Quasi-isometry rigidity of groups

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Contents

These notes represent a slightly modified version of the lectures given at the summer school "Géométries à courbure négative ou nulle, groupes discrets et rigidités" held from the 14-th of June till the 2-nd of July 2004 in Grenoble.

Many of the open questions formulated in the paper do not belong to the author and have been asked by other people before.

1 Preliminaries on quasi-isometries

Nota bene: In order to ensure some coherence in the exposition, some notions are not defined in the text, but in a Dictionary at the end of the text.

1.1 Basic definitions

A *quasi-isometric embedding* of a metric space (X, dist_X) into a metric space (Y, dist_Y) is a map $\mathfrak{q} : X \to Y$ such that for every $x_1, x_2 \in X$,

$$\frac{1}{L}\mathrm{dist}_X(x_1, x_2) - C \leq \mathrm{dist}_Y(\mathfrak{q}(x_1), \mathfrak{q}(x_2)) \leq L\mathrm{dist}_X(x_1, x_2) + C\,, \tag{1}$$

for some constants $L \geq 1$ and $C \geq 0$.

If X is a finite interval $[a, b]$ then $\mathfrak{q}$ is called *quasi-geodesic (segment)*. If $a = -\infty$ or $b = +\infty$ then $\mathfrak{q}$ is called *quasi-geodesic ray*. If both $a = -\infty$ and $b = +\infty$ then $\mathfrak{q}$ is called *quasi-geodesic line*. The same names are used for the image of $\mathfrak{q}$.

If moreover Y is contained in the C–tubular neighbourhood of $\mathfrak{q}(X)$ then $\mathfrak{q}$ is called *a quasi-isometry*. In this case there exists $\bar{\mathfrak{q}} : Y \to X$ quasi-isometry such that $\bar{\mathfrak{q}} \circ \mathfrak{q}$ and $\mathfrak{q} \circ \bar{\mathfrak{q}}$ are at uniformly bounded distance from the respective identity maps (see [GH_2] for a proof). We call $\bar{\mathfrak{q}}$ *quasi-converse of* $\mathfrak{q}$.

The objects of study are the finitely generated groups. We first recall how to make them into geometric objects. Given a group G with a finite set of generators S containing together with every element its inverse, one can construct the *Cayley graph* $\mathrm{Cayley}(G, S)$ as follows:

- its set of vertices is G;
- every pair of elements $g_1, g_2 \in G$ such that $g_1 = g_2 s$, with $s \in S$, is joined by an edge. The oriented edge (g_1, g_2) is labeled by s.

We suppose that every edge has length 1 and we endow $\mathrm{Cayley}(G, S)$ with the length metric. Its restriction to G is called *the word metric associated to* S and it is denoted by dist_S. See Figure 1 for the Cayley graph of the free group of rank two $\mathbb{F}_2 = \langle a, b\rangle$.

Remark 1.1. A Cayley graph can be constructed also for an infinite set of generators. In this case the graph has infinite valence in each point.

We note that if S and $\bar{S}$ are two finite generating sets of G then dist_S and $\mathrm{dist}_{\bar{S}}$ are bi-Lipschitz equivalent. In Figure 2 are represented the Cayley graph of $\mathbb{Z}^2$ with set of generators $\{(1, 0), (0, 1)\}$ and the Cayley graph of $\mathbb{Z}^2$ with set of generators $\{(1, 0), (1, 1)\}$.

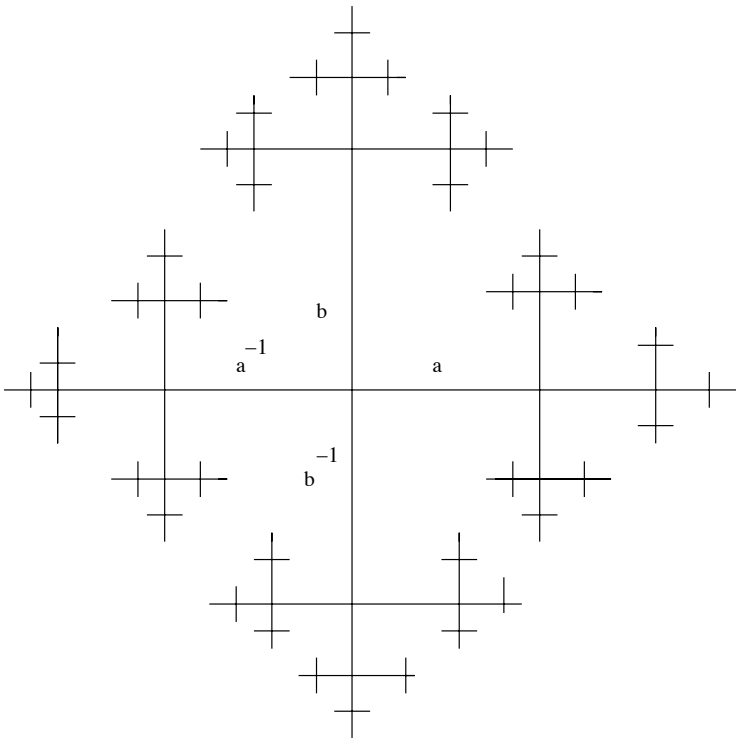

Figure 1: Cayley graph of $\mathbb{F}_2$.

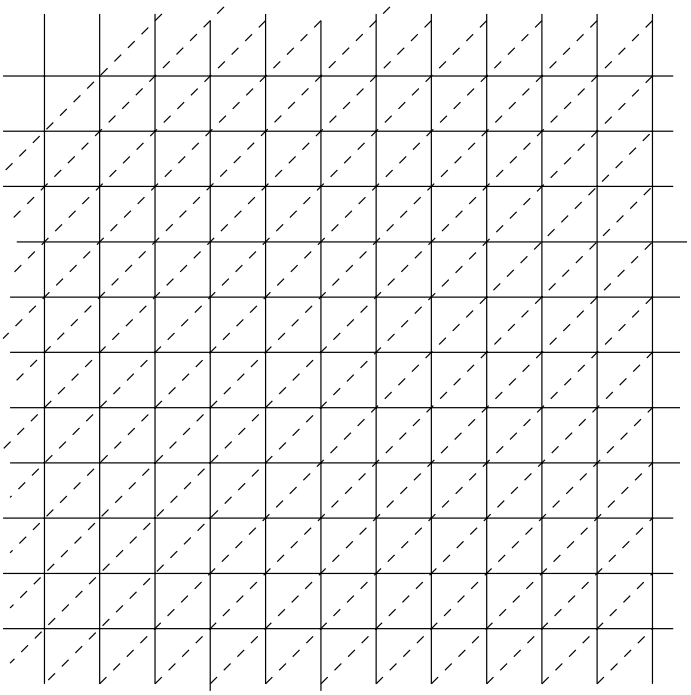

Figure 2: Cayley graph of $\mathbb{Z}^2$.

