularity at zero;

2. the choice of $q_1, \dots, q_N$ is unique.

An element $g \in GL(N, \mathbb{C})$ is a *diagonal symmetry of* W if g is a diagonal matrix of the form $\mathrm{Diag}(\lambda_1, \dots, \lambda_N)$ such that

$$W(\lambda_1 x_1, \dots, \lambda_N x_N) = W(x_1, \dots, x_N). \tag{II.1}$$

We will use $\mathrm{Aut}(W)$ to denote the group of all diagonal symmetries and we will refer to it as the maximal group of diagonal symmetries. It is easy to see that this group is finite. The group is also nontrivial since it contains the element $j_W = \mathrm{Diag}(e^{2\pi \mathrm{i} q_1}, \dots, e^{2\pi \mathrm{i} q_N})$.

The theory of Fan, Jarvis, and Ruan theory applies to a pair (W, G), where $G \subseteq \mathrm{Aut}(W)$. Two conditions will naturally arise in the rest of the paper; their role is specular in the sense of mirror symmetry.

A-admissibility. We will say that $G \subseteq \mathrm{Aut}(W)$ is *A-admissible* if j_W is contained in G.

B-admissibility. We will say that it is *B-admissible* if $G \subseteq SL(N, \mathbb{C})$; *i.e.* if G is included in $SL_W = SL(N, \mathbb{C}) \cap \mathrm{Aut}(W)$.

1.1 Lefschetz thimbles from the classical point of view

Consider $W\colon \mathbb{C}^N \to \mathbb{C}$. Let us recall some important facts on the relative homology of $(\mathbb{C}^N, W^{-1}(S_M^+))$ where S_M^+ is the half-plane $\{z \in \mathbb{C} \mid \mathrm{Re} z > M\}$ for $M > 0$. We denote it by

$$H_*(\mathbb{C}^N, W^{+\infty}; \mathbb{Z})$$

with $W^{+\infty} = W^{-1}(S_M^+)$ and, by abusing notation, we refer to it as the *space of Lefschetz thimbles.*

Remark 1.1.1. Due to the nondegeneracy condition, the origin is the only critical point of W and $\mathbb{C}^N \setminus W^{-1}(0)$ is a fiber bundle on $\mathbb{C}^\times$. For $N > 1$, since $\mathbb{C}^N$ is contractible, we can consider the relative cohomology group above as the homology (with compact support) of degree $N-1$ of the fiber over a point S_M^+ (see [107, 1.1]). We consider the Milnor fiber $W^{-1}(t)$ with $t \in S_M^*$; it is a Stein manifold with homology only in degree $N-1$ and 0. In fact $H_{N-1}(W^{-1}(t); \mathbb{C})$ is equipped with a canonical mixed Hodge structure and with a monodromy automorphism [117].

In this way, the above space of Lefschetz thimbles is concentrated in degree N and is equipped with a mixed Hodge structure. The monodromy automorphism is indeed the action of the diagonal matrix j_W; by [117], the j_W-invariants Lefschetz thimbles are are equipped with a pure Hodge structure (see also [48]).

Remark 1.1.2. A nondegenerate pairing can be defined as follows. We consider the relative homology

$$H_N(\mathbb{C}^N, W^{-\infty}; \mathbb{Z}),$$

where $W^{-\infty}$ denotes $W^{-1}(S_M^-)$ and S_M^- is the half-plane $\{z \in \mathbb{C} \mid \mathrm{Re} z < -M\}$ for $M > 0$. The intersection form for Lefschetz thimbles with boundaries in $W^{+\infty}$ and in $W^{-\infty}$ gives a well defined nondegenerate pairing

$$P\colon H_N(\mathbb{C}^N, W^{+\infty}; \mathbb{Z}) \times H_N(\mathbb{C}^N, W^{-\infty}; \mathbb{Z}) \longrightarrow \mathbb{Z}, \tag{II.2}$$

see [64, §8, Step 2] and [107].

1.2 The state space of (W, G)

In our setup the above facts can be used to define the state space as the space of Lefschetz thimbles for the *stack-theoretic* map

$$W\colon [\mathbb{C}^N/G] \longrightarrow \mathbb{C}$$

where G is an A-admissible group (*i.e.* a group of diagonal symmetries containing j_W). The quasihomogeneity condition yields a state space naturally equipped with a nondegenerate *inner* pairing.

Let us define the state space first; for the scheme-theoretic map $W\colon \mathbb{C}^N \to \mathbb{C}$ we considered the relative cohomology $H^*(\mathbb{C}^N, W^{+\infty}; \mathbb{C})$ which is concentrated in degree N and dual to the above space of Lefschetz thimbles. Since $[\mathbb{C}^N/G]$ is a stack, and the loci $W^{+\infty}$ and $W^{-\infty}$ (preimages of S_M^+ and S_M^-) are substacks, the suitable cohomology theory for this setup is orbifold cohomology (or Chen–Ruan cohomology).

Definition 1.2.1 (state space)**.** For any A-admissible group G, we set

$$\mathcal{H}_{W,G}^{a,b} := H_{\mathrm{CR}}^{a+q,b+q}([\mathbb{C}^N/G], W^{+\infty}; \mathbb{C}) \qquad q = \textstyle\sum_j q_j.$$

Remark 1.2.2. Because of its role in mirror symmetry, this is sometimes referred to as the *A model state space* (see next chapter).

Remark 1.2.3. By making the above definition explicit we may regard the state space as the direct sum over the elements $g \in G$ of the G-invariant cohomology classes

$$\mathcal{H}_{W,G} = \bigoplus_{g \in G} H^{N_g}(\mathbb{C}_g^N, W_g^{+\infty}; \mathbb{C})^G, \tag{II.3}$$

where N_g is the number of coordinates $x_1, \dots, x_N$ which are fixed by g, and $\mathbb{C}_g^N$ (and $W_g^{+\infty}$) denote the g-fixed subspaces of $\mathbb{C}^N$ (of $W^{+\infty}$). We recall that the G-invariant subspaces in $H^{N_g}(\mathbb{C}_g^N, W_g^{+\infty})$ are contained into the subspaces of j_W-invariant subspaces. By the remark 1.1.1, this guaranties that $H^{N_g}(\mathbb{C}_g^N, W_g^{+\infty}; \mathbb{C})^G$ is equipped by a pure Hodge structure of weight N_g. In this way, each class has bidegree $(p, N_g - p)$ in the standard cohomology. We now specify its bidegree in Chen–Ruan cohomology

$$(\deg^+, \deg^-) = (p, N_g - p) + (\mathrm{age}(g), \mathrm{age}(g)) - (q, q) \qquad (\text{with } q = \textstyle\sum q_j). \tag{II.4}$$

In the above formula age is the rational index defined [1] on the ring of finite order representations.

We will usually write $\mathcal{H}^{a,b}_{W,G}$ for the terms of bidegree (a,b) and we will write deg for the *total* degree $a+b$.

Remark 1.2.4. Consider the following map, naturally defined on $\mathcal{H}_{W,G}$

$$i_*\colon H^*_{\mathrm{CR}}([\mathbb{C}^N/G], W^{+\infty}) \to H^*_{\mathrm{CR}}([\mathbb{C}^N/G]).$$

Narrow states. In the decomposition II.3 i_* vanishes for all g with $N_g>0$ and is injective where N_g vanishes. The image of i_* is spanned in $\mathcal{H}_{W,G}$ by the summands in for which $N_g=0$. These are called *narrow* states (in Section 2, Remark 2.2.3, we see how this terminology is compatible with that of the previous chapter). A special case of narrow state is the fundamental class attached to j_W. By construction, we have $\deg(j_W)=0$. This element plays the role of the unit of $\mathcal{H}_{W,G}$ in the quantum cohomology ring (see [83]).

Broad states. In contrast with narrow states, the complementary space, *i.e.* the kernel of i_*, is usually referred to in [51] as the space of *broad* states. These are classes attached to diagonal symmetries fixing a nontrivial subspace of $\mathbb{C}^N$.

1.3 The inner pairing

We now define the nondegenerate inner pairing. The crucial fact is that the quasihomogeneity of the map W allows us to define an automorphism

$$I\colon [\mathbb{C}^N/G] \to [\mathbb{C}^N/G]$$

exchanging $[W^{+\infty}/G]$ with $[W^{-\infty}/G]$. Indeed we can set $I(x_1,\dots,x_N)=(e^{\pi \mathbf{i} q_1}x_1,\dots,e^{\pi \mathbf{i} q_N}x_N)$ satisfying

$$W(I(x_1,\dots,x_N)) = -W(x_1,\dots,x_N).$$

Recall that automorphisms of $[\mathbb{C}^N/G]$ are defined up to natural transformation (composition with elements of G). The automorphism I induces the nondegenerate *inner* pairing

$$\begin{array}{rcl}\langle\cdot,\cdot\rangle\colon H^N(\mathbb{C}^N,W^{+\infty};\mathbb{C})^G\times H^N(\mathbb{C}^N,W^{+\infty};\mathbb{C})^G & \longrightarrow & \mathbb{C}\\ (\alpha,\beta) & \mapsto & P(\alpha, I^*\beta)\end{array}$$

by passing to cohomology via the universal coefficient theorem and by using (II.2). Notice that I is defined up to a natural transformation; since we are working with G-invariant cohomology classes this still yields a well defined pairing.

There is an obvious identification ε between $H^{N_g}(\mathbb{C}^N_g, W_g^{+\infty};\mathbb{C})^G$ and $H^{N_h}(\mathbb{C}^{N_h}, W_h^{+\infty};\mathbb{C})^G$ as soon as $g=h^{-1}$ in G. This allows us to define a nondegenerate pairing between these two spaces via $\langle\cdot,\cdot\rangle_g=\langle\cdot,\varepsilon(\cdot)\rangle$ and, in turn, a nondegenerate pairing *globally* on $\mathcal{H}_{W,G}$.

Definition 1.3.1 (pairing for $\mathcal{H}_{W,G}$)**.** We have a nondegenerate inner product

$$\langle\cdot,\cdot\rangle\colon \mathcal{H}_{W,G}\times\mathcal{H}_{W,G}\to\mathbb{C}$$

pairing $\mathcal{H}^a_{W,G}$ and $\mathcal{H}^{2\widehat{q}_W-a}_{W,G}$, for

$$\widehat{q}_W = N-2q=\textstyle\sum_j(1-2q_j).$$

The index q_W is usually called the *central charge.*

The above formula follows from the well known relation

$$\mathrm{age}(g)+\mathrm{age}(g^{-1}) = N-N_g$$

from Chen–Ruan cohomology and the overall shift by q in Definition 1.2.1; it shows that the state space behaves like the cohomology of a variety of complex dimension $\widehat{q}_W$.

Assume that $\sum_j q_j=1$. Then the inner product pairs degree-a terms with degree-$(N-2-a)$ terms. This indicates that under the condition $\sum_j q_j$ the space $\mathcal{H}_{W,G}$ may be isomorphic to the cohomology of a smooth $(N-2)$-dimensional variety. Theorem III/3.1.2 shows that this is indeed the case.

[1] We define $\mathrm{age}(\alpha,V)\in\mathbb{Q}$ for any finite order autormorphism α of a vector space V, or — equivalently — for any representation of $\boldsymbol{\mu}_r$ for some $r\in\mathbb{N}$. Each character $\chi\colon\boldsymbol{\mu}_r\to\mathbb{C}^*$ is of the form $t\to t^k$ for a unique integer k with $0\le k\le r-1$ and, for these representations, we define the age of χ as k/r. Since these characters form a basis for the representation ring of $\boldsymbol{\mu}_r$, this extends to a unique additive homomorphism which we denote by age: $R\boldsymbol{\mu}_r\to\mathbb{Q}$.

2 The moduli space

The relevant moduli space is also defined starting from the pair (W, G) with an A-admissible group G.

2.1 The moduli stack associated to W

The first step is the definition of a moduli stack $W_{g,n}$ attached to the nondegenerate polynomial

$$W = \sum_{i=1}^{s} \gamma_i \prod_j x_j^{m_{i,j}}. \tag{II.5}$$

We provide an elementary definition, simplifying that of [51] without losing the essential geometric information needed to set up the intersection theory (see [33]). The moduli stack $W_{g,n}$ is an étale cover of a compactification of the usual moduli stack of l-stable orbifold curves $\overline{\mathsf{M}}_{g,n}^{l\text{-st}}$ for

$$l = \exp(\mathrm{Aut}(W));$$

in other words l equals the exponent of the group $\mathrm{Aut}(W)$ (the smallest integer l for which $g^l = 1$ for all $l \in \mathrm{Aut}(W)$).

