CONSTRUCTING RATIONAL CURVES ON K3 SURFACES

FEDOR BOGOMOLOV, BRENDAN HASSETT, AND YURI TSCHINKEL

Abstract. We develop a mixed-characteristic version of the Mori-Mukai technique for producing rational curves on K3 surfaces. We reduce modulo p, produce rational curves on the resulting K3 surface over a finite field, and lift to characteristic zero. As an application, we prove that all complex K3 surfaces with Picard group generated by a class of degree two have an infinite number of rational curves.

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1. Introduction

Let K be an algebraically closed field and S a K3 surface defined over K. It is known that S contains rational curves—see Mori-Mukai [16], as well as Theorem 7 and Proposition 17 below. In fact, an extension of the argument in [16] shows that the *general* K3 surface of given degree has infinitely many rational curves (see Theorem 9 and [7]). The idea is to specialize the K3 surface S to a K3 surface S_0 with Picard group of rank 2, where some multiple of the polarization can be expressed as a sum of linearly independent classes of smooth rational curves. The union of these rational curves deforms to an irreducible

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rational curve on S. This idea applies to K3 surfaces parametrized by points outside a countable union of subvarieties of the moduli space. In particular, *a priori* it doesn't apply to K3 surfaces over countable fields, such as $\overline{\mathbb{F}}_p$ and $\overline{\mathbb{Q}}$. Of course, there are also other techniques proving density of rational curves on special K3 surfaces, e.g., certain Kummer surfaces [16], surfaces with infinite automorphism groups [5, proof of Thm. 4.10], or with elliptic fibrations (see Remark 6 below). These K3 surfaces have Picard rank ≥ 2, and all except finitely many lattices in rank ≥ 3 correspond to K3 surfaces with infinite automorphisms or elliptic fibrations [17, 26].

Moreover, in [6] it is proved that, over $k = \overline{\mathbb{F}}_p$, every algebraic point on a Kummer K3 surface lies on an irreducible rational curve. The proof of this result uses the Frobenius endomorphism on the covering abelian surface.

The following is the main theorem of this paper:

Theorem 1. *Let S be a K3 surface over an algebraically closed field of characteristic zero with* $\mathrm{Pic}(S) = \mathbb{Z}$*, generated by a divisor of degree two. Then S contains infinitely many rational curves.*

The motivation for our argument comes from a result of Bogomolov and Mumford [16]: Let (S, f) be a general K3 surface of degree $2g-2$. We can degenerate S to a Kummer K3 surface (S_0, f), which has infinitely many rational curves. Indeed, we can produce examples where there are infinitely many (reducible) rational curves in $|Nf|, N \geq 1$, consisting of unions of smooth components meeting transversally. A deformation argument shows that these deform to infinitely many (irreducible) rational curves in nearby fibers. However, on subsequent specializations, distinct rational curves might collapse onto each other. If there were an infinite number of such collisions, the specialized K3 surfaces might only have a finite number of rational curves.

Here we emulate the argument in [16] in mixed characteristic. K3 surfaces over finite fields play the rôle of the Kummer surface; the 'general' K3 surface is a K3 surface over a number field with Picard group of rank one. The main technical issue is that we cannot assume *a priori* that the rational curves on the reduction mod $\mathfrak{p}$ have mild singularities. Thus we are forced to use more sophisticated deformation techniques.

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2. Guiding questions and examples

The following is well-known but hard to trace in the literature:

Conjecture 2 (Main conjecture)**.** Let K be an algebraically closed field of arbitrary characteristic and S a projective K3 surface over K. There exist infinitely many rational curves on S.

In characteristic zero, we can reduce this to the case of number fields:

Theorem 3. *Assume that for every K3 surface S_0 defined over a number field K_0, there are infinitely many rational curves in*

$$\overline{S_0} := S_0 \times_{\mathrm{Spec}(K_0)} \mathrm{Spec}(\overline{\mathbb{Q}}).$$

Then Conjecture 2 holds over fields of characteristic zero.

Proof. Let S be a K3 surface defined over a field of characteristic zero, which we may assume is the function field of a variety B defined over a number field F. Shrinking B as necessary, we obtain a smooth projective morphism

$$\pi : \mathcal{S} \to B$$

with generic fiber S.