1.2 Examples of quasi-isometries

1. The main example, which partly justifies the interest in quasi-isometries, is the following. Given M a compact Riemannian manifold, let $\widetilde{M}$ be its universal covering and let $\pi_1(M)$ be its fundamental group. The group $\pi_1(M)$ is finitely generated, in fact even finitely presented [BrH, Corollary I.8.11, p.137].

   The metric space $\widetilde{M}$ with the Riemannian metric is quasi-isometric to $\pi_1(M)$ with some word metric. This can be clearly seen in the case when M is the n-dimensional flat torus $\mathbb{T}^n$. In this case $\widetilde{M}$ is $\mathbb{R}^n$ and $\pi_1(M)$ is $\mathbb{Z}^n$. They are quasi-isometric, as $\mathbb{R}^n$ is a thickening of $\mathbb{Z}^n$.

2. More generally, if a group Γ acts properly discontinuously and with compact quotient by isometries on a complete locally compact length metric space (X, dist_ℓ) then Γ is finitely generated [BrH, Theorem I.8.10, p. 135] and Γ endowed with any word metric is quasi-isometric to $(X\,,\,\mathrm{dist}_\ell)$.

   Consequently two groups acting as above on the same length metric space are quasi-isometric.

3. Given a finitely generated group G and a finite index subgroup G_1 in it, G and G_1 endowed with arbitrary word metrics are quasi-isometric.

   In terms of Riemannian manifolds, if M is a finite covering of N then $\pi_1(M)$ and $\pi_1(N)$ are quasi-isometric.

4. Given a finite normal subgroup N in a finitely generated group G, G and G/N (both endowed with arbitrary word metrics) are quasi-isometric.

   Thus, in arguments where we study behaviour of groups with respect to quasi-isometry, we can always replace a group with a finite index subgroup or with a quotient by a finite normal subgroups.

5. All non-Abelian free groups of finite rank are quasi-isometric to each other. This follows from the fact that the Cayley graph of the free group of rank n with respect to a set of n generators and their inverses is the regular simplicial tree of valence $2n$.

   Now all regular simplicial trees of valence at least 3 are quasi-isometric. We denote by $\mathcal{T}_k$ the regular simplicial tree of valence k and we show that $\mathcal{T}_3$ is quasi-isometric to $\mathcal{T}_k$ for every $k \geq 4$.

   We define the map $\mathfrak{q} : \mathcal{T}_3 \to \mathcal{T}_k$ as in Figure 3, sending all edges drawn in thin lines isometrically onto edges and all paths of length $k-3$ drawn in thick lines onto one vertex. The map $\mathfrak{q}$ thus defined is surjective and it satisfies the inequality

   $$\frac{1}{k-2}\,\mathrm{dist}(x,y) - 1 \leq \mathrm{dist}(\mathfrak{q}(x), \mathfrak{q}(y)) \leq \mathrm{dist}(x,y)\,.$$

6. Let M be a non-compact hyperbolic two-dimensional orbifold of finite area. This is the same thing as saying that $M = \Gamma\backslash\mathbb{H}^2_{\mathbb{R}}$, where Γ is a discrete subgroup of $PSL_2(\mathbb{R})$ with fundamental domain of finite area.

   Nota bene: We assume that all the actions of groups by isometries on spaces are to the left, as in the particular case when the space is the Cayley graph. This means that the

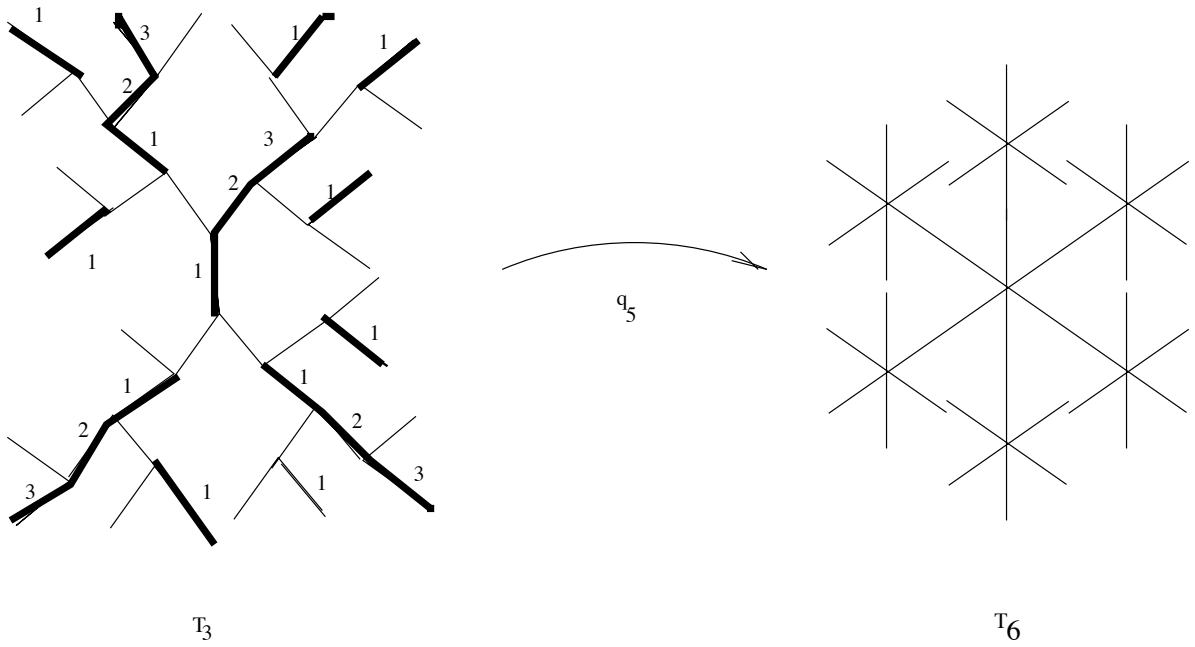

Figure 3: All regular simplicial trees are quasi-isometric.

quotient will be always taken to the left. We feel sorry for all people which are accustomed to the quotients to the right.