Definition 2.1.1. On an l-stable curve C, a *W-structure* is the datum of N (as many as the variables of W) level-l structures $\mathsf{L}_1, \dots, \mathsf{L}_N$ with respect to $\omega_{\log}^{\otimes lq_1}, \dots, \omega_{\log}^{\otimes lq_N}$

$$\big(\mathsf{L}_j\ ,\ \phi_j\colon \mathsf{L}_j^{\otimes l} \xrightarrow{\sim} \omega_{\log}^{\otimes lq_j}\big)_{j=1}^N$$

satisfying the following s conditions (as many as the monomials $W_1, \dots, W_s$). For each $i = 1, \dots, s$ and for $W_i(\mathsf{L}_1, \dots, \mathsf{L}_N) = \bigotimes_{j=1}^N \mathsf{L}_j^{\otimes m_{i,j}}$, the condition

$$W_i(\mathsf{L}_1, \dots, \mathsf{L}_N) \cong \omega_{\log} \tag{II.6}$$

holds. An n-pointed genus-g l-stable curve equipped with a W-structure is called an *n-pointed genus-g W-curve*. We denote by $W_{g,n}$ the moduli stack.

Remark 2.1.2. Let us point out a side issue which will not appear in the rest of this text. Since j_W is in $\mathrm{Aut}(W)$, it is automatic that lq_j is integer. On the other hand, the exponent l of $\mathrm{Aut}(W)$ is not the order $|j_W|$ of j_W. As a counterexample consider the D_4 singularity $x^3 + xy^2$: the order of j_W is 3 but the exponent l is 6. In the next pages we always study $W_{g,n}$ in cases where l equals $|j_W|$.

By definition, the stack $W_{g,n}$ is embedded into the fibred product of N copies of moduli stacks of level structures

$$\overline{\mathsf{R}}_{g,n}^{l,lq_1} \times_{\overline{\mathsf{M}}_{g,n}^{l\text{-st}}} \cdots \times_{\overline{\mathsf{M}}_{g,n}^{l\text{-st}}} \overline{\mathsf{R}}_{g,n}^{l,lq_N}.$$

It is a proper, smooth Deligne–Mumford stack; more precisely, it is étale and finite over $\overline{\mathsf{M}}_{g,n}^{l\text{-st}}$ which is a proper and smooth stack of dimension $3g - 3 + n$ (under the stability condition $2g - 2 + n > 0$)

$$W_{g,n} \longrightarrow \overline{\mathsf{M}}_{g,n}^{l\text{-st}}.$$

2.2 Decomposition of $W_{g,n}$

As a consequence of (I.9), the stack $W_{g,n}$ decomposes into several connected substacks defined by specifying the types of the roots $\mathsf{L}_1, \dots, \mathsf{L}_N$ at the points $\sigma_1, \dots, \sigma_n$. We organize these data into n multiindices $h_1, \dots, h_n$ each one with N entries.

Definition 2.2.1. Let us fix n multiindices with N entries

$$h_i = (e^{2\pi \mathtt{i} \Theta_1^i}, \dots, e^{2\pi \mathtt{i} \Theta_N^i}) \in U(1)^N \tag{II.7}$$

for $i = 1, \dots, n$ and $\Theta_j^i \in [0, 1[$. Then $W(h_1, \dots, h_n)_{g,n}$ is the stack of n-pointed genus-g W-curves for which the jth level strcuture L_j has type $(\Theta_j^1 l, \dots, \Theta_j^n l)$ at the n markings.

Proposition 2.2.2. *Let $n > 0$. The stack $W_{g,n}$ is the disjoint union*

$$W_{g,n} = \bigsqcup_{h_1,\dots,h_n \in U(1)^N} W(h_1,\dots,h_n)_{g,n}.$$

The stack $W(h_1,\dots,h_n)_{g,n}$ is nonempty if and only if

$$\begin{cases} h_i = (e^{2\pi \mathtt{i}\Theta_1^i},\dots,e^{2\pi \mathtt{i}\Theta_N^i}) \in \operatorname{Aut}(W) & i = 1,\dots,n; \\ q_j(2g-2+n) - \sum_{i=1}^n \Theta_j^i \in \mathbb{Z} & j = 1,\dots,N. \end{cases} \tag{II.8}$$

In this case, it has degree $\operatorname{Aut}(W)^{2g}/l^N$ over $\overline{\mathsf{M}}_{g,n}^{l\text{-}st}$. □

Remark 2.2.3. A marking of a W-curve is therefore attached with a multiindex $h = (h_1,\dots,h_N) \in \operatorname{Aut}(W)$. The case where all coordinates of h are nontrivial is special: the sections of the line bundles $\mathsf{L}_1,\dots,\mathsf{L}_N$ necessarily vanish at such a marking. In this sense the bundle at that marking is "narrow". Similarly, a narrow node in the sense of Section I/3.3 is a node whose multiplicities h and $h^{-1} \in \operatorname{Aut}(W)$ on the two branches are narrow. Again, as we pointed out in the previous chapter, sections necessarily vanish at such a node.

2.3 The moduli stack associated to W and G

We identify open and closed substacks of $W_{g,n,G}$ where the local indices h only belong to a given subgroup G of $\operatorname{Aut}(W)$. We always consider $G \ni j_W$ (A-admissibility condition). Then, G can be regarded as the group of diagonal symmetries $\operatorname{Aut}(Z)$ for a polynomial

$$Z = W(x_1,\dots,x_N) + \text{ extra quasihomog. terms in the variables } x_1,\dots,x_N.$$

The above statement is proven in [83] only if we allow negative exponents in the extra terms (we require that the extra monomials are distinct from those of W but involve the same variables $x_1,\dots,x_N$ with charges $q_1,\dots,q_N$).

In this way to each A-admissible subgroup G of $\operatorname{Aut}(W)$ we can associate a substack $W_{g,n,G}$ of $W_{g,n}$ whose object will be referred to as (W,G)-curves.

Definition 2.3.1. Let $W_{g,n,G}$ be the full subcategory of $W_{g,n}$ whose objects $(L_1,\dots,L_N)$ satisfy $Z_t(L_1,\dots,L_N) \cong \omega_{\log}$, where $Z = \sum_t Z_t$ is the sum of monomials Z_t satisfying $G = \operatorname{Aut}(Z)$.

Remark 2.3.2. As in Proposition 2.2.2, for $n > 0$, we have

$$W_{g,n,G} = \bigsqcup_{h_1,\dots,h_n \in G} W(h_1,\dots,h_n)_{g,n,G},$$

where $h_i \in G$ is the local index at the ith marked point.

Example 2.3.3. The case where $G = \langle j_W \rangle$ is easy to work out. The substack $W_{g,n,\langle j_W \rangle} \subseteq W_{g,n}$ can be easily identified to a stack of the form $\overline{\mathsf{R}}_{g,n}^{l}$. is the image of the stack of roots of ω of order l via the functor

$$(L,\varphi) \mapsto \big((L^{\otimes lq_1},\varphi^{\otimes lq_1}),\dots,(L^{\otimes lq_N},\varphi^{\otimes lq_N})\big)$$

We recall that we work under the assumption $l = |j_W|$ (see Remark 2.1.2).

3 The virtual cycle

The FJRW invariants of (W,G) fit in the the formalism of Gromov–Witten theory. Fix the genus g and the number of markings n (with $2g-2+n > 0$, stability condition); then, for any choice of nonnegative integers $a_1,\dots,a_n$ (associated to powers of psi classes $\psi_1^{a_1},\dots,\psi_1^{a_n}$) and any choice of elements $\alpha_1,\dots,\alpha_n \in \mathcal{H}_{W,G}$ we can define an invariant (a rational number)

$$\langle \tau_{a_1}(\alpha_1),\dots,\tau_{a_n}(\alpha_n)\rangle_{g,n}^{W,G}. \tag{II.9}$$

Once the "target" (W,G) is fixed, the procedure is — as in Gromov–Witten theory — as follows. An intrinsic mathematical object is attached to each genus g and each number of markings n: the so called "virtual cycle". Then the psi classes $\psi_1^{a_1},\dots,\psi_1^{a_n}$ and the state space entries $\alpha_1,\dots,\alpha_n \in \mathcal{H}_{W,G}$ naturally yield — via some form of intersection theory — a rational number, the FJRW invariant.

3.1 Existence of a virtual cycle on $\overline{\mathsf{R}}^{l}_{g,n}$

The general construction of the virtual cycle from [51, 52, 53] is analytic. See [109, 108, 25, 111] for constructions in algebraic geometry. The general idea is to equip the moduli space of level structures with respect to $\omega_{\log}$ with a cohomology class with suitable facotrization properties. We discussed in Section II/3.3 that this is a delicate issue. We showed that in genus zero the index K class on $\overline{\mathsf{R}}^{l,1}_{g,n}(\boldsymbol{m})$ is represented by a vector bundle as long as $m_1, \dots, m_n$ are (strictly) positive. Furthermore, it admits a simple factorization when narrow nodes occur, see Remark I/3.3.6.

The cases treated in this text can be reconstructed from the Landau–Ginzburg model $x \mapsto x^l$, where we have the following theorem which summarizes work of [28, 25, 26, 75, 73, 74, 76, 99, 109, 108] (see in particular the paper of Polishchuk [108] where some crucial factorization properties are proven).

Theorem 3.1.1. *For any $\boldsymbol{m} = (m_1, \dots, m_n)$ with $m_i \in \{0, \dots, l-1\}$, there is a cycle*

$$c^l_W(\boldsymbol{m}) \in H^{2D}(\overline{\mathsf{R}}^{l,1}_{g,n}(\boldsymbol{m}); \mathbb{Q}),$$

with $D = (g-1)(1-2/l) + \sum_{i=1}^n (m_i - 1)$ satisfying the following properties.
Factorization properties*:*
For any $i \in \{0, \dots, g\}$ and for any $I \subseteq [n]$, let $M = M(i, I)$. We have

$$(\mu_{i,I})_* (j_{i,I})^* \left(c^l_W(\boldsymbol{m})\right) = c^l_W(\boldsymbol{m}_I, M) \times c^l_W(\boldsymbol{m}_{I'}, M'). \tag{II.10}$$

For any $M \in \{0, \dots, l-1\}$, we have

$$(\mu^M_{\mathrm{irr}})_* (j^M_{\mathrm{irr}})^* \left(c^l_W(\boldsymbol{m})\right) = c^l_W(\boldsymbol{m}, M, M'). \tag{II.11}$$

Vanishing property*:*
If $m_i = 0$ for some $1 \le i \le n$, then we have

$$c^l_W(\boldsymbol{m}) = 0. \tag{II.12}$$

Concavity property*:*
If for every point pt *of $\overline{\mathsf{R}}^{l,1}_{g,n}(\boldsymbol{m})$ the corresponding level-l curve $(\mathsf{L}_{\mathrm{pt}} \to \mathsf{C}_{\mathrm{pt}})$ has no sections, then $R^1\mathsf{p}_*\mathsf{L}$ is a vector bundle and*

$$c_W = c_{\mathrm{top}}((R^1\mathsf{p}_*\mathsf{L})^\vee).$$

The morphism used above are those introduced in Section I/3

$$\overline{\mathsf{R}}^{l}_{i,|I|+1}(1; (\boldsymbol{m}_I, M)) \times \overline{\mathsf{R}}^{l}_{g-i,|I'|+1}(1; (\boldsymbol{m}_{I'}, M')) \xleftarrow{\mu_{i,I}} \mathbb{S}_{i,I} \xrightarrow{j_{i,I}} \overline{\mathsf{R}}^{l,1}_{g,n}(\boldsymbol{m})$$

for $i \in \{0, \dots, g\}$, $I \subseteq [n]$, $M = q(i, I)$, and

$$\overline{\mathsf{R}}^{l}_{g-1,n+2}(1; (\boldsymbol{m}, M, M')) \xleftarrow{\mu^M_{\mathrm{irr}}} \mathbb{S}^M_{\mathrm{irr}} \xrightarrow{j_{\mathrm{irr},M}} \overline{\mathsf{R}}^{l}_{g,n}(1; \boldsymbol{m})$$

for $M \in \{0, \dots, l-1\}$. □

3.2 Application to W-curves: Fermat-type potentials

In a wide range of cases, the virtual cycle c^l_W can be used to define a Gromov–Witten type theory attached to a nondegenerate quasi-homogeneous polynomial W and a group $G \ni J_W$. In general, a more involved construction is needed.

Here, we focus on the case of Fermat polynomials of the form

$$W = x_1^l + \cdots + x_N^l.$$

We impose no conditions on G apart from $G \ni j_W$.

Then, we can define a version of the theory of Fan, Jarvis, and Ruan by means of the above class. More precisely, the intersection theory defined here represents the invariants $\langle \tau_{a_1}(\alpha_1), \dots, \tau_{a_n}(\alpha_n)\rangle^{W,G}_{g,n}$ for $\alpha_1, \dots, \alpha_n$ lying in the subspace spanned by narrow states.

Having $\alpha_1, \dots, \alpha_n$ in the narrow state subspace simplifies our notation. In [51] the definition of (II.9) is given by extending linearly the treatment of the special case where the entries $\alpha_i \in \mathcal{H}_{W,G}$ lie within a single summand $H^{N_g}(\mathbb{C}_g^N, W^{+\infty};\mathbb{C})$ of (II.3). We denote by h_i the group element satisfying $\alpha_i \in H^{N_{h_i}}(\mathbb{C}_{h_i}^N, W^{+\infty};\mathbb{C})$. Note that, when α_i is narrow, we have

$$H^{N_{h_i}}(\mathbb{C}_g^N, W^{+\infty};\mathbb{C}) \cong \mathbf{1}_{h_i}\cdot\mathbb{C};$$

i.e. there is a canonical generator $\mathbf{1}_{h_i}$ and, by abuse of notation, we can write $\langle \tau_{a_1}(h_1),\dots,\tau_{a_n}(h_n)\rangle_{g,n}^{W,G}$ for all these invariants.