We claim there exists a point $b \in B(\overline{\mathbb{Q}})$ such that the specialization map to the fiber $S_b = \pi^{-1}(b)$

$$\mathrm{Pic}(S) \to \mathrm{Pic}(S_b)$$

is surjective. The argument is essentially the same as the proof of the main result of [10]. The only difference is that Ellenberg considers the Galois representation on the full primitive cohomology of a polarized K3 surface surface, whereas here we restrict to the representation on the transcendental cohomology of S, i.e., the orthogonal complement to $\mathrm{Pic}(S) \subset H^2(S, \mathbb{Z})$.

Our assumption is that S_b admits infinitely many rational curves. We claim each of these lifts to a rational curve of S, perhaps after a generically-finite base-change $\widetilde{B} \to B$. Suppose we have a morphism $\phi_b : \mathbb{P}^1 \to S_b$, birational onto its image. The class $\phi_{b*}[\mathbb{P}^1]$ remains algebraic in the fibers of $\mathcal{S} \to B$. Consider the normal bundle $\mathcal{N}_{\phi_b}$, which in this context is the cokernel of the differential

$$d\phi : \mathcal{T}_{\mathbb{P}^1} \to \phi_b^* \mathcal{T}_{S_b};$$

in Lemma 11 below, we use a more general formulation. Riemann-Roch shows that $\chi(\mathcal{N}_{\phi_b}) = -1$. Consider the space of morphisms

$$\mathrm{Mor}_B(\mathbb{P}^1 \times B, \mathcal{S}),$$

i.e., morphisms from $\mathbb{P}^1$'s into the fibers of $\mathcal{S}$ over B. Its dimension is at least $\dim(B) + \chi(\mathcal{N}_{\phi_b}) = \dim(B) - 1$ [15, II.2], but here we can do better: A result of Ran, which builds on earlier work of Voisin [27] and Bloch [3], shows the relative dimension is at least $\dim(B)+\chi(\mathcal{N}_{\phi_b})+1 = \dim(B)$. See [21, Cor. 3.2 and 3.3] for the general statement and [21, §5] for the computation of the relevant parameters when the source is a curve. K3 surfaces are not uniruled in characteristic zero, therefore ϕ_b admits no deformations in S_b and ϕ_b lifts to a morphism $\phi : \mathbb{P}^1 \to S$. $\square$

Remark 4. We do not know whether the positive-characteristic case of Conjecture 2 can be reduced to the case of finite fields.

Example 5. Here we show that any Kummer K3 surface over an arbitrary algebraically-closed field of characteristic $\neq 2$ admits an infinite number of rational curves.

Let A be an abelian surface with Kummer surface S:

$$\begin{array}{ccc} & & S \\ & & \downarrow \\ A & \to & A/\pm \end{array}$$

Note that A is isogenous to the Jacobian J of a genus two curve C. (Every abelian surface is isogenous to a principally-polarized surface, which is either a Jacobian or a product $E_1 \times E_2$ of elliptic curves. In the latter case, if we express E_1 and E_2 as branched coverings on $\mathbb{P}^1$ at $\{0, \infty, \alpha_1, \beta_1\}$ and $\{0, \infty, \alpha_2, \beta_2\}$ with the α_i and β_i distinct, then the genus-two double cover $C \to \mathbb{P}^1$ branched at $\{0, \infty, \alpha_1, \beta_1, \alpha_2, \beta_2\}$ will work. Indeed, E_1 and E_2 are Prym varieties of C.)

We choose the embedding $C \hookrightarrow J$ such that a Weierstrass point is mapped to zero. Then the images of $n \cdot C$ in $A/\pm$ are distinct rational curves. Indeed, multiplication-by-n commutes with $\pm$, and acts on C via the hyperelliptic involution.

Remark 6. Elliptic complex K3 surfaces always have infinitely many rational curves: see [5, Thm. 1.8] or [13, Cor. 8.12, Prop. 9.10, Rem. 9.7] and Example 5 above for the degenerate case where the elliptic surface arises from a Kummer construction.