We can apply the following result.

Lemma 1.2 (Selberg's Lemma). *A finitely generated group which is linear over a field of characteristic zero has a torsion free subgroup of finite index.*

We recall that *torsion free group* means a group which does not have finite non-trivial subgroups. For an elementary proof of Selberg's Lemma see [Al].

We conclude that Γ has a finite index subgroup Γ_1 which is torsion free. It follows that $N = \Gamma_1 \backslash \mathbb{H}^2_{\mathbb{R}}$ is a hyperbolic surface which is a finite covering of M, hence it is of finite area but non-compact. On the other hand, it is known that the fundamental group of such a surface is a free group of finite rank (see for instance [Mass]).

Conclusion: The fundamental groups of all hyperbolic two-dimensional orbifolds are quasi-isometric to each other.

At this point one may start thinking that the quasi-isometry is too weak a relation for groups, and that it does not distinguish too well between groups with different algebraic structure. It goes without saying that we are discussing here only infinite finitely generated groups, because we need a word metric and because finite groups are all quasi-isometric to the trivial group.

We can start by asking if the result in Example 6 is true for any rank one symmetric space.

Question 1.3. Given M and N orbifolds of finite volume covered by the same rank one symmetric space, is it true that $\pi_1(M)$ and $\pi_1(N)$ are quasi-isometric ?

It is true if N is obtained from M by means of a sequence of operations obviously leaving the fundamental group $\Gamma = \pi_1(M)$ in the same quasi-isometry class :

- going up or down a finite covering, which at the level of fundamental groups means changing Γ with a finite index subgroup or a finite extension;
- replacing a manifold with another one isometric to it, which at the level of groups means changing Γ with a conjugate Γ^g, where g is an isometry of the universal covering.

Above we have used the following

Notation: For A an element or a subgroup in a group G and $g \in G$, we denote by A^g its image gAg^{-1} under conjugacy by g.

In the sequel we also use the following

Convention: In a group G we denote its neutral element by id if we consider an action of the group on a space, and by 1 otherwise.

2 Rigidity of non-uniform rank one lattices

It turns out that the answer to the Question 1.3 is "very much negative", so to speak, that is: apart from the exceptional case of two dimensional hyperbolic orbifolds, in the other cases finite volume rank one locally symmetric spaces which are not compact have quasi-isometric fundamental groups if and only if the locally symmetric spaces are obtained one from the other by means of a sequence of three of the operations described previously. More precisely, the theorem below, formulated in terms of groups, holds.

2.1 Theorems of Richard Schwartz

We recall that a discrete group of isometries Γ of a symmetric space X such that $\Gamma\backslash X$ has finite volume is called a *lattice*. If $\Gamma\backslash X$ is compact, the lattice is called *uniform*, otherwise it is called *non-uniform*.

Theorem 2.1 (R. Schwartz, [Sch$_1$]). *(1) (quasi-isometric lattices) Let G_i be a non-uniform lattice of isometries of the rank one symmetric space X_i, $i = 1, 2$. Suppose that G_1 is quasi-isometric to G_2. Then $X_1 = X_2 = X$ and one of the following holds:*

(a) $X = \mathbb{H}^2_{\mathbb{R}}$;

(b) there exists an isometry g of X such that $G_1^g \cap G_2$ has finite index both in G_1^g and in G_2.

(2) (finitely generated groups quasi-isometric to lattices) Let Λ be a finitely generated group and let G be a non-uniform lattice of isometries of a rank one symmetric space $X \neq \mathbb{H}^2_{\mathbb{R}}$. If Λ is quasi-isometric to G then there exists a non-uniform lattice G_1 of isometries of X and a finite group F such that one has the following exact sequence:

$$0 \to F \to \Lambda \to G_1 \to 0\,.$$

The notion of commensurability is recalled in Section 8. The particular case of commensurability described in Theorem 2.1, (1), (b), means that the locally symmetric spaces $G_1\backslash X$ and $G_2\backslash X$ have isometric finite coverings.

We note that Theorem 2.1 is in some sense a much stronger result than Mostow Theorem. Mostow Theorem requires the isomorphism of fundamental groups - which is an algebraic relation between groups, also implying their quasi-isometry. Theorem 2.1 only requires that the groups

are quasi-isometric, which is a relation between "large scale geometries" of the two groups, and has *a priori* nothing to do with the algebraic structure of the groups.

Since Mostow rigidity holds for all kinds of lattices, a first natural question to ask is:

Question 2.2. Are the two statements in Theorem 2.1 also true for uniform lattices ?

Concerning statement (1), the following can be said: given G_i uniform lattices of isometries of the rank one symmetric spaces X_i, $i = 1, 2$, G_1 quasi-isometric to G_2 implies that $X_1 = X_2 = X$.

Now one can ask if in case $X \neq \mathbb{H}^2_{\mathbb{R}}$ there exists an isometry g of X such that $G_1^g \cap G_2$ has finite index both in G_1^g and in G_2 ? In other words is it true that *all* uniform lattices of isometries of the same rank one symmetric space $X \neq \mathbb{H}^2_{\mathbb{R}}$ are commensurable ?

A weaker variant of the previous question is whether all *arithmetic* uniform lattices of isometries of $X \neq \mathbb{H}^2_{\mathbb{R}}$ are commensurable.

The answer to both questions is negative, as shown by the following counter-example.

Counter-example:

All the details for the statements below can be found in [GPS].

Let Q be a quadratic form of the type $\sqrt{2}x_{n+1}^2 - a_1x_1^2 - \cdots - a_nx_n^2$, where a_i are positive rational numbers. The set

$$\mathbb{H}_Q = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid Q(x_1, \ldots, x_{n+1}) = 1\,,\ x_{n+1} > 0\}$$

is a model of the hyperbolic n-dimensional space. Its group of isometries is $SO_{\mathrm{Id}}(Q)$, the connected component containing the identity of the stabilizer of the form Q in $SL(n+1, \mathbb{R})$. The discrete subgroup $G_Q = SO_{\mathrm{Id}}(Q) \cap SL(n+1, \mathbb{Z}(\sqrt{2}))$ is a uniform lattice. Now if two such lattices G_{Q_1} and G_{Q_2} are commensurable then there exist $g \in GL(n+1, \mathbb{Q}[\sqrt{2}])$ and $\lambda \in \mathbb{Q}[\sqrt{2}]$ such that $Q_1 \circ g = \lambda Q_2$. In particular, if n is odd then the ratio between the discriminant of Q_1 and the discriminant of Q_2 is a square in $\mathbb{Q}[\sqrt{2}]$. It now suffices to take two forms such that this is not possible, for instance (like in [GPS]):

$$Q_1 = \sqrt{2}x_{n+1}^2 - x_1^2 - x_2^2 - \cdots - x_n^2 \text{ and } Q_2 = \sqrt{2}x_{n+1}^2 - 3x_1^2 - x_2^2 - \cdots - x_n^2\,.$$

Statement (2) of Theorem 2.1, on the other hand, also holds for uniform lattices. See the discussion in the beginning of Section 3.