Then, $W_{g,n,G}$ lies in the N-fold fibred product

$$W_{g,n,G} \subseteq \overline{\mathsf{R}}_{g,n}^{l,1} \times_{\overline{\mathsf{M}}^{l\text{-st}}} \cdots \times_{\overline{\mathsf{M}}^{l\text{-st}}} \overline{\mathsf{R}}_{g,n}^{l,1},$$

where the products are fibred over $\overline{\mathsf{M}}^{l\text{-st}}$. Each nonempty component $W_{g,n,G}(h_1,\dots,h_n)$ satisfying the conditions of Proposition 2.2.2 can be naturally projected to one of the factors via the forgetful morphism

$$\begin{aligned}\mathrm{pr}_j\colon W_{g,n,G}(h_1,\dots,h_n) &\to \overline{\mathsf{R}}_{g,n}^{l}(l\Theta_j^1,\dots,l\Theta_j^n) \qquad j=1,\dots,N,\\ (\mathsf{L}_1,\dots,\mathsf{L}_N)&\mapsto \mathsf{L}_j,\end{aligned}$$

where $h_1,\dots,h_N$ are multiindexes of complex numbers belonging to the circle group $U(1)$ as in (II.7). Then, we set

$$\langle \tau_{a_1}(h_1),\dots,\tau_{a_n}(h_n)\rangle_{g,n}^{W,G} = \int_{W_{g,n,G}} \prod_{i=1}^N \mathrm{pr_i}^* c_W^l(l\Theta_j^1,\dots,l\Theta_j^n)\prod_{h=1}^n \psi_h^{a_h}.$$

3.3 The special case of W paired with $G=\langle j_W\rangle$

Let us focus on a case where $G=\langle j_W\rangle$:

$$W=\sum_{j=1}^N x_j^l \qquad \text{and} \qquad G=\langle j_W\rangle.$$

The moduli space is simple: the line bundles $L_1,\dots,\mathsf{L}_N$ are equal to each other.

The genus-zero theory either vanishes or falls into the concave case. Indeed, on $W_{g,n,\langle J_W\rangle}(h_1,\dots,h_n)$ with

$$h_i=(e^{2\pi\mathtt{i}\frac{m_i}{l}},e^{2\pi\mathtt{i}\frac{m_i}{l}},\dots,e^{2\pi\mathtt{i}\frac{m_i}{l}},e^{2\pi\mathtt{i}\frac{m_i}{l}})\in\langle j_W\rangle$$

the virtual cycle is

$$\prod_{j=1}^N \mathrm{pr}_j^* c_W^l(m_i,\dots,m_n) = (c_W^l(\boldsymbol{m}))^N.$$

Furthermore, by Theorem 3.1.1 the above product, either vanishes, or can be written as

$$\langle \tau_{a_1}(h_1),\dots,\tau_{a_n}(h_n)\rangle_{g,n}^{W,G} = \int_{W_{g,n,G}} c_{\mathrm{top}}((R^1\mathsf{p}_*L)^\vee)^N\prod_{h=1}^n \psi_h^{a_h}. \tag{II.13}$$

This allows explicit computations. We have

$$c_{\mathrm{top}} = \exp\left(\sum\nolimits_k s_k\mathrm{ch}_k\right),$$

where ch_k is the degree-$2k$ term of the Chern character from Theorem 3.2.1 and the coefficients s_k are equal to $(k-1)!$.

For the Calabi–Yau case of the quintic degree-5 polynomial in 5 variables these numbers can be reduced to

$$\int_{\overline{\mathsf{R}}^5_{0,3+5k}(\mathbf{2})} (c_W^l)^5 \qquad \forall k\geq 0 \text{ in genus } 0. \tag{II.14}$$

(This is a consequence of the analogue to the topological recursion relations in Gromov–Witten theory [31].)

3.4 A quasihomogeneous polynomial W with $G = \langle j_W \rangle$.

Our final example is a slight generalization of the previous homogeneous Fermat polynomial W. We consider N divisors of l: $w_1, \dots, w_N$. Let

$$W = x_1^{l/w_1} + \dots + x_N^{l/w_N} \qquad \text{and} \qquad G = \langle j_W \rangle.$$

A W-structure is given by $(\mathsf{L}_1, \dots, \mathsf{L}_N)$ where L_j is an l/w_jth root of $\omega_{\log}$. Again, in $W_{g,n,\langle j_W \rangle}$the line bundles $\mathsf{L}_1, \dots, \mathsf{L}_N$ are strictly related to each other. They are determined by a single level-l structure L (with respect to $\omega_{\log}$): we have

$$\mathsf{L}_1 = \mathsf{L}^{\otimes w_1}, \dots, \mathsf{L}_N = \mathsf{L}^{\otimes w_N}.$$

We make the forgetful morphism explicit. It maps

$$(\mathsf{L}_1, \dots, \mathsf{L}_N) \in W_{g,n,\langle j_W \rangle}(h_1, \dots, h_n)$$

to each factor. First recall that each h_i is in $\langle j_W \rangle$; *i.e.* it is of the form

$$h_i = (e^{2\pi \mathtt{i} m_i \frac{w_1}{l}}, e^{2\pi \mathtt{i} m_i \frac{w_2}{l}}, \dots, e^{2\pi \mathtt{i} m_i \frac{w_N}{l}}) \in \langle j_W \rangle. \tag{II.15}$$

Then, we may regard the projection pr_j as

$$W_{g,n,\langle j_W \rangle}(h_1, \dots, h_n) \to \overline{\mathsf{R}}^{l/w_j,1}_{g,n} \left(\frac{l}{w_j} \left\langle \frac{w_j}{l} m_1 \right\rangle, \dots, \frac{l}{w_j} \left\langle \frac{w_j}{l} m_n \right\rangle \right)$$

where the operation

$$m \mapsto \frac{l}{w_j} \left\langle \frac{w_j}{l} m \right\rangle$$

is the standard reduction of $m \in \{0, \dots, l-1\}$ to the corresponding element of $\{0, \dots, \frac{l}{w_j} - 1\}$ via

$$\mathbb{Z}/l \mapsto w_j \mathbb{Z}/l\mathbb{Z} \cong \mathbb{Z}/\tfrac{l}{w_j}\mathbb{Z}.$$

An object of $W_{g,n,\langle j_W \rangle}(h_1, \dots, h_n)$ is determined by a level-l structure L, and the image via pr_j is given by $\mathsf{L} \mapsto \mathsf{L}^{\otimes w_j}$. The virtual cycle is

$$\prod_{j=1}^{N} \mathrm{pr}_j^* c_W^{l/w_j} \left(\frac{l}{w_j} \left\langle \frac{w_j}{l} m_1 \right\rangle, \dots, \frac{l}{w_j} \left\langle \frac{w_j}{l} m_n \right\rangle \right).$$

By the vanishing property of Theorem 3.1.1 it vanishes as soon as one of the entries $m_1, \dots, m_n \in \{0, \dots, l-1\}$ is a multiple of l/w_j for some j. In other words as soon as one of the n states $h_1, \dots, h_n$ of the form (II.15) attached to the markings is *broad*.

Example 3.4.1. Consider the explicit example

$$W(x_1, x_2, x_3, x_4) = x_1^6 + x_2^4 + x_3^4 + x_4^3,$$

a quasihomogeneous polynomial of degree 12 in four variables of weight $2, 3, 3, 4$. The 12 rays represent the elements $m \in \langle j_W \rangle \cong \mathbb{Z}/12$.

m	$\frac{l}{w_1}\langle \frac{w_1}{l} m \rangle$	$\frac{l}{w_2}\langle \frac{w_2}{l} m \rangle$	$\frac{l}{w_3}\langle \frac{w_3}{l} m \rangle$	$\frac{l}{w_4}\langle \frac{w_4}{l} m \rangle$
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	0
4	4	0	0	1
5	5	1	1	2
6	0	2	2	0
7	1	3	3	1
8	2	0	0	2
9	3	1	1	0
10	4	2	2	1
11	5	3	3	2

,

The virtual cycle c_W on $W_{0,n}(h_1, \dots, h_n)$ vanishes as soon as one of the entries $h_1, \dots, h_n \in \mathbb{Z}/12$ yields one zero via one of the functions $\frac{l}{w_j}\langle \frac{w_j}{l} h_i \rangle$ illustrated in the diagram above.

The diagram of figure (III.1) also provides a useful bookkeeping device and will serve to illustrate the LG-CY correspondence in some examples laters. The twelve rays correspond to the elements of $\langle j_W \rangle$. The rays carrying a black dot are the broad states, whereas the empty rays are the narrow states. The dots are obtained by drawing as many circles as the number of variables and by placing a dot on the rays whose angles correspond to w_jth roots of unities.

Chapter III

Cohomology

State spaces lie at the starting point of the definition of Gromov–Witten theory and of Fan–Jarvis–Ruan–Witten theory. At their level, we can provide an exhaustive picture featuring LG-CY correspondence as well as mirror symmetry.

Schematically, the theory presented in this text attaches invariants to any pair of nondegenerate polynomial W with a group containing j_W (A-admissible) and included in $SL_W = \mathrm{Aut}(W) \cap SL(N, \mathbb{C})$ (B-admissible). The invariants are of two sorts A-model invariants and B-model invariants. In this chapter they will simply amount to cohomology groups, or — more explicitly — to double graded vector spaces. We already saw the definition of the A-model invariant: it is the double graded space $\mathcal{H}_{W,G}$, its double grading will be from now on written as $(\deg_A^+, \deg_A^-)$. The first section is devoted to a B-model construction $(\mathcal{Q}_{W,G}, \deg_B^+, \deg_B^-)$ paralleling the construction of $(\mathcal{H}_{W,G}, \deg_A^+, \deg_A^-)$.

The second section will present an elementary mirror duality associating a mirror object $(W^\vee, G^\vee)$ to the object (W, G) (and preserving the admissibility conditions). We will state Berglund–Hübsch–Krawitz mirror theorem: the A-model $(\mathcal{H}_{W,G}, \deg_A^+, \deg_A^-)$ is isomorphic to the B-model $(\mathcal{Q}_{W,G}, \deg_B^+, \deg_B^-)$.

The third section focuses on a natural question: the application of this theorem to the geometric setup where (W, G) is interpreted as the hypersurface $X_W = \{W = 0\}$ modulo the group G. We will see that, under a condition insuring that $[X_W/G]$ is Calabi–Yau there is a Landau–Ginzburg mirror symmetry theorem stating that the cohomology of $[X_W/G]$ is isomorphic to $\mathcal{H}_{W,G}$ (with matching double gradings).

1 B model state space

Our discussion parallels Section II/1. The definition is based on the *local algebra* (also known as the *chiral ring* or the *Milnor ring*)

$$\mathcal{Q}_W := \mathbb{C}[x_1, \ldots, x_N] / \mathrm{Jac}(W),$$

where $\mathrm{Jac}(W)$ is the Jacobian ideal generated by partial derivatives:

$$\mathrm{Jac}(W) := (\partial_1 W, \ldots, \partial_N W).$$

We present the state space paralleling the discussion of the A model state space.

1.1 Local algebra from the classical point of view

The algebra $\mathcal{Q}_W$ is a graded algebra whose grading is determined by $x_j \mapsto q_j$. There is a unique element

$$\mathrm{hess}(W) = \det(\partial_i \partial_j W)$$

whose degree is maximal and equals the central charge $\widehat{q}_W = \sum_j (1 - 2q_j)$ already found in Definition 1.3.1. This is a fundamental invariant for singularities. The singularities with $\widehat{q}_W < 1$ are called *simple singularities* and are classified by the ADE sequence

- $A_l = x^{l+1}$ $(l \geq 1)$,
- $D_l = x^{l-1} + xy^2$ $(l \geq 4)$,
- $E_6 = x^3 + y^4$, $E_7 = x^3 + xy^3$, $E_8 = x^3 + y^5$.

When $\sum_i q_i = 1$, the corresponding hypersurface $X_W = \{W = 0\}$ is of CY type in the sense that ω is trivial. Note that in this case $\widehat{q}_W = N - 2$ is integral and $\mathcal{Q}$ may well be related to the cohomology of a variety. One of the main results of this chapter is the proof that this is indeed the case.

The dimension of the local algebra is given by the formula

$$\mu(W) = \prod_i \left(\frac{1}{q_i} - 1\right).$$

The dimension h_i of the subspace of $\mathcal{Q}_W$ of elements of degree λ_i can be easily computed, because $\sum_i h_i t^{\lambda_i}$ equals

$$P(t^d, W) = \prod_{j=1}^{N} \frac{1 - t^{d-w_j}}{1 - t^{w_j}},$$

where d is the order of j_W; *i.e*, the least common denominator of the charges q_i satisfying $q_j = w_j/d$.