3. Background results

A polarized K3 surface (S, f) consists of a K3 surface and an ample divisor f that is primitive in the Picard group. Its degree is the positive even integer $f \cdot f$. Let $\mathcal{K}_g, g \geq 2$ denote the moduli space (stack) of complex polarized K3 surfaces of degree $2g-2$, which is smooth and connected of dimension 19.

The following result was initially presented by the first author in October 1981 at Mori's seminar at IAS; the proof was based on deformation-theoretic ideas developed several years earlier. A different argument was presented in [16]; Mori and Mukai indicate that Mumford also had a proof.

Theorem 7. [16] *Every complex projective K3 surfaces contains a rational curve.*

Remark 8. The argument yields a more precise statement [13, 6.10]: Let S be a complex projective K3 surface, D a nonzero effective divisor on S, such that D is indecomposable in the effective monoid, i.e., it cannot be written as a sum of two nonzero effective divisors. Then there exists a rational curve in $|D|$.

The following theorem has been known to experts; the first published proof is in [7]. (Xi Chen attributes a special case to S. Nakatani.)

Theorem 9. *Fix $N \geq 1$. Then for a generic $(S, f) \in \mathcal{K}_g$ there exists an irreducible rational curve in $|Nf|$.*

Corollary 10. *A very general K3 surface of degree $2g-2$ contains an infinite number of rational curves.*

The proof in [7] involves specializing the K3 surface to a union of two rational normal scrolls, meeting transversely along an elliptic curve. Xi Chen identifies reducible rational curves on this surface that can be deformed back to irreducible rational curves on a general K3 surface.

For our purposes this argument is not sufficiently flexible. Our technique entails analyzing the reductions $\pmod p$ of a K3 surface defined over a number field. We cannot expect these reductions to be unions of rational normal scrolls.

4. Deformation results for stable maps

In this section, we work over a field of arbitrary characteristic. Let Y be a smooth projective variety and β a curve class on Y.

Consider the open substack

$$\overline{\mathcal{M}}_0^{\circ}(Y,\beta) \subset \overline{\mathcal{M}}_0(Y,\beta)$$

corresponding to maps $\phi : T \to Y$ that are generic embeddings, i.e., there exists a dense open subset $T' \subset T$ over which ϕ is an embedding. Note that $\overline{\mathcal{M}}_0^{\circ}(Y,\beta)$ is a quasi-projective scheme, since generic embeddings have trivial automorphism groups.

This open set dominates the locus Ξ in the Chow variety consisting of cycles

$$C = C_1 \cup \ldots \cup C_r \subset Y, \quad \sum_{i=1}^{r} [C_i] = \beta,$$

of rational curves C_i with each component having multiplicity one; the induced morphism is finite-to-one. Indeed, there is a finite collection of connected seminormal curves T factoring

$$C' := \coprod_{j=1}^{r} \mathbb{P}^1 \to T \to C,$$

where C' is the disjoint union of the normalizations of $C_1, \ldots, C_r$.

The obstruction theory over $\overline{\mathcal{M}}_0^{\circ}(Y,\beta)$ takes a particularly simple form: Given a stable map $\phi : T \to Y$, first-order deformations and obstructions are given by

$$\mathbb{H}^i(\mathbb{R}\mathcal{H}om_{\mathcal{O}_T}(\Omega_\phi^\bullet, \mathcal{O}_T)), i = 1, 2,$$

where $\Omega_\phi^\bullet$ is the complex

$$d\phi^t : \phi^*\Omega_Y^1 \to \Omega_T^1$$

supported in degrees -1 and 0 [12, p. 61].

We shall require the following result; our analysis is similar to the discussion in [12], except that we work under slightly less restrictive assumptions:

Lemma 11. *Let $\phi : T \to Y$ be a stable map to a smooth variety that is unramified at the generic point of each irreducible component of T. Then the complex*

$$\mathbb{R}\mathcal{H}om_{\mathcal{O}_T}(\Omega_\phi^\bullet, \mathcal{O}_T)$$

is quasi-isomorphic to $\mathcal{N}_\phi[-1]$ for some sheaf $\mathcal{N}_\phi$.