A main step in the proof of Theorem 2.1 is the following rigidity result, interesting by itself.

Theorem 2.3 (Rigidity Theorem [Sch$_1$]). *Let Γ and G be two non-uniform lattices of isometries of $\mathbb{H}^n_{\mathbb{F}} \neq \mathbb{H}^2_{\mathbb{R}}$. An (L, C)–quasi-isometry $\mathfrak{q}$ between Γ and G is at finite distance from an isometry g of $\mathbb{H}^n_{\mathbb{F}}$ with the property that $\Gamma^g \cap G$ has finite index in both Γ^g and G.*

The meaning of the statement "$\mathfrak{q}$ *is at finite distance from* g" is the following:

For every compact $\mathcal{K}$ in $G\backslash\mathbb{H}^n_{\mathbb{F}}$ there exists $D = D(L, C, \mathcal{K}, \Gamma, G)$ such that for every x_0 with $Gx_0 \in \mathcal{K}$, one has

$$\mathrm{dist}(\mathfrak{q}(\gamma)x_0\,,\, g\gamma x_0) \leq D,\ \forall \gamma \in \Gamma\,.$$

As it is, it does not look very enlightening. We shall come back to this statement in Section 2.3, after recalling what is the structure of finite volume real hyperbolic manifolds in Section 2.2. Also in Section 2.3 we shall give an outline of the proof of Theorem 2.3 in the particular case when $\mathbb{H}^n_{\mathbb{F}} = \mathbb{H}^3_{\mathbb{R}}$. All the ideas of the general proof are already present in this particular case, and we avoid some technical difficulties that are irrelevant in a first approach.

According to Selberg's Lemma, we may suppose, without loss of generality, that both Γ and G are without torsion.

2.2 Finite volume real hyperbolic manifolds

Let M be a finite volume real hyperbolic manifold, that is a manifold with universal covering $\mathbb{H}^n_{\mathbb{R}}$, for some $n \geq 2$. Let Γ be its fundamental group.

Given a point $x \in M$ denote by $r(x)$ the injectivity radius of M at x.

For every $\varepsilon > 0$ the manifold can be decomposed into two parts:

- the *ε-thick part*: $M_{\geq \varepsilon} = \{x \in M \mid r(x) \geq \varepsilon\}$;
- the *ε-thin part*: $M_{<\varepsilon} = \{x \in M \mid r(x) < \varepsilon\}$.

The following theorem describes the structure of M. We refer to [Th, §4.5] for details.

Theorem 2.4. *(1) There exists a universal constant $\varepsilon_0 = \varepsilon_0(n) > 0$ such that for every complete manifold M of universal covering $\mathbb{H}^n_{\mathbb{R}}$ and of fundamental group Γ, the ε_0-thin part $M_{<\varepsilon_0}$ is a disjoint union of*

- *tubular neighbourhoods of short closed geodesics;*
- *neighbourhoods of cusps, that is sets of the form $\Gamma_\alpha \backslash Hbo(\alpha)$, where $Hbo(\alpha)$ is an open horoball of basepoint $\alpha \in \partial_\infty \mathbb{H}^n_{\mathbb{R}}$ and Γ_α is the stabilizer of α in Γ.*

(2) A complete hyperbolic manifold M has finite volume if and only if for every $\varepsilon > 0$ the ε-thick part $M_{\geq \varepsilon}$ is compact.

Note that the fact that $M_{\geq \varepsilon_0}$ is compact implies that

- *$M_{<\varepsilon_0}$ has finitely many components;*
- *for every neighbourhood of a cusp, $\Gamma_\alpha \backslash Hbo(\alpha)$, its boundary $\Gamma_\alpha \backslash H(\alpha)$, where H_α is the boundary horosphere of $Hbo(\alpha)$, is compact.*

Let now M be a finite volume real hyperbolic manifold and $\Gamma = \pi_1(M)$. Consider the finite set of tubular neighbourhoods of cusps

$$\{\Gamma_{\alpha_i} \backslash Hbo(\alpha_i) \mid i \in \{1, 2, \ldots, m\}\}.$$

According to Theorem 2.4, the set

$$M_0 = M \setminus \coprod_{i=1}^{m} \Gamma_{\alpha_i} \backslash Hbo(\alpha_i)$$

is compact. The pre-image of each cusp $\Gamma_{\alpha_i} \backslash Hbo(\alpha_i)$ is the Γ–orbit of $Hbo(\alpha_i)$ and the open horoballs composing this orbit are pairwise disjoint. Thus, the space

$$X_0 = \mathbb{H}^3_{\mathbb{R}} \setminus \coprod_{i=1}^{m} \coprod_{\gamma \in \Gamma / \Gamma_{\alpha_i}} \gamma Hbo(\alpha_i)$$

satisfies $\Gamma \backslash X_0 = M_0$.

Remarks 2.5. The group Γ endowed with a word metric dist_w is quasi-isometric to the space X_0 with the length metric dist_ℓ according to the example (2) of quasi-isometry given in Section 1.2. In particular, since for every $x_0 \in X_0$, Γx_0 is a net in X_0, we also have that $(\Gamma\,, \mathrm{dist}_w)$ is quasi-isometric to $(\Gamma x_0\,, \mathrm{dist}_\ell)$.