For $f, g \in \mathcal{Q}_W$, the residue pairing $\langle f, g\rangle$ is determined by writing fg in the form

$$fg = \langle f, g\rangle \frac{\operatorname{hess}(W)}{\mu(W)} + \text{terms whose degree is less than } \widehat{q}_W.$$

This pairing is well defined, it is nondegenerate, and endows the local algebra with the structure of a Frobenius algebra (*i.e.* $\langle fg, h\rangle = \langle f, gh\rangle$). For more details, see [10].

1.2 The state space of (W, G)

From the modern point of view, the local algebra is regarded as a part of the B model theory of singularities. For its application, it is important to orbifold the construction by G.

Let us define the G action on $\mathcal{Q}$. The elements of $\mathcal{Q}_W$ are identified to the N-forms of $\Omega^N/dW \wedge \Omega^{N-1}$ via

$$\alpha \longmapsto \alpha dx_1 \wedge dx_2 \wedge \cdots \wedge dx_N \tag{III.1}$$

The group G naturally operates on $\Omega^N/dW \wedge \Omega^{N-1}$ and, via the above identification, on $\mathcal{Q}_W$. Note that this action on $\mathcal{Q}_W$ differs from the G-action induced on $\mathcal{Q}_W$ by $\mathbb{C}[x_1, \ldots, x_N]$. We have

$$\operatorname{Diag}(\alpha_1, \ldots, \alpha_N) \cdot \left(\prod\nolimits_j x_j^{m_j}\right) \bigwedge_j dx_j = \left(\prod\nolimits_j \alpha_j^{m_j+1} x_j^{m_j}\right) \bigwedge_j dx_j.$$

The orbifold B model graded vector space with pairing $(\mathcal{Q}_{W,G}, \langle\ \rangle)$ was essentially worked out by the physicists Intriligator and Vafa [70] (see [77] for a mathematical account). The ring structure was constructed later by Kaufmann [78] and Krawitz [83] in the case of the so called "invertible" W and B-admissible group $G \subseteq \operatorname{Aut}(W)$.

For each $g \in G$, we write as usual $\mathbb{C}^N_g$ for the points of $\mathbb{C}^N$ fixed by g. We write W_g for the restriction of W to $\mathbb{C}^N_g$. In this way W_g is a quasihomogeneous singularity in a subspace of $\mathbb{C}^N$ and admits a local algebra $\mathcal{Q}_{W_g}$ with a natural G-action.

Definition 1.2.1 (B model state space)**.** For any B-admissible group G, we set

$$\mathcal{Q}_{W,G} = \bigoplus_{g \in G} (\mathcal{Q}_{W_g})^G,$$

where $(\cdot)^G$ denots the G-invariant subspace.

Remark 1.2.2. The state space $\mathcal{Q}_{W,G}$ is clearly a module over $(\mathcal{Q}_W)^G$.

Remark 1.2.3. The B model state space is isomorphic to the A state space of Definition 1.2.1. On the other hand, the space is equipped with a different Hodge bigrading as follows. For a G-invariant form α of degree p in $\mathcal{Q}_{W_g} \cong \Omega^{N_g}/dW_g \wedge \Omega^{N_g-1}$, the bidegree $(\deg^+_B(\alpha), \deg^-_B(\alpha))$ is defined as follows

$$(\deg^+_B(\alpha), \deg^-_B(\alpha)) = (p, p) + (\operatorname{age}(g), \operatorname{age}(g^{-1})) - (q, q) \qquad (\text{with } q = \textstyle\sum q_j).$$

We will usually write $\mathcal{Q}^{a,b}_{W,G}$ for the terms of bidegree (a, b) and $\deg_B$ for the *total* degree $a + b$.

1.3 The inner pairing

Notice that $\mathcal{Q}_g$ is canonically isomorphic to $\mathcal{Q}_{g^{-1}}$. The pairing of $\mathcal{Q}_{W,G}$ is the direct sum of residue pairings

$$\langle\cdot,\cdot\rangle\colon \mathcal{Q}_g\times\mathcal{Q}_{g^{-1}}\to\mathbb{C}$$

via the pairing of the local algebra.

Definition 1.3.1 (pairing for $\mathcal{Q}_{W,G}$)**.** We have a nondegenerate inner product

$$\langle\cdot,\cdot\rangle\colon \mathcal{Q}_{W,G}\times\mathcal{Q}_{W,G}\to\mathbb{C}$$

pairing $\mathcal{Q}^a_{W,G}$ and $\mathcal{Q}^{2\widehat{q}_W-a}_{W,G}$.

2 Mirror symmetry between LG models

Berglund and Hübsch [13] consider polynomials in N variables having N monomials

$$W(x_1,\dots,x_N)=\sum_{i=1}^N\prod_{j=1}^N x_j^{m_{i,j}}. \tag{III.2}$$

Note that each of the N monomials has coefficient one; indeed, since the number of variables equals the number of monomials, even when we start from a polynomial of the form $\sum_{i=1}^N\gamma_i\prod_{j=1}^N x_j^{m_{i,j}}$, it is possible to reduce to the above expression by conveniently rescaling the N variables (this uses the nondegeneracy condition). In this way assigning a polynomial W as above amounts to specifying its exponent square matrix

$$E_W=(m_{i,j})_{1\le i,j\le N}.$$

The polynomials studied in [13] are called "invertible" because the matrix E_W is an invertible $N\times N$ matrix as a consequence of the uniqueness of the charges $q_1,\dots,q_N$ (nondegeneracy of W). There is a strikingly simple classification of invertible nondegenerate singularities by Kreuzer and Skarke [85].

An invertible potential W is nondegenerate if and only if it can be written, for a suitable permutation of the variables, as a sum of invertible potentials (with disjoint sets of variables) of one of the following three types:

$W_{\text{Fermat}}=x^a.$

$W_{\text{loop}}=x_1^{a_1}x_2+x_2^{a_2}x_3+\dots+x_{N-1}^{a_{N-1}}x_N+x_N^{a_N}x_1.$

$W_{\text{chain}}=x_1^{a_1}x_2+x_2^{a_2}x_3+\dots+x_{N-1}^{a_{N-1}}x_N+x_N^{a_N}.$

One can compute the charges $q_1,\dots,q_N$ by simply setting

$$q_i=\textstyle\sum_j m^{i,j}, \tag{III.3}$$

the sum of the entries on the ith line of $E_W^{-1}=(m^{i,j})_{1\le i,j\le N}$.

Each column $(m^{1,j},\dots,m^{N,j})$ of the matrix E_W^{-1} can be used to define the diagonal matrix

$$\rho_j=\operatorname{Diag}(\exp(2\pi\mathtt{i}m^{1,j}),\dots,\exp(2\pi\mathtt{i}m^{N,j})). \tag{III.4}$$

In fact these matrices satisfy the following properties $\rho_j^*W=W$; *i.e.* W is invariant with respect to ρ_j. Furthermore the group $\operatorname{Aut}(W)$ of diagonal matrices α such that $\alpha^*W=W$ is generated by the elements $\rho_1,\dots,\rho_N$:

$$\operatorname{Aut}(W):=\{\alpha=\operatorname{Diag}(\alpha_1,\dots,\alpha_N)\mid\alpha^*W=W\}=\langle\rho_1,\dots,\rho_N\rangle.$$

For instance, the above mentioned matrix j_W whose diagonal entries are $\exp(2\pi\mathtt{i}q_1),\dots,$ and $\exp(2\pi\mathtt{i}q_N)$ lies in $\operatorname{Aut}(W)$ and is indeed the product $\rho_1\cdots\rho_N$. Recall that

$$SL_W=\operatorname{Aut}(W)\cap SL(N,\mathbb{C}),$$

the matrices with determinant 1; in Berglund and Hübsch's construction we consider groups G containing j_W (A-admissible) and included in SL_W (B-admissible). We write $\widetilde{G}$ for the quotient $G/\langle j_W\rangle$.

The geometric side of the LG-CY correspondence is an orbifold or Deligne–Mumford stack. More precisely, let d be the least common denominator of $q_1=w_1/d,\dots,q_N=w_N/d$ (*i.e.* $d=|j_W|$). Then $X_W=\{W=0\}$ is a degree d hypersurface of the weighted projective space $\mathbb{P}(w_1,\dots,w_N)$. Then, W is nondegenerate (*i.e.* W

has a single critical point at the origin) if and only if X_W is a smooth Deligne-Mumford stack. Let W be a nondegenerate invertible potential of charges $q_1, \dots, q_N$ satisfying the *Calabi–Yau condition*

$$\textstyle\sum_j q_j = 1. \tag{III.5}$$

The geometrical meaning of this condition is that $X_W = \{W = 0\}$ is of Calabi–Yau type in the sense that ω is trivial (adjuction formula: $d = \sum_j w_j$).

Another important geometric condition is the Gorenstein condition of the ambient weighted projective space $\mathbb{P}(w_1, \dots, w_N)$. The Gorenstein condition corresponds to a numerical condition $w_i | \sum_j w_j$. For weighted projective spaces, the Gorenstein condition is equivalent to another well known condition, namely that the associated toric variety is reflexive. In the Calabi–Yau case we have $d = \sum_j w_j$. Then X_W is a Calabi–Yau hypersurface of Gorenstein weighted projective space if and only if $w_j | d$. If W is a Fermat polynomial, the ambient weighted projective stack is Gorenstein. Otherwise, the ambient weighted projective stack is not Gorenstein in general. Among all the known examples of Calabi–Yau hypersurfaces, Fermat polynomials constitute only a small fraction.

2.1 The polynomial $W^\vee$

Following Berglund–Hübsch, we consider the transposed polynomial $W^\vee$ defined by the property

$$E_{W^\vee} = (E_W)^\vee.$$

Namely, the polynomial $W^\vee$ is defined by transposing the matrix $(m_{i,j})$:

$$W^\vee(x_1, \dots, x_N) = \sum_{i=1}^{N} \prod_{j=1}^{N} x_j^{m_{j,i}}. \tag{III.6}$$

This construction preserves the type of polynomial sending Fermat to Fermat, loop to loop and chain to chain. This shows that $W^\vee$ is nondegenerate if and only if W is nondegenerate. Recall that q_j is the sum of the entries of the jth column of the inverse matrix E_W^{-1}. Hence, the charges $\overline{q}_1, \dots, \overline{q}_N$ of $W^\vee$ are the sums of the rows of E_W^{-1}. Therefore,

$$\textstyle\sum_j q_j = \sum_j \overline{q}_j.$$

In this way, $W^\vee$ is of Calabi–Yau type if and only if W is of Calabi–Yau type.

The striking idea of Berglund and Hubsch is that W and $W^\vee$ should be related by mirror symmetry. Clearly this is not true in the naive way: the mirror of a Fermat quintic three-fold X_W is not the quintic itself as one would get by transposing the corresponding exponent matrix. Instead, as already discussed in the introduction, the mirror $X_W^\vee$ is the quotient of X_W by the automorphism group $(\mathbb{Z}/5)^3$. It was already understood by Berglund–Hübsch that the correct statement should read

$$(W, G) \text{ mirror to } (W^\vee, G^\vee)$$

for a conjectural dual group $G^\vee$. Many examples of dual groups have been constructed in the literature. The general construction was given only recently by Krawitz [83].

2.2 The group $G^\vee$

The group $G^\vee$ is contained in $\mathrm{Aut}(W^\vee)$. Recall that $\mathrm{Aut}(W^\vee)$ is spanned by the diagonal symmetries $\rho_1^\vee, \dots, \rho_N^\vee$ determined by the columns of $(E_W^\vee)^{-1}$ as in (III.4):

$$\mathrm{Aut}(W^\vee) = \langle \rho_1^\vee, \dots, \rho_N^\vee \rangle.$$

Then $G^\vee$ is the subgroup defined by

$$G^\vee = \Big\{ \textstyle\prod_{j=1}^N (\rho_i^\vee)^{a_i} \mid \text{ if } \prod_{j=1}^N x_i^{a_i} \text{ is } G\text{-invariant} \Big\}. \tag{III.7}$$

Alternatively, we express any $g \in G$ as $g = \rho_1^{k_1} \dots \rho_N^{k_N}$ and $h \in G^\vee$ as $h = (\rho_1^\vee)^{l_1} \dots (\rho_N^\vee)^{l_N}$. Then, $G^\vee$ is determined by imposing within $\mathrm{Aut}(W^\vee)$ the following conditions for all $g = \rho_1^{k_1} \dots \rho_N^{k_N} \in G$

$$\begin{bmatrix} k_1 & \dots & k_N \end{bmatrix} E_W^{-1} \begin{bmatrix} l_1 \\ \vdots \\ l_N \end{bmatrix} \in \mathbb{Z}.$$

We have the following properties: transposition is an involution $(G^\vee)^\vee = G$, it is inclusion-reversing ($H \subseteq K \Rightarrow H^\vee \supseteq K^\vee$), it sends the trivial subgroup of $\mathrm{Aut}(W^\vee)$ to the total group $\mathrm{Aut}(W)$, and it exchanges $\langle j_W \rangle$ and $SL_{W^\vee}$.