The sheaf $\mathcal{N}_\phi$ is called the *normal sheaf* of ϕ. In the special case where the domain T is smooth, $\mathcal{N}_\phi$ is the cokernel of the differential $d\phi : \mathcal{T}_T \to \phi^*\mathcal{T}_Y$. First order deformations of ϕ are given by $H^0(\mathcal{N}_\phi)$; obstructions are given by $H^1(\mathcal{N}_\phi)$.

Proof. Let T be a nodal projective curve. Suppose that $p \in T$ is a node expressed as $xy = 0$ in local étale/analytic coordinates. Then

$$\Omega^1_T = (\mathcal{O}_T dx + \mathcal{O}_T dy) / \langle y dx + x dy \rangle$$

and Ω^1_T admits a local resolution in $\mathcal{O}_T$-modules

$$0 \to \mathcal{E}_1 \xrightarrow{f_1} \mathcal{E}_0 \to \Omega^1_T \to 0,$$

where $\mathcal{E}_1$ is invertible and $\mathcal{E}_0$ is locally free of rank two.

Given a bounded complex of $\mathcal{O}_T$-modules

$$\mathcal{E}^\bullet = \{0 \cdots \mathcal{E}^{-p-1} \to \mathcal{E}^{-p} \to \mathcal{E}^{-p+1} \to \cdots 0\}$$

we compute $\mathbb{R}\mathcal{H}om_{\mathcal{O}_T}(\mathcal{E}^\bullet, \mathcal{O}_T)$ using the spectral sequence

$$E_1^{p,q} = \mathcal{E}xt^q_{\mathcal{O}_T}(\mathcal{E}^{-p}, \mathcal{O}_T) \Rightarrow \mathcal{E}xt^{p+q}_{\mathcal{O}_T}(\mathcal{E}^\bullet, \mathcal{O}_T).$$

Note that

- $\mathcal{E}xt^q_{\mathcal{O}_T}(\phi^*\Omega^1_Y, \mathcal{O}_T) = 0$ for $q > 0$ as Ω^1_Y is locally free;
- $\mathcal{E}xt^q_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T) = 0$ for $q > 1$ by the explicit resolution above.

In particular, only the following terms can be nonzero

$$\mathcal{H}om_{\mathcal{O}_T}(\phi^*\Omega^1_Y, \mathcal{O}_T), \mathcal{H}om_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T), \mathcal{E}xt^1_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T).$$

We focus on the unique interesting arrow

$$\begin{array}{rcl} d_1 : E_1^{0,0} & \to & E_1^{1,0} \\ d\phi : \mathcal{H}om_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T) & \to & \mathcal{H}om_{\mathcal{O}_T}(\Omega^1_Y, \mathcal{O}_T). \end{array}$$

Since $\mathcal{H}om_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T)$ is torsion-free, $d\phi$ is injective if and only if it is injective at generic points of T, which was one of our assumptions.

Thus we have

$$E_2^{1,0} = \mathcal{H}om_{\mathcal{O}_T}(\phi^*\Omega^1_Y, \mathcal{O}_T)/\mathcal{H}om_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T)$$

and

$$E_2^{0,1} = \mathcal{E}xt^1_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T).$$

Consequently

$$\mathbb{R}\mathcal{H}om_{\mathcal{O}_T}(\Omega^\bullet_\phi, \mathcal{O}_T)$$

is supported in degree one, and the associated sheaf $\mathcal{N}_\phi$ fits into an exact sequence

$$0 \to \mathcal{H}om_{\mathcal{O}_T}(\phi^*\Omega^1_Y, \mathcal{O}_T)/\mathcal{H}om_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T) \to \mathcal{N}_\phi \to \mathcal{E}xt^1_{\mathcal{O}_T}(\Omega^1_T, \mathcal{O}_T) \to 0.$$

Note that the first term corresponds to deformations that leave the nodes of T unchanged; the last term is the local versal deformation space of these nodes. $\square$

Remark 12. In fact, $\mathcal{N}_\phi$ is locally-free if ϕ is unramified (see, for example, [12, §2])). Conversely, if ϕ is ramified at a smooth point then $\mathcal{N}_\phi$ necessarily has torsion.