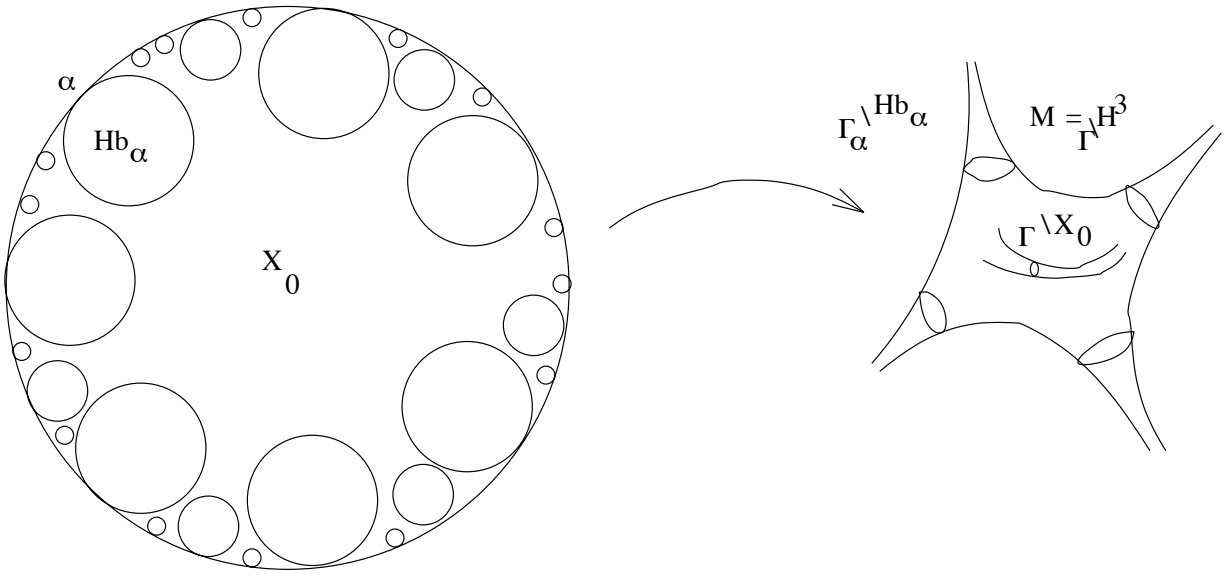

Figure 4: Finite volume hyperbolic manifold, space X_0.

2.3 Proof of Theorem 2.3

According to Section 2.2, there exists X_0 complementary set in $\mathbb{H}^3_{\mathbb{R}}$ of countably many pairwise disjoint open horoballs such that $\Gamma\backslash X_0$ is compact. Consequently, Γ with any word metric is quasi-isometric to X_0 with the length metric. Similarly, one can associate to G a complementary set Y_0 of countably many pairwise disjoint open horoballs such that $G\backslash Y_0$ is compact and such that G with a word metric is quasi-isometric to Y_0 with the length metric.

The quasi-isometry $\mathfrak{q} : \Gamma \to G$ induces a quasi-isometry between $(\Gamma x_0, \mathrm{dist}_\ell)$ and $(Gy_0, \mathrm{dist}_\ell)$, for every $x_0 \in X_0$, $y_0 \in Y_0$, hence also between X_0 and Y_0 (each quasi-isometry having different parameters L and C). For simplicity, we denote all these quasi-isometries by $\mathfrak{q}$ and all their constants by L and C.

In these terms, the conclusion of Theorem 2.3 means that $\mathfrak{q}$ seen as a quasi-isometry between X_0 and Y_0 is at distance at most D from the restriction to X_0 of an isometry g in $Comm(\Gamma, G)$, where $D = D(L, C, X_0, Y_0)$.

We now give an outline of proof of Theorem 2.3.

Step 1. The following general statement holds.

Lemma 2.6 (Quasi-Flat Lemma [Sch$_1$], §3.2). *Let Γ be a non-uniform lattice of isometries of $\mathbb{H}^3_{\mathbb{R}}$. For every $L \geq 1$ and $C \geq 0$ there exists $M = M(L, C)$ such that every quasi-isometric embedding*

$$\mathfrak{q} : \mathbb{Z}^2 \to \Gamma$$

has its range in $\mathcal{N}_M(\gamma\Gamma_\alpha)$, where Γ_α is a cusp group and $\gamma \in \Gamma$.

We shall come back to the proof of Lemma 2.6 later (when discussing Theorem 5.8). For the moment let us apply it to the (L, C)–quasi-isometry $\mathfrak{q}$ from Γ to G, and to its quasi-converse $\bar{\mathfrak{q}} : G \to \Gamma$.

- For every $\gamma \in \Gamma$ and $\alpha \in \partial_\infty \mathbb{H}^3_{\mathbb{R}}$ corresponding to a cusp of $\Gamma\backslash\mathbb{H}^3_{\mathbb{R}}$,

  $$\mathfrak{q}(\gamma\Gamma_\alpha) \subset \mathcal{N}_M(gG_\beta)\,,$$

  for some $g \in G$ and $\beta \in \partial_\infty \mathbb{H}^3_{\mathbb{R}}$ corresponding to a cusp of $G\backslash\mathbb{H}^3_{\mathbb{R}}$;

- For every $g \in G$ and $\beta \in \partial_\infty \mathbb{H}^3_{\mathbb{R}}$ corresponding to a cusp of $G\backslash\mathbb{H}^3_{\mathbb{R}}$,

$$\bar{\mathfrak{q}}(gG_\beta) \subset \mathcal{N}_M(\gamma'\Gamma_{\alpha'})\,,$$

for some $\gamma' \in \Gamma$ and $\alpha' \in \partial_\infty \mathbb{H}^3_{\mathbb{R}}$ corresponding to a cusp of $\Gamma\backslash\mathbb{H}^3_{\mathbb{R}}$.

Combining both and noticing that if the left class $\gamma\Gamma_\alpha$ is contained in the tubular neighbourhood of another left class $\gamma'\Gamma_{\alpha'}$ then the two coincide, a bijection is obtained, between left classes $\gamma\Gamma_\alpha$ and left classes gG_β, such that

$$\mathrm{dist}_H(\mathfrak{q}(\gamma\Gamma_\alpha), gG_\beta) \leq M'\,. \tag{2}$$

Here dist_H denotes the Hausdorff distance (see the Dictionary for a definition).
The situation is represented in Figure 5.