2.3 Mirror symmetry conjectures between LG models

Now, we can state two mirror symmetry conjectures. Here, "mirror" means that the A model and the B model are exchanged. The first one is the Berglund–Hübsch–Krawitz mirror symmetry of the form LG|ᘓ⅃.

Conjecture 2.3.1 (mirror symmetry LG|ᘓ⅃)**.** *Suppose that W is a nondegenerate invertible polynomial. Then the Landau–Ginzburg models (W, G) and $(W^\vee, G^\vee)$ mirror each other.*

Let W be invertible and of Calabi–Yau type. We say $G \subseteq \mathrm{Aut}(W)$ is of *Calabi–Yau type* if $\langle j_W \rangle \subseteq G \subseteq SL_W$ (the fact that j_W is contained in SL_W follows from the Calabi–Yau condition (III.5)). In this case $\widetilde{G}$ acts on X_W faithfully and the quotient $[X_W/\widetilde{G}]$ is still an orbifold with trivial canonical bundle (Calabi–Yau type). The properties listed above for the construction associating $G^\vee$ to G show that G is of Calabi–Yau type if and only if $G^\vee$ is of Calabi–Yau type. Then, within the Calabi–Yau category, we obtain a mirror symmetry conjecture of type CY|YƆ.

Conjecture 2.3.2 (mirror symmetry CY|YƆ)**.** *Suppose that W and G satisfy the Calabi–Yau condition (automatically the same holds for $W^\vee$ and $G^\vee$). Then the stack $[X_W/\widetilde{G}]$ is the mirror of $[X_{W^\vee}/\widetilde{G}^\vee]$.*

Remark 2.3.3. Since we have not given a precise meaning to to the notion of mirror, the above conjectures should be viewed as a guideline instead of a mathematical statement. In the next section the above conjectures are turned into more precise mathematical statements. Here, we only interpret them as relations in terms of state spaces. Then, they may be regarded as saying: the A model state space of (W, G) is isomorphic to the B model state space to $(W^\vee, G^\vee)$. Even if it is elementary, the claim is nontrivial. For example it does not fit in Borisov–Batyrev duality of Gorenstein cones [12].. This happens systematically when W is not Fermat as was first noted in [42]. It was proven by Krawitz.

Theorem 2.3.4 (cohomological mirror symmetry LG|ᘓ⅃, [83])**.** *Suppose that W is invertible. Then, there is a bigraded vector space isomorphism*

$$\mathcal{H}_{W,G} \cong \mathcal{Q}_{W^\vee,G^\vee}.$$

Remark 2.3.5. It is worth mentioning that Krawitz [83] and Kaufmann [78] have also develloped the ring structure on $\mathcal{Q}$ and proven some cases of ring isomorphism. Furthermore, recently, Borisov has found [17] a new proof of the theorem above via vertex algebras. This approach may actually lead to a unified setup including both Berglund–Hübsch and Borisov–Batyrev duality.

3 LG-CY correspondence

3.1 The correspondence and mirror symmetry

With both state spaces of the LG model and the GW theory established, the simplest conjecture from the LG-CY correspondence is the following *cohomological LG-CY correspondence conjecture.*

Conjecture 3.1.1 (cohomological LG-CY correspondence)**.** *Suppose that the pair (W, G) is of Calabi–Yau type;* i.e. *W is nondegenerate (not necessarily invertible) with $\sum_j q_j = 1$ (Calabi–Yau condition) and G contains $\langle j_W \rangle$ and lies in SL_W. Then, there is a bigraded vector space isomorphism*

$$\mathcal{H}^{p,q}_{W,G} \cong H^{p,q}_{\mathrm{CR}}\left([X_W/\widetilde{G}]; \mathbb{C}\right), \qquad \forall p, q, \tag{III.8}$$

where the right-hand side is Chen–Ruan orbifold cohomology of the stack $[X_W/\widetilde{G}]$ with $\widetilde{G} = G/\langle j_W \rangle$.

This conjecture is certainly not true without assuming that W satisfied the Calabi–Yau condition $\sum_j q_j = 1$. For instance a quartic polynomial in five variables provides an immediate counterexample. The Calabi–Yau condition plays a crucial in the proof of the correspondence. Even if the formula in the statement above makes sense even for $G \not\subseteq SL_W$, this indicates that the isomorphism may fail without imposing a Calabi–Yau condition to G. Surprisingly we found that the above statement still holds when G is not contained in SL_W.

Theorem 3.1.2 (cohomological LG-CY correspondence, [40])**.** *Suppose that W is of Calabi–Yau type and that G contains j_W (no upper bound for G). Then the above cohomological LG-CY correspondence* (III.8) *holds.*

The main application is the folllowing *classical mirror symmetry*, which is a direct consequence of the cohomological LG-CY correspondence and Krawitz's mirror symmetry theorem of type LG|ᘓ⅃.

Corollary 3.1.3 (cohomological mirror symmetry CY|YƆ, [40]). *Suppose that W is invertible and that the pair (W, G) is of Calabi–Yau type (*i.e. *$\sum_j q_j = 1$ and $G \in SL_W$). Automatically, also the pair $(W^\vee, G^\vee)$ is of Calabi–Yau type. Furthermore, the Calabi–Yau orbifold $[X_W/\widetilde{G}]$ and the Calabi–Yau orbifold $[X_{W^\vee}/\widetilde{G}^\vee]$ form a mirror pair in the classical sense;* i.e. *we have the following isomorphism of Chen–Ruan orbifold cohomologies*

$$H^{p,q}_{\mathrm{orb}}\left([X_W/\widetilde{G}];\mathbb{C}\right) \cong H^{N-2-p,q}_{\mathrm{orb}}\left([X_{W^\vee}/\widetilde{G}^\vee];\mathbb{C}\right).$$

Corollary 3.1.4. *Assume that the quotient schemes $X_W/\widetilde{G}$ and $X_{W^\vee}/\widetilde{G}^\vee$ admit crepant resolutions Z and $Z^\vee$. Then the above statement yields a statement in ordinary cohomology:*

$$h^{p,q}(Z;\mathbb{C}) = h^{N-2-p,q}(Z^\vee;\mathbb{C}).$$

In the case where w_j divides d, Corollary 3.1.3 can be deduced from Borisov and Batyrev's construction of mirror pairs in toric geometry [12]. As already mentioned, the general case does not fit into polar duality because the associated toric variety is not reflexive. The following example illustrates this well.

Example 3.1.5. We consider the quintic hypersurface in $\mathbb{P}^4$ defined as the vanishing locus of

$$W = x_1^4x_2 + x_2^4x_3 + x_3^4x_4 + x_4^4x_5 + x_5^5.$$

This is a chain-type Calabi–Yau variety X whose Hodge diamond is clearly equal to that of the Fermat quintic and is well known: $h^{1,1} = 1$, $h^{0,3} = 1$, $h^{1,2} = 101$. The mirror Calabi–Yau is given by the vanishing of the polynomial

$$W^\vee(x_1, x_2, x_3, x_4, x_5) = x_1^4 + x_1x_2^4 + x_2x_3^4 + x_3x_4^4 + x_4x_5^5,$$

which may be regarded as defining a degree-256 hypersurface $X^\vee$ inside $\mathbb{P}(64, 48, 52, 51, 41)$. This is a degree-256 hypersurface of Calabi–Yau type (256 is indeed the sum of the weights). In this case, the ambient weighted projective stack is no longer Gorenstein (all weights but 64 do not divide the total weight 256). Note that the group SL coincides with $\langle j\rangle$ on both sides; therefore, Corollary 3.1.3 reads

$$h^{p,q}_{\mathrm{CR}}(X;\mathbb{C}) = h^{3-p,q}_{\mathrm{CR}}(X^\vee;\mathbb{C}).$$

Indeed, the Hodge diamond of $X_{W^\vee}$ satisfies $h^{1,1} = 101$, $h^{0,3} = 1$, $h^{1,2} = 1$ matching (IV.3).

Let us explain the role of the Gorenstein condition. Let us call the hypersurface $X_W \subset \mathbb{P}(w_1, \dots, w_N)$ *transverse* if the intersection of X_W with every coordinate subspace of the form $\mathbb{P}(w_{i_1}, \dots, w_{i_k})$ is either empty or a hypersurface. The transversality of X_W amounts essentially to the ambient space being Gorenstein. In another words, if $\mathbb{P}(w_1, \dots, w_N)$ is not Gorenstein, X_W will contain some coordinate subspace. The presence of these coordinate subspaces makes it more difficult to study X_W and its quotients. For instance, it is well known that the enumerative geometry of rational stable maps for these coordinate subspaces is an open problem in Gromov–Witten theory (this is due to the behavior of the virtual fundamental cycle). Initially, we thought that nonGorenstein cases such as loop and chain polynomials may provide counterexamples for the classical mirror symmetry conjecture. We actually found out that the cohomological LG-CY correspondence as well as the classical mirror symmetry conjecture hold in full generality. Similar issues arise in the enumerative geometry of curves; we will discuss this issue in §3.

3.2 A combinatorial model

To illustrate the idea of the proof, it is instructive to work out the case of the quintic three-fold.

Example 3.2.1. Consider $W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$ and the cyclic group $G = \langle j\rangle$ of order 5. For each element $j^m = (e^{2\pi i m/5}, \dots, e^{2\pi i m/5}) \in G$ with $m = 0, \dots, 4$ we compute $\mathcal{H}_{W,G} = \bigoplus_{g\in G} H^{N_g}(\mathbb{C}^N_g, W_g^{+\infty};\mathbb{C})^G$ and the total degree of its elements.

Let $m \neq 0$ and consider the elements of the summands corresponding to j^m. These are the narrow states where $H^{N_g}(\mathbb{C}^N_g, W_g^{+\infty};\mathbb{C})^G$ is isomorphic to $\mathbf{1}_g\mathbb{C}$. The total degree of $\mathbf{1}$ is $2m-2$. We obtain four elements of degree $0, 2, 4$ and 6; they correspond to the generators of $H^0(X_W,\mathbb{C})$, $H^2(X_W,\mathbb{C})$, $H^4(X_W,\mathbb{C})$ and $H^6(X_W,\mathbb{C})$.

Finally consider the remaining states which are not narrow and lie in $H^N(\mathbb{C}^N_1, W_1^{+\infty},\mathbb{C})^G$. This space is isomorphic to the degree-3 cohomology group of X_W. This holds in full generality as a consequence of the isomorphism between the G-invariant part of the local algebra and the primitive cohomology. The total degree of these elements is 3. Therefore, we recover the desired degree-preserving vector space isomorphism.

We learn from this example that the dichotomy determined by narrow and broad states within the Landau–Ginzburg state space corresponds to the well known dichotomy on the Calabi–Yau side between fixed classes and variable (or primitive) classes. In the orbifold setting, each sector of X_W lies in some subweighted projective coordinate space of the form $\mathbb{P}(w_{i_1}, \dots, w_{i_k})$. Therefore, this dichotomy applies to each sector. We say that an orbifold cohomological class is variable (or primitive) if it comes from a variable (or primitive) cohomology classe of some sector. It is straightforward to match the broad sector with variable classes. But it is far trickier to do so for narrow group elements versus fixed classes. We match these classes via a combinatorial construction based on an earlier model for Chen–Ruan orbifold cohomology of weighted projective spaces due to Boissière, Mann, and Perroni [16].

Example 3.2.2. In the case of $W(x_1, \dots, x_4) = x_1^6 + x_2^4 + x_3^4 + x_4^3$ the combinatorial model uses the diagram which appeared already in Example II/3.4.1.

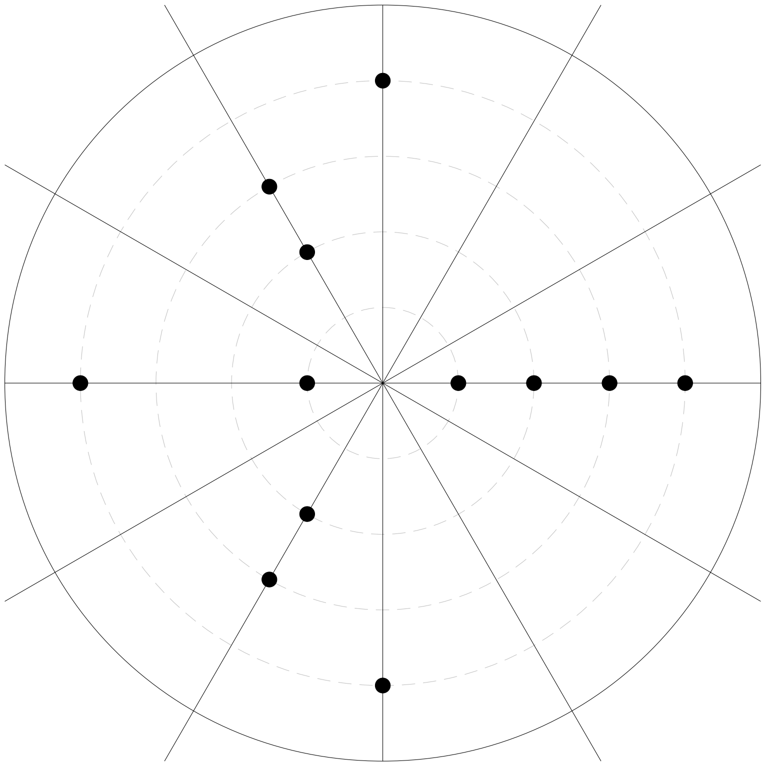


Figure III.1: Diagram of $\{x_1^6 + x_2^4 + x_3^4 + x_4^3 = 0\}$ inside $\mathbb{P}(2, 3, 3, 4)$.