5. K3 surfaces over finite fields

For general background and definitions, we refer the reader to [22].

Let S_0 be a K3 surface over a finite field $\mathbb{F}_q$, and $\overline{S_0}$ the resulting surface over $\overline{\mathbb{F}_q}$. Consider the Picard group $\mathrm{Pic}(\overline{S_0})$ and the ℓ-adic cohomology group $H^2(\overline{S_0}, \mathbb{Q}_\ell(1))$, which are related by the cycle-class map

$$\mathrm{Pic}(\overline{S_0}) \to H^2(\overline{S_0}, \mathbb{Q}_\ell(1)).$$

Frobenius acts on both these groups compatibly with this map, and preserving the intersection form.

The following application of the Tate conjecture is well-known to experts (and was ascribed to Swinnerton-Dyer in [2, p. 544]) but we are not aware of a reference:

Theorem 13. *Let S_0 be a non-supersingular K3 surface over a finite field of characteristic ≥ 5. Then $\mathrm{Pic}(\overline{S_0})$ has even rank.*

Proof. We refer the reader to [2] for the definition of 'supersingular'.

The Frobenius action on $H^2(\overline{S_0}, \mathbb{Q}_\ell(1))$ is diagonalizable over $\overline{\mathbb{Q}}_\ell$ with eigenvalues $\alpha_1, \dots, \alpha_{22}$ [9]. Since this factors through the orthogonal group, if α appears as an eigenvalue then α^{-1} also appears. Consequently, we conclude that the following sets have an even number of elements

- the eigenvalues that are not roots of unity;
- the eigenvalues that are roots of unity but are not equal to ± 1;
- the total number of times ± 1 appears as an eigenvalue.

Results of Nygaard and Ogus on the Tate conjecture for K3 surfaces [18, 19] imply Galois-invariant cohomology classes are in the image of the cycle map, at least rationally. □

Remark 14. Shioda proposes an alternate definition [22, §5]: $\overline{S_0}$ is supersingular if $\mathrm{Pic}(\overline{S_0})$ has rank twenty two. This condition implies supersingularity in the sense of Artin [22, §9, Prop. 2]; the converse remains open.

We shall use the following result:

Proposition 15. *Let X be a smooth projective surface over an algebraically closed field with Néron-Severi group* $\mathrm{NS}(X)$*. If X is uniruled then*

$$\operatorname{rank} \mathrm{NS}(X) = b_2(X),$$

where $b_2(X)$ is the second Betti number. In particular, if X is a K3 surface then

$$\operatorname{rank} \operatorname{Pic}(X) = \operatorname{rank} \mathrm{NS}(X) = 22$$

and thus is supersingular.

This follows from [22, §2 Prop. 2]; while the argument there assumes unirational rather than uniruled, the weaker condition suffices.

Suppose S is a K3 surface over a number field F with an integral model

$$\mathcal{S} \to \operatorname{Spec}(\mathfrak{o}_F),$$

which is smooth away from a finite set of primes. For each finite extension F'/F, consider the set of primes

$$\operatorname{Ord}_{F'}(S) = \{\mathfrak{p} \in \operatorname{Spec}(\mathfrak{o}_{F'}) : S_{\mathfrak{p}} \text{ is smooth and ordinary } \}.$$

After passing to a suitably large finite extension F'/F, the set $\operatorname{Ord}_{F'}(S)$ has Dirichlet density one. This is due to Tankeev [25] in the special case where the Hodge group of $S_{\mathbb{C}}$ is semisimple, and to Joshi-Rajan [14, §6] and Bogomolov-Zarhin [4] in general.

Finally, ordinary K3 surfaces are not supersingular [22, §9] and thus not uniruled.

6. Curves on K3 surfaces in positive and mixed characteristic

Let k be an algebraically closed field of characteristic $p > 0$. We shall require some general results on lifting to characteristic zero. In particular, we shall need to set up the formalism in order to discuss stable maps in the relative context of a universal family over the versal deformation space.