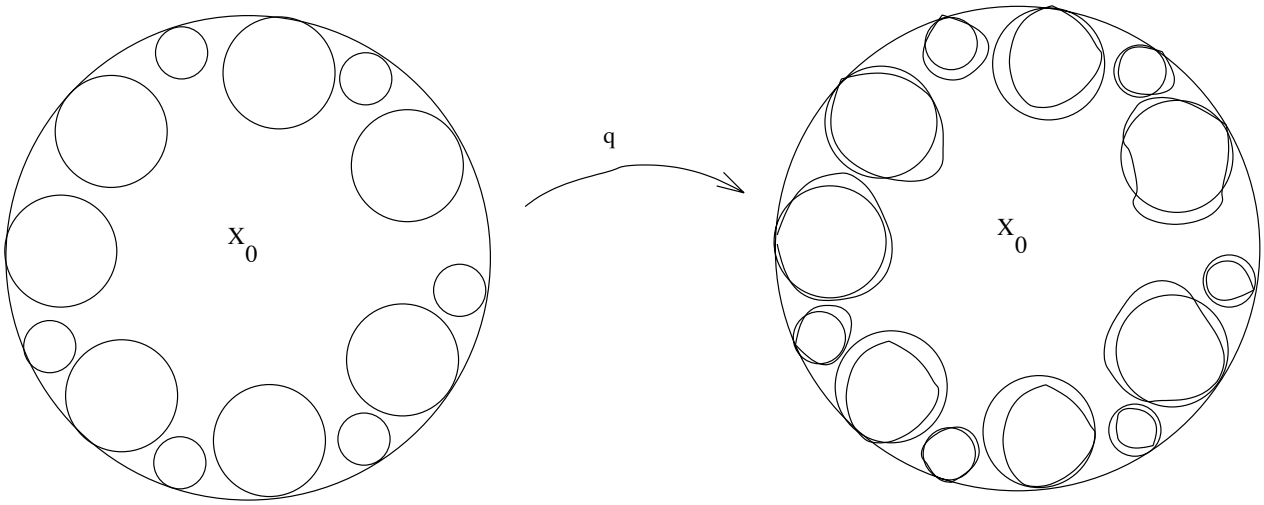

Figure 5: Quasi-isometric embeddings of $\mathbb{Z}^2$.

Step 2. The map $\mathfrak{q}$ seen as a quasi-isometry between Γx_0, net in X_0 and Gy_0, net in Y_0, is extended to a quasi-isometry $\mathfrak{q}_e$ between a net N_1 in $\mathbb{H}^3_{\mathbb{R}}$ and a net N_2 in $\mathbb{H}^3_{\mathbb{R}}$. This is done horoball by horoball. Let $\gamma\Gamma_\alpha$ and gG_β satisfying (2). Let γHb_α and gHb_β be the corresponding horoballs. We divide each of them into strips of constant width, by means of countably many horospheres. We note that $\gamma\Gamma_\alpha x_0$ is δ-separated, and that the horosphere γH_α is contained in $\mathcal{N}_\epsilon(\gamma\Gamma_\alpha x_0)$, for some $\delta > 0$ and $\epsilon > 0$.

We project $\gamma\Gamma_\alpha x_0$ onto the first horosphere H_1. We get a $(\delta'\,,\,\epsilon')$–net, $\mathcal{N}_1^{(1)}$, for some $\delta' < \delta$ and $\epsilon' < \epsilon$. We choose a maximal δ-separated subset $N_1^{(1)}$ in $\mathcal{N}_1^{(1)}$, hence a (δ, δ)–net in $\mathcal{N}_1^{(1)}$ and a $(\delta, \delta + \epsilon')$–net in H_1. We extend $\mathfrak{q}$ to $N_1^{(1)}$ by $\mathfrak{q}(n_1) = \pi \circ \mathfrak{q} \circ \pi^{-1}(n_1)$, where π denotes the projection in each of the spaces onto the first horosphere H_1.

We repeat the argument and extend $\mathfrak{q}$ to a net in H_2, H_3, etc. A global quasi-isometry is thus obtained. Indeed, given two points $A \in H_n$ and $B \in H_m$, with $m \geq n$, if B' is the projection of B onto H_n, the distance $\mathrm{dist}(A, B)$ is bi-Lipschitz equivalent to $\mathrm{dist}(A, B') + \mathrm{dist}(B', B)$.

We finally obtain a quasi-isometry $\mathfrak{q}_e$ between nets of $\mathbb{H}^3_{\mathbb{R}}$, hence a quasi-isometry of $\mathbb{H}^3_{\mathbb{R}}$.

Nota bene: In the case of the Mostow rigidity theorem also a quasi-isometry of the whole space is obtained, but it has the extra property that it is *equivariant with respect to a given isomorphism between the two groups* Γ *and* G. Here, the property of equivariance is replaced

with the extra geometric information that $\mathfrak{q}$ sends the space X_0 at uniformly bounded distance from Y_0, by sending each boundary horosphere at bounded distance from a boundary horosphere.

The quasi-isometry $\mathfrak{q}_e$ extends, according to the Theorem of Efremovitch-Tikhomirova to a map between boundaries

$$\partial\mathfrak{q}_e : \mathbb{S}^2_\infty \to \mathbb{S}^2_\infty ,$$

which is a quasi-conformal homeomorphism.

Next, two theorems are used.

Theorem 2.7 (Rademacher-Stepanov, see [LV]). *Every quasi-conformal homeomorphism between open sets in $\mathbb{S}^2$ is differentiable almost everywhere.*

Theorem 2.8 ([LV]). *A quasi-conformal homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that almost everywhere its differential is a similarity is a Möbius transformation.*

Step 3. It is shown that in almost every point in the set of differentiability of $\partial\mathfrak{q}$ the differential is a similarity.

Let Ω_1 be the set of points ξ in $\mathbb{S}^2_\infty$ such that the geodesic ray $[x_0, \xi)$ returns in X_0 infinitely often. The set of such ξ has full Lebesgue measure in $\mathbb{S}^2_\infty$. This can be seen for instance by projecting onto $\Gamma\backslash\mathbb{H}^3_{\mathbb{R}}$ and noting that almost every locally geodesic ray in it is equidistributed, hence it returns infinitely often in the compact $\Gamma\backslash X_0$.

Likewise, let Ω_2 be the set of points ζ in $\mathbb{S}^2_\infty$ such that the geodesic ray $[y_0, \zeta)$ returns in Y_0 infinitely often.