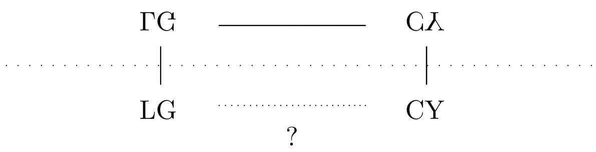
In fact, in [16], this diagram represents the sectors of the weighted projective stack $\mathbb{P}(2, 3, 3, 4)$; indeed, the dotted rays correspond to the so called "sectors"; *i.e.*, loci with nontrivial stabilizers. The number of dots lying on one ray corresponds to the dimension of the cohomology of the corresponding sector (which, in turn, is a weighted projective stack). If we consider the hypersurface where $W(x_1, \dots, x_4) = 0$ vanishes we can use the same diagram. The rays should be regarded as hypersurfaces lying inside the sectors of the ambient weighted projective stack. In the surface above we actually have six dotted rays corresponding to the sectors of the ambient projective stack. When the ray carries a single dot, the hypersurface is empty. When the ray carries two dots the hypersurface is 0-dimensional. Hence, in the example there are only four nonempty sectors corresponding to $J^0 = 1, J^{-4}, J^{-6}$, and J^{-8}. In general n dots on one ray correspond to an $(n-2)$-dimensional hypersurface: the first $n - 1$ dots counting from the origin are the classes cut out by $\mathbf{1}, p, \dots, p^{n-2}$ (where p is the hyperplane class), whereas the extremal dot corresponds to the contribution from primitive cohomology.

In this way, Example II/3.4.1 illustrates how the diagram above represents the elements of the state space $\mathcal{H}_{W,\langle j \rangle}$ on the LG side. The present example shows how the same diagram represents the cohomological classes of $H^*_{\mathrm{CR}}(X_W; \mathbb{C})$ on the CY side. In [32] we show how the gradings on both sides can be read off the diagram (above, we have decorated each dot with its total degree). The correspondence — at least in the Gorenstein case — can be proven using this observation. We refer to [32] for the argument in full generality.

Chapter IV

Global mirror symmetry

This chapter consists of three sections. In the first section, we detail the problem of providing a global formulation to mirror symmetry. For sake of clarity, we focus on the case of the quintic three-fold and we will only sketch the case of quotients of finite groups acting on Calabi–Yau hypersurfaces (we refer to [33] for a more detailed discussion). In the second section, we state a result obtained in collaboration with Yongbin Ruan in the case of the quintic three-fold. In the third section, we illustrate work in collaboration with Iritani and Ruan allowing us to rely the geometry of Calabi–Yau varieties to the Landau–Ginzburg model without using the mirror symmetry framework.



1 A global formulation

In the previous chapter we have provided a precise statement of mirror symmetry in terms of cohomology (*i.e.* in terms of state spaces). A deeper statement involves on the A-side informations such as the Kähler structure and Gromov–Witten invariants, and on the B side moduli of complex structures and period integrals. From a global point of view, this framework is at the same time deeper and somewhat incomplete. In fact the topology of the moduli space of complex structures on the B side is nontrivial, whereas the moduli space of Kähler structures is not. We illustrate this point in explicit terms.

We get back to

$$X_W = \{x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0\} \subset \mathbb{P}^4 \tag{IV.1}$$

and to the quotient stack

$$X_W^\vee = [X_W/(\mathbb{Z}/5)^3] \tag{IV.2}$$

where $(\mathbb{Z}/5)^3$ is defined as SL_W modulo the cyclic group of order 5 generated by $j_W = (\xi_5, \xi_5, \xi_5, \xi_5, \xi_5)$. The cohomology classes whose total degree is odd of $X_W^\vee$ form a four-dimensional subspace with a particularly simple filtration: the Hodge numbers $h^{p,q}$ with odd total index $p+q$ are given by $(h^{3,0}, h^{2,1}, h^{1,2}, h^{0,3}) = (1,1,1,1)$ and reflect the four powers $\mathbf{1}, p, p^2, p^3$ of the cycle p defined by the hyperplane section of the projective hypersurface X_W. This follows from the statement of Corollary 3.1.3

$$h^{p,q}(X_W) = h^{\dim - p,q}(X_W^\vee). \tag{IV.3}$$

1.1 Local mirror symmetry

We now illustrate a deeper mirror symmetry statement in this case ; we will focus on the differences from the global point of view between the two sides. On one side, for X_W, we consider the (complexified) moduli space of Kähler structures. This is a contractible complex space of dimension one ; we can consider it as an A side invariant denoted by

$$\mathcal{A}_{X_W}.$$

We can think of $\mathcal{A}_{X_W}$ as an open and contractible analytic neighborhood of the origin of $H^{1,1}(X_W;\mathbb{C})$. On the other side of the mirror, we consider a B side invariant: the deformations (of the complex structure) of $[X_W/(\mathbb{Z}/5)^3]$. These are indeed the deformations of X_W which are invariant with respect to the action of $(\mathbb{Z}/5)^3$. We obtain in this way the Dwork family already discussed in the introduction

$$X_{W,\psi} = \left\{ x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + 5\psi \prod_{i=1}^{5} x_i = 0 \right\},$$

on the projective line $\mathbb{P}^1$. We discuss in more detail the geometry of this family. The group $(\mathbb{Z}/5)^3 = SL_W/\langle j_W \rangle$ acts on the fibers of the family ; the quotient stack obtained by modding out $(\mathbb{Z}/5)^3$ gives rise to a family of Calabi–Yau stacks $[X_W/(\mathbb{Z}/5)^3]_\psi$ fibered over a Zariski open set in $\mathbb{P}^1_\psi$ (the complement of the divisor where the singularities arise). In fact, for all $i \in \{1, \ldots, 5\}$, the diagonal symmetry $x_i \mapsto \xi_5 x_i$ (fixing all remaining coordinates) acts on this family ; this action identifies the fibre $[X_W/(\mathbb{Z}/5)^3]_\psi$ with the fiber $[X_W/(\mathbb{Z}/5)^3]_{\xi_5\psi}$. In this way, the Dwork family induces a family of Calabi–Yau three-dimensional stacks on $[\mathbb{P}^1/(\mathbb{Z}/5)]$. Let us set $t = \psi^5$; then, the new family is smooth away from $t = \infty$ et $t = 1$. These two points, as well as the point $t = 0$ with nontrivial stabilizer in $[\mathbb{P}^1/(\mathbb{Z}/5)]$, are usually called *special limit points*; more specifically, $0, \infty$, et 1 are the *Gepner point*, the *large complex structure point*, and the *conifold point*. Unlike the Kähler moduli space, the moduli space of complex structures is *non contractible*. For this reason, mirror symmetry has been studied as a local identification between the contractible Kähler moduli space $\mathcal{A}_{X_W}$ and a contractible neighborhood of the point at infinity $t = \infty$

$$\mathcal{B}^{\infty}_{X_W^\vee}.$$

This leads to a formulation of mirror symmetry as a local statement matching the A mode and the B model restricted to a neighborhood of the large complex structure point.

We consider the vector bundle on $\mathcal{B}^{\infty}_{X_W^\vee}$ whose fibre on $t \in \mathcal{B}^{\infty}_{X_W^\vee}$ is given by $H^3([X_W/(\mathbb{Z}/5)^3]_t;\mathbb{C})$. The local system determined by $H^3([X_W/(\mathbb{Z}/5)^3]_t;\mathbb{Z})$ in $H^3([X_W/(\mathbb{Z}/5)^3]_t;\mathbb{C})$ can be regarded as a flat connection, the Gauss–Manin connection. Dubrovin showed how to use Gromov–Witten invariants to define a flat connection on the rank-four vector bundle defined as $H^{\mathrm{ev}}(X_W;\mathbb{C}) \otimes \mathcal{O}$ on $\mathcal{A}_{X_W}$. Under a suitable identification (mirror map)

$$\begin{array}{c} \mathcal{B}^{\infty}_{X_W^\vee} \\ \updownarrow \cong \\ \mathcal{A}_{X_W} \end{array} \tag{IV.4}$$

the two structures are identified (Givental [56], Lian–Liu–Yau [92]). This local point of view dominated the mathematical study of mirror symmetry for the last twenty years.

1.2 The global point of view

On the other hand, it is natural to study the entire complex moduli space near every special point. This *global* point of view underlies a large part of the recent physical literature and naturally gives rise to important predictions such as the holomorphic anomaly equation [15] and the above mentioned predictions by Huang–Klemm–Qackenbush [69] for genus $g \leq 52$. In the early 90's, a physical solution was proposed to complete the Kähler moduli space by including other *phases* [101, 125]. For the quintic three-fold, two phases arise in the A model: the CY geometry and the LG phase.

Whereas the CY geometry of the quintic has already been identified by mirror symmetry to a neighborhood of the large complex structure limit point $\mathcal{B}^{\infty}_{X_W^\vee}$, the LG model of Chapter II is expected to be mirror to the neighborhood of the Gepner point at 0

$$\mathcal{B}^{0}_{X_W^\vee}.$$

In this framework, the LG model and the Gromov–Witten theory of the quintic are related to each other via an analytic continuation from the Gepner point to the large complex structure point. From this point of view, the so called LG-CY correspondence should be viewed as a step towards global mirror symmetry.

On $\mathcal{B}^{0}_{X_W^\vee}$ consider the bundle with fibre $H^3([X_W/\widetilde{G}]_0,\mathbb{C})$ over the point t. There is again the flat Gauss–Manin connection induced by the local system $H^3([X_W/\widetilde{G}]_t;\mathbb{Z}) \subset H^3([X_W/\widetilde{G}]_\infty;\mathbb{C})$. The work of Fan, Jarvis, and Ruan [51] yields — via Dubrovin connection — a flat connection on a vector bundle on a contractible one-dimensional space

$$\mathcal{A}_{W,\mathbb{Z}/5}$$

(see the next section for further details (IV.9), and [71] for the abstract formalism of Dubrovin connection). The fibre of this bundle is the four-dimensional state space $\mathcal{H}_{W,\mathbb{Z}/5}$ attached to the singularity $(W,\mathbb{Z}/5)$ (see Chapter II). Under a suitable identification (mirror map)

$$\begin{array}{c} \mathcal{B}^0_{X_W^\vee} \\ \big\updownarrow \cong \\ \mathcal{A}_{W,\mathbb{Z}/5} \end{array} \tag{IV.5}$$

the two structures are identified [31].

Now, the correspondence between the Gromov–Witten theory of the quintic and the Landau–Ginzburg model of the singularity $W\colon \mathbb{C}^5 \to \mathbb{C}$ can be carried out via (IV.4) and (IV.5) on the B side via the local system induced by the family of CY orbifolds $[X_W/\widetilde{G}]_t$ with t varying in $(\mathbb{P}^1)^\times = \mathbb{P}^1_t \setminus \{0,1,\infty\}$.

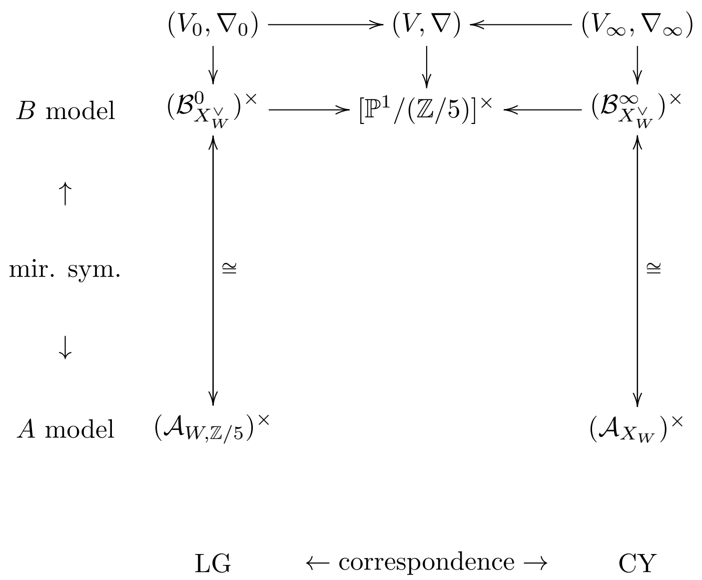


Figure IV.1: Casting LG-CY correspondence within the global mirror symmetry framework. For each U the notation $U^\times$ stands for $U \setminus \{\text{special points}\}$, the horizontal maps to $(\mathbb{P}^1)^\times$ are the natural inclusions, and V, V_0 and V_∞ are the four-dimensional bundles with fibre $H^3([X_W/\widetilde{G}]_t;\mathbb{C})$ equipped with the respective Gauss–Manin connections ∇.