We first review deformation and lifting results for K3 surfaces, established in [8]:

Theorem 16. *Let S_0 be a K3 surface defined over k. Then the versal deformation space of S_0 is smooth over the Witt vectors $W(k)$ of relative dimension twenty.*

Let (S_0, f) be a polarized K3 surface over an algebraically closed field k of characteristic p. Then (S_0, f) admits a lifting to a polarized K3 surface (S, f), after passing to an extension of $W(k)$. Moreover, the *locus Σ_f in the formal versal deformation space, corresponding to K3 surfaces admitting f as a polarization, is a Cartier divisor in the versal deformation space, not contained in the closed fiber over* $\mathrm{Spf}(W(k))$.

Ogus [20, §2] has more precise lifting results for ordinary K3 surfaces. These require finer analysis of Chern classes and crystalline cohomology.

Proposition 17. *Let S_0 be a projective K3 surface over an algebraically closed field of characteristic p. Then S_0 contains a rational curve.*

More precisely, let D be a nonzero effective divisor on S_0 that is indecomposable in the effective monoid. Assume that D is ample or S_0 is ordinary. Then S_0 contains a rational curve in $|D|$.

Proof. The first statement is a corollary of Theorems 16 and 7. Lift S_0 to a projective K3 surface S in characteristic zero; rational curves on S specialize to unions of rational curves on S_0.

We recall a result of Saint-Donat [23, 2.6, 2.7]: Let D be a nonzero indecomposable effective divisor on a K3 surface S_0. Then the higher cohomology of D vanishes.

Assume first that D is ample. In the application of Theorem 16, we may assume that D lifts to a polarization on S. Now assume S_0 is ordinary. Take S to be the canonical lift of S_0 to characteristic zero; specialization induces an isomorphism $\mathrm{Pic}(S) \xrightarrow{\sim} \mathrm{Pic}(S_0)$ [18, 1.8]. In either case, since $H^i(\mathcal{O}_{S_0}(D)) = 0$ for $i > 0$ semicontinuity implies D lifts to an effective divisor on S. This clearly remains indecomposable. By Remark 8, S admits a rational curve with class $[D]$, which specializes to a rational curve in S_0. □

Theorem 18. *Let (S_0, f) be a polarized K3 surface over k. Suppose that*

$$C = C_1 + \ldots + C_r$$

is a connected union of rational curves $C_i \subset S_0$, such that $[C] = Nf$. Let (S, f) be a polarized K3 surface over the Witt vectors $W(\overline{k})$ reducing to (S_0, f). Assume S_0 is not uniruled and the C_i are distinct. Then there exists a relative curve $R \subset S$, defined over a finite extension of $W(k)$, such that R reduces to C and each irreducible component of R is rational.

Proof. Consider the formal versal deformation space of S_0

$$\mathcal{S} \to B,$$

where $B \simeq \mathrm{Spf}(W(k)[[x_1, \dots, x_{20}]])$, i.e., a smooth formal scheme of dimension 20 over $W(k)$. Let $b \in B$ denote the distinguished closed point. For each $N \geq 1$, consider the relative stable map space

$$\overline{\mathcal{M}}_0^\circ(\mathcal{S}/B, Nf) \to B,$$

consisting of generic embeddings. This is a formal *scheme* over B, as generic embeddings have only trivial automorphisms. We refer the reader to [1, §8] for a discussion of the construction of moduli spaces of stable maps for proper (but not necessarily projective) schemes. (We apply this over systems of Artinian local rings approximating B.)

General deformation-theoretic arguments (cf. [15, I.2.15]) show that the relative dimension of $\overline{\mathcal{M}}_0^\circ(\mathcal{S}/B, Nf)$ over B at ϕ is at least

$$\chi(T, \mathcal{N}_\phi) + \dim(B) = \dim(B) - 1.$$

Take T to be a nodal connected curve of genus zero factoring through the normalization

$$C^\nu \to T \to C.$$

The induced $\phi : T \to S_0$ is a stable map unramified at the generic point of each component, i.e., $\phi \in \overline{\mathcal{M}}_0^\circ(\mathcal{S}/B, Nf)$. Since S is not uniruled, we conclude that ϕ does not deform to another genus-zero stable map to S_0.