Let Ω be the set of points in $\Omega_1 \cap \partial\mathfrak{q}_e^{-1}(\Omega_2)$ in which $\partial\mathfrak{q}_e$ is differentiable. It is shown that in every $\xi \in \Omega$ the differential of $\partial\mathfrak{q}_e$ is a similarity. Denote by ζ the image $\partial\mathfrak{q}_e(\xi)$. Also, denote by ξ' the image of ξ under the geodesic symmetry of center x_0 and by ζ' the image of ζ under the geodesic symmetry of center y_0. In the sequel consider the two stereographic projections of $\mathbb{H}^3_{\mathbb{R}}$ sending (ξ, ξ', x_0) and respectively (ζ, ζ', y_0) onto $(O, \infty, (0, 0, 1))$. We work in the corresponding half-space models of $\mathbb{H}^3_{\mathbb{R}}$ for the domain and the range of $\mathfrak{q}_e$, respectively. In these models, $\partial\mathfrak{q}_e(O) = O$, $\partial\mathfrak{q}_e(\infty) = \infty$ and the differential $d_\xi\partial\mathfrak{q}_e$ becomes $d_O\partial\mathfrak{q}_e$. The goal is to show that the latter is a similarity.

Let (x_n) be a sequence of points on $[x_0, \xi) \cap X_0$ diverging to ξ. Let t_n be the hyperbolic isometry of axis containing $[x_0, \xi)$ such that $t_n(x_0) = x_n$.

Similarly, let (y_n) be a sequence of points on $[y_0, \zeta) \cap Y_0$ diverging to ζ. Let τ_n be the hyperbolic isometry of axis containing $[y_0, \zeta)$ such that $\tau_n(y_0) = y_n$.

Consider the sequence of (L, C)–quasi-isometries $\mathfrak{q}_n = \tau_n^{-1} \circ \mathfrak{q} \circ t_n : t_n^{-1}(X_0) \to \tau_n^{-1}(Y_0)$.

Since $t_n^{-1}(X_0)$ are isometric copies of X_0 containing x_0, by Ascoli Theorem they converge to an isometric copy X_1 of X_0. A similar argument can be done for $\tau_n^{-1}(Y_0)$, which converge to an isometric copy Y_1 of Y_0, therefore $\mathfrak{q}_n$ converges to an (L, C)–quasi-isometry $\tilde{\mathfrak{q}} : X_1 \to Y_1$. Also, the extensions $\mathfrak{q}_n^e = \tau_n^{-1} \circ \mathfrak{q}_e \circ t_n$ of $\mathfrak{q}_n$ to $\mathbb{H}^3_{\mathbb{R}}$ converge to an extension $\tilde{\mathfrak{q}}_e$ of $\tilde{\mathfrak{q}}$.

On the other hand, since t_n and τ_n restricted to $\mathbb{C} \subset \partial_\infty\mathbb{H}^3_{\mathbb{R}}$ are homotheties of center O, the restrictions of the boundary maps $\partial\mathfrak{q}_n^e : \mathbb{C} \to \mathbb{C}$ converge to the differential $d_O\partial\mathfrak{q}_e$. Thus, $d_O\partial\mathfrak{q}_e$ is the restriction to $\mathbb{C}$ of $\partial\tilde{\mathfrak{q}}_e$. From this it can be deduced that $d_O\partial\mathfrak{q}_e$ is a similarity. We give the sketch of proof below. The full proof is more elaborate and can be found in [Sch$_1$].

The argument in Step 1 implies that $\tilde{\mathfrak{q}}$ sends every boundary horosphere of X_1 at uniformly bounded distance of a boundary horosphere of Y_1. From this it can be deduced that all horospheres having a certain Euclidean height h are sent at uniformly bounded distance from horospheres having an Euclidean height in $[h/\lambda, \lambda h]$ (for some constant $\lambda \geq 1$ depending on the constant M given by Step 1 for $\tilde{\mathfrak{q}}$). Note that the basepoints of horospheres of heights at least

h in X_1 compose nets $\mathbf{N}_h$ in $\mathbb{C}$ with the corresponding constants δ and ϵ smaller and smaller as h decreases to zero. Thus, $d_O\partial\mathfrak{q}_e$ sends each of these nets $\mathbf{N}_h$ of $\mathbb{C}$ into a net $\mathbf{N}_{h/\lambda}$ of $\mathbb{C}$. Up to now, nothing surprising, since $d_O\partial\mathfrak{q}_e$ is a linear map.

Now we change stereographic projection, and in both the domain and the range model reverse O with ∞. In the new models, we again have $\tilde{\mathfrak{q}} : X_1 \to Y_1$ extended to $\tilde{\mathfrak{q}}_e : \mathbb{H}^3_{\mathbb{R}} \to \mathbb{H}^3_{\mathbb{R}}$ such that $\partial\tilde{\mathfrak{q}}_e$ fixes both O and ∞ and such that its restriction to $\mathbb{C} \setminus \{O\}$ coincides with $I \circ d_O\partial\mathfrak{q}_e \circ I$, where I is the inversion with respect to the unit circle. An argument as above implies that $I \circ d_O\partial\mathfrak{q}_e \circ I$ sends nets $\mathbf{N}_h$ of $\mathbb{C}$ into nets $\mathbf{N}_{h/\lambda}$ of $\mathbb{C}$ for every h. This implies that $I \circ d_O\partial\mathfrak{q}_e \circ I$ is also linear. But this can happen only if $d_O\partial\mathfrak{q}_e$ is a similarity.

Step 4. Theorem 2.8 and Step 3 imply that there exists an isometry g of $\mathbb{H}^3_{\mathbb{R}}$ such that $\partial g = \partial\mathfrak{q}_e$. It follows that g and $\mathfrak{q}_e$ are at uniformly bounded distance from each other. In particular g restricted to X_0 is at uniformly bounded distance from $\mathfrak{q}$. Next it is shown that g is in the commensurator $Comm(\Gamma, G)$ of Γ into G.

The argument is by contradiction. Suppose that $\Gamma^g \cap G$ has infinite index both in Γ^g and in G. It follows that there exists a sequence g_n of elements in G such that $g_n(\Gamma^g)$ are distinct left cosets.

Let β be a basepoint of a boundary horosphere of Y_0. It is an easy exercise to show that, up to taking a subsequence, there exists a sequence γ_n in Γ and another basepoint α of a boundary horosphere of Y_0 such that $g_n\gamma_n^g(\alpha) = \beta$ and $g_n\gamma_n^g(y_0) \in B(y_0, R)$, where R is a constant.