On the CY side, *i.e.* on $\mathcal{A}_{X_W}$, the isomorphism (IV.4) and the study of the variation of the Hodge structure of $X^\vee_{W,t}$ on $\mathcal{B}^\infty_{X_W^\vee}$ allow us to associate to a given basis of $H^{\mathrm{ev}}(X_W)$ a basis of multivalued functions from $\mathcal{A}_{X_W}$ to $H^{\mathrm{ev}}(X_W)$ which are flat with respect to Dubrovin connection. This amounts to solving Gromov–Witten theory for X_W in genus zero. The analogous problem occurs on $\mathcal{A}_{W,\mathbb{Z}/5}$ on the LG side; it is solved via (IV.5) and amounts to compute Fan–Jarvis–Ruan–Witten theory for $(W,\mathbb{Z}/5)$ in genus zero. Furthermore, via analytic continuation, we can extend the bases of flat sections globally on $[\mathbb{P}^1/(\mathbb{Z}/5)]^\times$ and find a change of bases matrix

$$\mathbb{U}_{\text{LG-CY}}. \tag{IV.6}$$

This is explicitly computed in [31] and illustrated in the next section. We provide an interpretation of this linear transformation via Orlov's equivalence in the last section of this chapter.

Remark 1.2.1 (global mirror symmetry via Berglund–Hübsch)**.** In [33] we provide a conjectural picture where Figure IV.1 is generalized. Schematically, the setup is the following. The one-parameter Dwork family is replaced by a deformation of the form

$$t_0 W^\vee + \sum_{i=1}^{l} t_i M_i,$$

with $M_1 = \prod_j x_j$ and $M_i \in [\mathcal{Q}_{W^\vee}]^{\deg=1}$ (here $\mathcal{Q}_{W^\vee}$ denotes the Milnor ring as in the previous chapter). The basis is the projective space $\mathbb{P}^l$ modulo the group $Z = \mathrm{Aut}(W^\vee)/G^\times$. (We notice that when $G = \langle j_W \rangle$ and W is the quintic polynomial, this yields $\mathbb{Z}/5$.) The special points 0 and ∞ are the points $0 = (t_0 \neq 0)$ and

$$
\begin{array}{cccccc}
B \text{ model} & (\mathcal{B}^0_{[X_W/\widetilde{G}]^\vee})^\times & \longrightarrow & [\mathbb{P}^l/Z] & \longleftarrow & (\mathcal{B}^\infty_{[X_W/\widetilde{G}]^\vee})^\times \\
\uparrow & \Big\updownarrow \cong & & & & \Big\updownarrow \cong \\
\text{mir. sym.} & & & & & \\
\downarrow & & & & & \\
A \text{ model} & (\mathcal{A}_{W,G})^\times & & & & (\mathcal{A}_{[X_W/\widetilde{G}]})^\times \\
 & \text{LG} & & \leftarrow \text{correspondence} \rightarrow & & \text{CY}
\end{array}
$$

Figure IV.2: The generalized framework: $[X_W/\widetilde{G}]^\vee = [X_{W^\vee}/\widetilde{G}^\vee]$.

$\infty = (t_1 \neq 0)$. The local system $H^3(X^\vee_{W,t};\mathbb{C})$ is replaced by the analogue space of primitive cohomology classes in H^{N-2}. The restriction to primitive cohomology classes on the B side has a specular counterpart in the conjecture: namely, we only consider the $\mathrm{Aut}(W)$-invariant part of quantum theory on the A-side (the isomorphism from Theorem 2.3.4 sets precisely the equivalence involving $\mathrm{Aut}(W)$-invariant states, see [83] and [33]).

2 LG-CY via global mirror symmetry

In this section we give a precise statement of the LG-CY correspondence for the quintic threefold. We follow [31].

The Fan–Jarvis–Ruan–Witten genus-zero invariants

$$\langle \tau_{a_1}(\phi_{h_1}), \dots, \tau_{a_{n-1}}(\phi_{h_{n-1}}), \tau_{a_n}(\phi_{h_n})\rangle^{\mathrm{FJRW}}_{0,n}$$

of the quintic polynomial W with respect to the group $\langle j\rangle$ have been defined precisely in II.13. We use the notation $\langle \dots \rangle^{\mathrm{FJRW}}_{0,n}$ in order to distinguish them from the analogue genus-zero Gromov–Witten invariants of the W-hypersurface (CY side)

$$\langle \tau_{a_1}(\varphi_{h_1}), \dots, \tau_{a_{n-1}}(\varphi_{h_{n-1}}), \tau_{a_n}(\varphi_{h_n})\rangle^{\mathrm{GW}}_{0,n,\delta}.$$

Both sets of invariants are defined for any $(a_1, \dots, a_n) \in \mathbb{N}^n$ and for any entry of the even-dimensional state spaces of the two respective theories: $H_{\mathrm{FJRW}} = \mathcal{H}_{W,\langle j\rangle} = \oplus_h \phi_h \mathbb{C}$ and $H_{\mathrm{GW}} = H^{\mathrm{ev}}(X_W;\mathbb{C}) = \oplus_h \varphi_h \mathbb{C}$ (in both cases these are four-dimensional spaces, and we set h between 0 and 3).

These two sets of numbers, defined in very different ways, can be incorporated into the Fan–Jarvis–Ruan–Witten generating function (also called partition function) and into the Gromov–Witten generating function, which, by standard techniques, can be reconstructed from the generating functions of one-point descendants: the invariants $\langle \tau_0(\phi_{h_1}), \dots, \tau_0(\phi_{h_{n-1}}), \tau_a(\phi_{h_n})\rangle^{\mathrm{FJRW}}$ and $\langle \tau_0(\varphi_{h_1}), \dots, \tau_0(\varphi_{h_{n-1}}), \tau_a(\varphi_{h_n})\rangle^{\mathrm{GW}}$ with not more than one entry $\tau_a(\phi_h)$ and $\tau_a(\varphi_h)$ having $a \neq 0$. In other words, the two theories are determined by the J-functions $J_{\mathrm{FJRW}}(\sum_h t_0^h \phi_h, z)$ equal to

$$z\phi_0 \quad + \quad \sum_h t_0^h \phi_h \quad + \quad \sum_{\substack{n\geq 0 \\ (h_1,\dots,h_n)}} \sum_{\epsilon,k} \frac{t_0^{h_1}\cdots t_0^{h_n}}{n! z^{k+1}} \langle \tau_0(\phi_{h_1}), \dots, \tau_0(\phi_{h_n}), \tau_k(\phi_\epsilon)\rangle^{\mathrm{FJRW}}_{0,n+1} \phi^\epsilon, \tag{IV.7}$$

and $J_{\mathrm{GW}}(\sum_h t_0^h \varphi_h, z)$ equal to

$$z\varphi_0 \quad + \quad \sum_h t_0^h \varphi_h \quad + \quad \sum_{\substack{n\geq 0 \\ \delta\geq 0 \\ (h_1,\dots,h_n)}} \sum_{\epsilon,k} \frac{t_0^{h_1}\cdots t_0^{h_n}}{n! z^{k+1}} \langle \tau_0(\varphi_{h_1}), \dots, \tau_0(\varphi_{h_n}), \tau_k(\varphi_\epsilon)\rangle^{\mathrm{GW}}_{0,n+1,\delta} \varphi^\epsilon, \tag{IV.8}$$

which can be regarded as terms of $H_{\mathrm{FJRW}}((z^{-1}))$ and $H_{\mathrm{GW}}((z^{-1}))$, *i.e.* Laurent series with coefficients in H_{FJRW} and H_{GW}. The techniques allowing to reconstruct the entire theories from these two functions hold in general and are based on the string and dilaton equations (see [9]).

In fact, for $W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$, further simplifications occur. In Fan–Jarvis–Ruan–Witten theory, all computations are reduced to

$$J_{\mathrm{FJRW}}(t_1\phi_1, z) = z\phi_0 + t_1\phi_1 + \sum_{n\geq 0}\sum_{\epsilon,k} \frac{(t_1)^n}{n!z^{k+1}} \langle \tau_0(\phi_1), \dots, \tau_0(\phi_1), \tau_k(\phi_\epsilon)\rangle_{0,n+1}^{\mathrm{FJRW}} \phi^\epsilon.$$

The analogue one-parameter expression for the J-function also holds on the line $t_1\varphi_1$ in Gromov-Witten theory; we refer to the discussion in [31].

On a neighbouhood of 0 in $t_1 \in \mathbb{C}$, consider the trivial bundles whose fiber equals H_{FJRW}. A nontrivial connection may be defined as follows. Recall that $\langle \dots \rangle_{0,n}^{\mathrm{FJRW}}$ naturally defines a family of products $\circ_{t_1}$ depending on a the parameter t_1. Then, set

$$\nabla_{t_1}(X) = \frac{\partial}{\partial t_1} + \phi_1 \circ_{t_1} X. \qquad \text{(IV.9)}$$

In this way, we have the rank-four local system on $\mathcal{A}_{W,\mathbb{Z}/5}$. This parallels the setup of Dubrovin connection in Gromov–Witten theory yielding the rank-four local system on $\mathcal{A}_{X_W}$. As discussed in the previous section two mirror map play a crucial role: (IV.4) and (IV.5).

2.1 Mirror symmetry on the CY side

On the CY side, Givental's mirror symmetry theorem [56] for the quintic three-fold sets an equivalence between the above J-function and the $H_{\mathrm{GW}}((z^{-1}))$-valued I-function

$$I_{\mathrm{GW}}(q,z) = \sum_{d\geq 0} zq^{p/z+d} \frac{\prod_{k=1}^{5d}(5p+kz)}{\prod_{k=1}^{d}(p+kz)^5},$$

where H is the cohomology class corresponding to the hyperplane section, $q^{p/z}$ should be read as the expansion of $\exp(p\log(q)/z)$ in the cohomology ring, and $q = \exp(t_1)$ parametrizes the line $\mathbb{C}\varphi_1$ as already mentioned. Expanded in the variable p, the I-function decomposes in the form $f_{0,\mathrm{GW}} + f_{1,\mathrm{GW}} + f_{2,\mathrm{GW}} + f_{3,\mathrm{GW}}$, the sum of the period integrals spanning the space of solutions of Picard–Fuchs equation

$$\left[{D_q}^4 - 5q\prod_{m=1}^{4}(5D_q + mz)\right] I_{\mathrm{GW}} = 0 \qquad \left(\text{for } D_q = zq\frac{\partial}{\partial q}\right).$$

Via an explicit change of variables

$$\log q' = \frac{f_{1,\mathrm{GW}}(q)}{f_{0,\mathrm{GW}}(q)} \qquad (\text{with } f_{0,\mathrm{GW}} \text{ and } f_{1,\mathrm{GW}}\ \mathbb{C}\text{-valued and } f_{0,\mathrm{GW}} \text{ invertible})$$

the A-model of the quintic (*i.e.* J_{GW}), matches the B-model of the quintic (*i.e.* I_{GW}), via a mirror map

$$\frac{I_{\mathrm{GW}}(q,z)}{f_{0,\mathrm{GW}}(q)} = J_{\mathrm{GW}}(q', z). \qquad \text{(IV.10)}$$

In other words, up to an identification between the base spaces, the four-dimensional local system on $\mathcal{A}_{X_W}$ matches the four-dimensional local system on $\mathcal{B}^{\infty}_{X_W^\vee}$ as in (IV.4).

2.2 Mirror symmetry on the LG side

We provide the same picture on the LG side.

Theorem 2.2.1 ([31]). *Consider the $H_{\mathrm{FJRW}}((z^{-1}))$-valued function (where $[a]_n = a(a+1)\dots(a+n-1)$)*

$$I_{\mathrm{FJRW}}(t,z) = z\sum_{k=1,2,3,4} \frac{1}{\Gamma(k)} \sum_{l\geq 0} \frac{([\frac{k}{5}]_l)^5\ t^{k+5l}}{[k]_{5l}\ z^{k-1}} \phi_{k-1}.$$

The four summands $f_{0,\mathrm{FJRW}}, f_{0,\mathrm{FJRW}}, f_{0,\mathrm{FJRW}}, f_{0,\mathrm{FJRW}}$ span the solution space of the Picard–Fuchs equation

$$\left[{D_t}^4 - 5^5 t^{-5}\prod_{m=1}^{4}(D_t - mz)\right] I_{\mathrm{FJRW}} = 0 \qquad \left(\text{for } D_t = zt\frac{\partial}{\partial t}\right)$$

and coincide with the period integrals at the Gepner point computed by Huang, Klemm, and Quackenbush [69]. The above I-function and the J-function of FJRW*-theory are related by an explicit change of variables (mirror map)*

$$t' = \frac{f_{1,\mathrm{FJRW}}(t)}{f_{0,\mathrm{FJRW}}(t)} \qquad \text{(with } f_{0,\mathrm{FJRW}} \text{ and } f_{1,\mathrm{FJRW}} \ \mathbb{C}\text{-valued and } f_{0,\mathrm{FJRW}} \text{ invertible)}$$

satisfying

$$\frac{I_{\mathrm{FJRW}}(t,z)}{f_{0,\mathrm{FJRW}}(t)} = J_{\mathrm{FJRW}}(t',z). \tag{IV.11}$$

Remark 2.2.2. Equation IV.10 is the mirror identification (IV.5).