In either case, the dimensions of $\overline{\mathcal{M}}_0^\circ(\mathcal{S}/B, Nf)$ and its image in B are at least 20. On the other hand, this image is contained in the locus

$$\Sigma_{Nf} \subset B$$

parametrizing K3 surfaces admitting Nf as a polarization. Indeed, in each fiber

$$(\phi_t)_* \mathcal{T}_t = Nf.$$

By Theorem 16, the formal scheme Σ_{Nf} has dimension 20 and is not contained in the fiber over the closed point of $\mathrm{Spf}(W(k))$. The same must hold for $\overline{\mathcal{M}}_0^\circ(\mathcal{S}/B, Nf)$, so there are formal lifts of $\phi : T \to S$ to genus-zero maps in characteristic zero.

It remains to show these formal deformations are algebraic. For this purpose, we restrict to the polarized deformation space

$$\mathcal{S}_{\Sigma_{Nf}} \to \Sigma_{Nf},$$

which is projective in the sense that it admits a formal embedding into a projective space $\mathbb{P}^d_{\Sigma_{Nf}}, d = \chi(S, \mathcal{O}_S(Nf)) - 1$. This deformation is algebraizable by standard results of Grothendieck (see [24, 2.5.13], for example.) It follows that the associated moduli spaces of stable maps are algebraizable as well. Indeed, moduli spaces of stable maps into projective schemes are proper stacks with projective coarse moduli spaces. □

7. Proof of the Main Theorem

In light of Theorem 3, it suffices to restrict to S defined over a number field F. Let

$$\mathcal{S} \to \operatorname{Spec}(\mathfrak{o}_F)$$

be an integral model. For each prime $\mathfrak{p}$, let $S_{\mathfrak{p}}$ denote the reduction modulo $\mathfrak{p}$ and $\overline{S}_{\mathfrak{p}}$ its basechange to the algebraic closure of the corresponding finite field.

Assume $\operatorname{Pic}(S)$ is generated by an ample class f, of arbitrary degree. Suppose S admits only a finite number of rational curves $R_1, \dots, R_s$ with classes $[R_i] = m_i f$ and write $m = \max\{m_1, \dots, m_s\}$.

Lemma 19. *There are only a finite number of primes $\mathfrak{p}$ such that there exists a curve $C \subset \overline{S}_{\mathfrak{p}}$ with $[C] \notin \mathbb{Z}f$ and*

$$C \cdot f \leq mf \cdot f.$$

Proof. Consider the Hilbert scheme $\mathcal{H} \to \operatorname{Spec}(\mathfrak{o}_F)$ parametrizing curves of degree $\leq m$ in fibers of $\mathcal{S} \to \operatorname{Spec}(\mathfrak{o}_F)$; this is projective over $\operatorname{Spec}(\mathfrak{o}_F)$. For each irreducible component dominating $\operatorname{Spec}(\mathfrak{o}_F)$, the corresponding curves have class Nf for some $N > 0$. Curves C with $[C] \notin \mathbb{Z}f$ are therefore contained in many 'fibral components' of $\mathcal{H}$, i.e., components supported over primes $\mathfrak{p} \in \operatorname{Spec}(\mathfrak{o}_F)$. However, there can be at most finitely many such components, lying over finitely many primes. □

Now we assume S has degree two. Let $\iota : S \to S$ denote the involution associated to the branched double cover $S \to \mathbb{P}^2$. It acts on the primitive cohomology of S via multiplication by -1. We shall derive a contradiction by producing an irreducible rational curve in a class Nf for some $N > m$.

Choose $\mathfrak{p}$ to a prime satisfying the following conditions:

1. $S_{\mathfrak{p}}$ is of good and ordinary reduction (see Section 5), so that $S_{\mathfrak{p}}$ is not uniruled by Proposition 15;
2. $f_{\mathfrak{p}}$, the restriction of f to $S_{\mathfrak{p}}$, remains a polarization;
3. $\iota_{\mathfrak{p}}$, the reduction of ι mod $\mathfrak{p}$, remains an involution fixing $f_{\mathfrak{p}}$ and acting via multiplication by -1 on the primitive part of Picard group;
4. $\mathfrak{p}$ is not in the finite set of primes specified in Lemma 19.

Since $\mathrm{Pic}(\overline{S}_{\mathfrak{p}})$ has rank ≥ 2, there exist numerous effective divisors not proportional to $f_{\mathfrak{p}}$; choose an indecomposable class in the effective monoid with this property. Proposition 17 implies there exists an irreducible rational curve $C_{\mathfrak{p}} \subset \overline{S}_{\mathfrak{p}}$ with this class. Write $C'_{\mathfrak{p}} = \iota(C_{\mathfrak{p}})$, which is distinct from $C_{\mathfrak{p}}$; since $C_{\mathfrak{p}} + C'_{\mathfrak{p}}$ is invariant under $\iota_{\mathfrak{p}}$, we have

$$[C_{\mathfrak{p}} + C'_{\mathfrak{p}}] = N f_{\mathfrak{p}}$$

for some N. Note that $N > 2m$ by our assumption on $\mathfrak{p}$ and Lemma 19.

We claim $C_{\mathfrak{p}} \cup C'_{\mathfrak{p}}$ lifts to an irreducible rational curve R over $\overline{\mathbb{Q}}$. Consider the chain of two $\mathbb{P}^1$'s

$$T = \{xy = 0\} \subset \mathbb{P}^2$$

and choose a birational morphism $\phi : T \to C_{\mathfrak{p}} \cup C'_{\mathfrak{p}}$. Let $j : T \to S_{\mathfrak{p}}$ be the induced morphism; then T is the specialization of a rational curve over the Witt vectors by Theorem 18. Since this curve does not deform—$S_{\mathfrak{p}}$ is not uniruled—it can be lifted to $\overline{\mathbb{Q}}$. $\square$

While Theorem 1 applies to general degree two K3 surfaces, it does not apply to every such surface. A crucial aspect of the proof is that, for K3 surfaces with Picard group of rank one (or odd rank), the Picard groups of the reductions mod $\mathfrak{p}$ jump. The resulting surfaces over finite fields have additional curve classes, which are necessary for our lifting argument.

We give an example of a degree two K3 surface with Picard group of rank two, for which we do not know whether there are infinitely many rational curves, over $\overline{\mathbb{Q}}$ or $\overline{\mathbb{F}}_p$.

Example 20. Let S be a K3 surface over an algebraically closed field such that the Picard group is generated over $\mathbb{Q}$ by smooth rational curves C_1 and C_2 satisfying

$$\begin{array}{c|cc} & C_1 & C_2 \\ \hline C_1 & -2 & 6 \\ C_2 & 6 & -2 \end{array} \qquad (7.1)$$

and generated over $\mathbb{Z}$ by C_1 and $f = (C_1 + C_2)/2$. Note that (S, f) can be realized geometrically as the double cover of $\mathbb{P}^2$ branched over a plane sextic curve that admits a six-tangent conic.

Surfaces of this type can be defined over $\mathbb{F}_3$ [11, Ex. 6.1], e.g.,

$$w^2 = (y^3 - x^2y)^2 + (x^2 + y^2 + z^2)(2x^3y + x^3z + 2x^2yz + x^2z^2 + 2xy^3 + 2y^4 + z^4).$$

The technique of [10] can be used to obtain examples over $\mathbb{Q}$. Indeed, the moduli space of lattice-polarized K3 surfaces of type (7.1) is unirational: The sextic plane curves six-tangent to a fixed conic plane curve D are parametrized by a $\mathbb{P}^{15}$-bundle over $\mathbb{P}^6$, and these dominate our moduli space. We can apply Ellenberg's Hilbert irreducibility argument [10] directly to this rational variety.

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Courant Institute, New York University, New York, NY 10012, USA
E-mail address: bogomolo@cims.nyu.edu

Department of Mathematics, Rice University, MS 136, Houston, Texas 77251-1892, USA
E-mail address: hassett@rice.edu

Courant Institute, New York University, New York, NY 10012, USA
E-mail address: tschinkel@cims.nyu.edu