Remark 2.2.3. The proof of the theorem above may be regarded as an application of Theorem I/3.2.1.

2.3 The correspondence via mirror symmetry

The Picard–Fuchs equation in the above statement coincides with that of the quintic three-fold for $q = t^{-5}$. After this identification of the coordinate patch at $t = 0$ with the coordinate patch at $q = \infty$, the two I-functions are solutions of the same Picard–Fuchs equation. Since I_{GW} and I_{FJRW} take values in two isomorphic state spaces, we can compute the analytic continuation of I_{GW} and obtain two different bases spanning the space of solutions of the same Picard–Fuchs equation. Therefore, we have the following corollary.

Corollary 2.3.1. *There is a degree-preserving symplectic transformation $\mathbb{U}_{LG\text{-}CY}$ mapping I_{FJRW} to the analytic continuation of I_{GW} near $t = 0$.*

Remark 2.3.2. Since, the I-functions I_{FJRW} and I_{GW} encode the generating functions for the genus-zero part of the corresponding theories, the symplectomorphism $\mathbb{U}_{\text{LG-CY}}$ establishes an equivalence between the computation of the Gromov–Witten genus-zero invariants of the quintic and the intersection theory setup in Chapter II

Remark 2.3.3 (higher genus conjecture)**.** We have explicitly computed $\mathbb{U}$ using the Mellin–Barnes method for analytic continuation. In [31], we conjecture that the quantization of $\mathbb{U}_{\text{LG-CY}}$ yields the full higher genus Gromov–Witten generating function of the quintic three-fold when applied to the full higher genus Fan–Jarvis–Ruan–Witten generating function.

3 LG-CY shortcircuiting the mirror

In [30], in collaboration with Hiroshi Iritani and Yongbin Ruan, we describe a transition going directly from Gromov–Witten theory of the Calabi–Yau variety X_W to Fan–Jarvis–Ruan–Witten theory of the Landau–Ginzburg model $W\colon [\mathbb{C}^5/(\mathbb{Z}/5)] \to \mathbb{C}$. This avoids the use of the local system $(V, R^3\pi_*\mathbb{Z})$ on $\mathbb{P}^1 \setminus \{0,1,\infty\}$. The idea is to reach a point of view, which does not depend on mirror symmetry but rather on tools that are intrinsically related to X_W and $W\colon [\mathbb{C}^5/(\mathbb{Z}/5)] \to \mathbb{C}$. Of course, at the same time, this effort allows us to complete and provide more structure to the mirror symmetry framework: it shows a path going directly from $\mathcal{A}_{X_W}$ to $\mathcal{A}_{W,\mathbb{Z}/5}$ which is specular to the local system $(V, R^3\pi_*\mathbb{Z})$.

We point out a direct application. The correspondence $\mathbb{U}_{\text{LG-CY}}$ was not proven in order to further understand genus-zero Gromov–Witten theory. Indeed this issue has been already completely elucidated by Givental and Lian–Liu–Yau. On the other hand, as mentioned in the previous section, one can regard $\mathbb{U}_{\text{LG-CY}}$ as an operator which conjecturally relates the higher genus Gromov–Witten theory to level structures. In order to phrase this conjecture, one needs the language of quantization, which applies to symplectic operators $\mathbb{U}_{\text{LG-CY}}$. The fact that this map is symplectic was proven via analytic continuation and direct calculation; in [31] we were lacking a conceptual explanation. The direct approach illustrated here presents $\mathbb{U}_{\text{LG-CY}}$ as a cohomological application deriving from Orlov's equivalence. This simplifies the computation of $\mathbb{U}_{\text{LG-CY}}$ and explains why it is symplectic.

We have proven this result for all hypersurfaces of CY type within a Gorenstein weighted projective space (each weight divides the total weight $\sum_i w_j$). Again, for sake of clarity, we will limit ourselves to the case of the quintic three-fold.

3.1 Geometric invariant theory

Via GIT, we describe the passage from X_W to the corresponding Landau–Ginzburg model. This purely mathematical approach was described first by Witten in [125].

Consider the $\mathbb{C}^*$-action on $V = \operatorname{Spec}\mathbb{C}[x_1, x_2, x_3, x_4, x_5, p]$ with weights $(1, 1, 1, 1, 1, -5)$. We are interested in representing geometrically the space parametrizing the orbits of this action; in other words we would like to produce some sort of geometric quotient. We are faced with the problem of the existence of nonclosed orbite, which prevents us from constructing a quotient space. GIT allows to identify all open and $\mathbb{C}^\times$-invariant subspaces of V, having only closed orbits, and admitting a (separated) quotient space. Even if we require that the open subset is maximal, there is not a unique possibility. GIT identifies the two possible cases (they correspond to two types of choices of polarisations on $[V/\mathbb{C}^*]$).

1. We consider the open subset Ω_1 defined by $(x_1, x_2, x_3, x_4, x_5) \neq \mathbf{0}$. We have a quotient of $(\mathbb{C}^5 \setminus \{\mathbf{0}\}) \times \mathbb{C}$ by the free $\mathbb{C}^\times$-action. The quotient space is the total space of the line bundle $\mathcal{O}(-5)$ on $\mathbb{P}^4$.

2. We consider Ω_2 defined by $p \neq 0$. We obtaint the $\mathbb{C}^*$-action on $\mathbb{C}^5 \times \mathbb{C}^*$. This setup is represented in an equivalent way by the $\mathbb{Z}/5$-action on $\mathbb{C}^5$ (given by ξ_5 times the identity matrix). We obtain the Deligne–Mumford quotient stack $[\mathbb{C}^5/(\mathbb{Z}/5)]$.

We can enhance the above discussion by introducing the function $W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$. In order to lift this function to a $\mathbb{C}^\times$-invariant function on V we consider $\overline{W} = p\sum_{j=1}^5 x^5$.

Then, $\overline{W}$ descends to

$$\overline{W}\colon \mathcal{O}(-5) \to \mathbb{C}$$

(the composite of $\mathcal{O}(-5) \to \mathcal{O}$ and of the projection of the total space $\mathcal{O}$ on the fiber). We obtain in this way a geometric model essentially equivalent to the quintic three-fold (the critical locus of $\overline{W}\colon \mathcal{O}(-5) \to \mathbb{C}$).

On the other side $\overline{W}$ reduces to what we have called so far Landau–Ginzburg model: the morphisms

$$W\colon [\mathbb{C}^5/\langle j_W\rangle] \to \mathbb{C},$$

with an isolated singularity at the origin.

From this point of view, both sides of the LG-CY correspondence stem from

$$\overline{W}\colon [V/\mathbb{C}^*] \longrightarrow \mathbb{C} \qquad \text{(pour } V = \mathbb{C}^5 \times \mathbb{C}). \tag{IV.12}$$

3.2 Matrix factorization and Orlov's equivalence

The above construction yields the equivalence of categories proven by Orlov in [105] (this GIT perspective is extensively treated in [65]).

On the CY side, for X_W, we consider the bounded derived category $\mathcal{D}^b(X_W)$ of coherent sheaves on X_W. On the LG side, for $W\colon [\mathbb{C}^5/(\mathbb{Z}/5)] \to \mathbb{C}$, we consider the triangulated category $\mathrm{MF}^{\mathrm{gr}}(W)$ of graded matrix factorizations.

We recall that a *matrix factorization* of W is the datum of

$$(E, \delta_E) = \left(E^0 \underset{\delta_0}{\overset{\delta_1}{\rightleftarrows}} E^1\right),$$

where $E = E^0 \oplus E^1$ is a finitely generated free module on $R = \mathbb{C}[x_1, \dots, x_N]$ with a $\mathbb{Z}/2$-graduation, and $\delta_E \in \operatorname{End}^1_R(E)$ is an odd endomorphism of E satisfying the condition

$$\delta^2 = W \cdot \mathrm{id}_E\,.$$

There is a natural $\mathbb{Z}$-graded version; it gives rise to a triangulated category $\mathrm{MF}^{\mathrm{gr}}(W)$.

In [110], Polishchuk an Vaintrob have shown how to apply the general formalism of Chern characters of triangulated categories, to the special case of $\mathrm{MF}^{\mathbb{Z}/d}(W)$, the category of $\mathbb{Z}/d$-equivariant matrix factorizations. We have

$$\mathrm{ch}\colon K(\mathrm{MF}^{\mathbb{Z}/d}(W)) \to \mathrm{HH}(\mathrm{MF}^{\mathbb{Z}/d}(W)),$$

where HH denotes the Hochschild cohomology of $\mathrm{MF}^{\mathbb{Z}/d}(W)$. This yields a homomorphisms

$$K(\mathrm{MF}^{\mathrm{gr}}(W)) \to \mathcal{H}_{W,\mathbb{Z}/d}$$

via the composition with the natural function from $\mathrm{MF}^{\mathrm{gr}}(W)$ to $\mathbb{Z}/d$-equivariant matrix factorizations and the natural isomorphism

$$\mathrm{HH}(\mathrm{MF}^{\mathbb{Z}/d}(W)) \cong \mathcal{H}_{W,\langle j_W \rangle}$$

shown by [110]. Hence, Orlov equivalences from [105, §2.2]

$$\widetilde{\Phi}_a \colon \mathrm{MF}^{\mathrm{gr}}(W) \xrightarrow{\sim} \mathcal{D}^b(X_W)$$

for $a \in \mathbb{Z}$, induce, by passage to K-theory and to cohomology, isomorphisms between the state spaces $H^*(X_W;\mathbb{C})$ and $\mathcal{H}_{W,\mathbb{Z}/5}$ involved. We will denote these homomorphisms by

$$\Phi_a \colon H_{\mathrm{FJRW}} \longrightarrow H_{\mathrm{GW}}. \tag{IV.13}$$

Serre's functor yields a symplectic form on both cohomologies; since Orlov's equivalence is compatible with this structure, for all a, Φ_a is symplectic. We also point out that Φ_a does not respect the bigraduation of chapter III.

3.3 The direct path

Corollary 2.3.1 establishes a LG-CY correspondence LG-CY via two mirror maps (IV.10) et (IV.11). These two mirror maps identify two bases of of multivalued flat sections of $V = R^3\pi_*(\mathbb{Z}) \otimes \mathcal{O}$: $\sigma = (\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ near $t = \infty$ and $\gamma = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ near $t = 0$. Via analytic continuation we can extend these bases on the entire open set $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

We illustrate in more detail why these bases are multivalued. In fact, on an open contractible subspace $U \subset [\mathbb{P}^1/(\mathbb{Z}/5)]^\times$, each multivalued basis $\gamma = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ can be expresses as $\bigoplus_{i \in I} \gamma_i$, where $\gamma_i = (\Gamma_1^i, \Gamma_2^i, \Gamma_3^i, \Gamma_4^i)$ is a basis of singlevalued flat section taking values in $V_U \to U$. The basis γ_i cannot be extended on the entire space $\mathbb{P}^1 \setminus \{0, 1, \infty\}$; indeed when we transport γ_i along a closed path around $t = \infty$, we obtain four cycles $T(\Gamma_1^i, \Gamma_2^i, \Gamma_3^i, \Gamma_4^i)$ for a given nontrivial monodromy operator T.

Note also that the local data (IV.10) near ∞ and (IV.11) neat 0 yielding σ and γ are determined by choices of bases $H_{\mathrm{GW}} = \oplus_{h=0}^3 \varphi_h \mathbb{C}$ and $H_{\mathrm{FJRW}} = \oplus_{h=0}^3 \phi_h \mathbb{C}$. We have defined $\mathbb{U}_{\text{LG-CY}}$ as the change of basis matrix. We point out here that, since σ and γ are multivalued, the matrix $\mathbb{U}_{\text{LG-CY}}$ is determined up to conjugation with T. We get in this way a set of symplectic transformations

$$\{\mathbb{U}_a = T^a \mathbb{U}_{\text{LG-CY}} T^{-a} \mid a \in \mathbb{Z}\}$$

yielding an equivalence between the computation of Gromov–Witten genus-zero invariants and that of genus-zero Fan–Jarvis–Ruan–Witten invariants.

In collaboration with Iritani and Ruan we have identified these linear application $\mathbb{U}_a$.

Theorem 3.3.1 (Chiodo–Iritani–Ruan, [30]). *For all $a \in \mathbb{Z}$ the symplectic operation $\mathbb{U}_a$ identifying I_{FJRW} to the analytic continuation of I_{GW} at a neighborhood of $t = 0$ can be directly computed via Orlov's equivalence. We have*

$$\Phi_a = \mathbb{U}_a$$

for all $a \in \mathbb{Z}$.

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