Consider the respective stereographic projections sending (α, y_0) to $(\infty, (0, 0, 1))$ on the $\mathbb{H}^3_{\mathbb{R}}$ of definition and (β, y_0) to $(\infty, (0, 0, 1))$ on the range $\mathbb{H}^3_{\mathbb{R}}$. In these new half-space models of $\mathbb{H}^3_{\mathbb{R}}$ the isometry $g_n\gamma_n^g$ fixes ∞ and sends $(0, 0, 1)$ at distance at most R from itself. Also, since γ_n^g are isometries of $g(X_0)$, g_n are isometries of Y_0 and Y_0 is at uniformly bounded distance from $g(X_0)$, it follows that $g_n\gamma_n^g(Y_0)$ is at Hausdorff distance at most D from Y_0, where D is a constant independent of n.

By Ascoli Theorem, $g_n\gamma_n^g$ converges to an isometry $\hat{g}$ such that $Y_1 = \hat{g}(Y_0)$ is at distance at most D from Y_0.

Let $\mathbf{N}_h$ be the set of basepoints of boundary horospheres of Y_0 of Euclidean height at least h. Note that G_∞, the stabilizer in G of the point ∞, acts on $\mathbb{C}$ such that $G_\infty\backslash\mathbb{C}$ is a flat torus. Let $D \subset \mathbb{C}$ be a fundamental domain (quadrangle) projecting on this torus. The number of horoballs of Y_0 of Euclidean height at least h and with basepoints in D is finite. Let $\mathbf{B}_h$ be the finite set of their basepoints.

Then $\mathbf{N}_h = \coprod_{b\in\mathbf{B}_h} G_\infty b$ is a finite union of grids in $\mathbb{C}$, hence a net in $\mathbb{C}$.

The previous considerations imply that $\mathbf{N}_h^n = g_n\gamma_n^g(\mathbf{N}_h)$ is a net contained in $\mathbf{N}_{h/\lambda}$ and likewise for the net $\hat{\mathbf{N}}_h = \hat{g}(\mathbf{N}_h)$. On the other hand $\mathbf{N}_h^n$ converges to $\hat{\mathbf{N}}_h$ in the compact-open topology. The only way in which this convergence can occur, both nets being in the larger net $\mathbf{N}_{h/\lambda}$, is that they coincide on larger and larger subsets.

An isometry of $\mathbb{H}^3_{\mathbb{R}}$ fixing four points on $\partial_\infty\mathbb{H}^3_{\mathbb{R}}$ which are not on the same circle is the identity isometry[1]. Hence two isometries which coincide on four points on $\partial_\infty\mathbb{H}^3_{\mathbb{R}}$ not on the same circle are equal. It follows that the sequence $g_n\gamma_n^g$ becomes stationary, for n large enough. This contradicts the hypothesis that $g_n(\Gamma^g)$ are distinct left cosets. □

[1]Elementary proof: an isometry fixing three distinct points in the boundary has to fix a point in $\mathbb{H}^3_{\mathbb{R}}$ [BP, Proposition A.5.14]. Thus the isometry can be identified with a matrix in $SO(3)$ fixing four tangent vectors-the vectors pointing towards the four fixed points in $\partial_\infty\mathbb{H}^3_{\mathbb{R}}$. If the four vectors were in the same plane then the corresponding points in the boundary would be on the same circle. Therefore three of the four vectors are linearly independent and the isometry has to be the identity.

2.4 Proof of Theorem 2.1

A non-uniform lattice of isometries of $\mathbb{H}^n_{\mathbb{F}}$ has infinitely many ends if and only if $\mathbb{H}^n_{\mathbb{F}} = \mathbb{H}^2_{\mathbb{R}}$. On the other hand, having infinitely many ends is a quasi-isometry invariant. Thus, either both X_1 and X_2 are $\mathbb{H}^2_{\mathbb{R}}$ or both differ from it. Suppose we are in the second case. The quasi-isometry $\mathfrak{q} : G_1 \to G_2$ induces as in the previous section a quasi-isometry $\mathfrak{q} : X_1 \to X_2$, hence a quasi-conformal homeomorphism between the boundaries at infinity $\partial_\infty X_1$ and $\partial_\infty X_2$. Elementary dimension and structure arguments imply that $X_1 = X_2$.

The rest of the statement (1) follows from the Theorem 2.3.

We now get to the proof of statement (2). The following standard fact is needed.

Lemma 2.9. *Let Λ and G be finitely generated groups and let $\mathfrak{q} : \Lambda \to G$, $\bar{\mathfrak{q}} : G \to \Lambda$ be two quasi-converse (L_0, C_0)–quasi-isometries. Then to every $\lambda \in \Lambda$ one can associate an (L, C)–quasi-isometry of G, $\mathfrak{q}_\lambda = \mathfrak{q} \circ L_\lambda \circ \bar{\mathfrak{q}}$, where L_λ denotes the isometry on Λ determined by the action of λ to the left, and (L, C) can be effectively computed from (L_0, C_0). Moreover the map $\lambda \to \mathfrak{q}_\lambda$ defines a*

- ***quasi-action of Λ on G:*** *there exists $D = D(L_0, C_0)$ so that for every $\lambda, \eta \in \Lambda$ the following holds:*

$$\begin{aligned} \mathrm{dist}(\mathfrak{q}_\lambda \circ \mathfrak{q}_\eta \,,\, \mathfrak{q}_{\lambda\eta}) &\leq D\,, & (3)\\ \mathrm{dist}(\mathfrak{q}_\lambda \circ \mathfrak{q}_{\lambda^{-1}} \,,\, \mathrm{id}) &\leq D\,. & (4) \end{aligned}$$

- ***which moreover is quasi-transitive:*** *for every $g, g' \in G$ there exists $\lambda \in \Lambda$ such that*

$$\mathrm{dist}(\mathfrak{q}_\lambda(g), g') \leq C_1\,,$$

  where $C_1 = C_1(L_0, C_0)$;

- ***and of finite quasi-kernel:*** *for every $K > 0$ the set of λ such that $\mathrm{dist}(\mathfrak{q}_\lambda, \mathrm{id}) \leq K$ is finite.*

The proof of the lemma is left as an exercise to the reader.

In the particular case considered, Lemma 2.9 and Theorem 2.3 imply that if Λ is quasi-isometric to G non-uniform lattice of isometries of $X = \mathbb{H}^n_{\mathbb{F}} \neq \mathbb{H}^2_{\mathbb{R}}$, then there exists a morphism of finite kernel

$$\phi : \Lambda \to Comm(G)\,.$$

The fact that the image G_1 of Λ under ϕ is discrete can be proved by contradiction. Suppose it is not discrete, hence there are infinitely many elements in $\phi(\Lambda)$ in the neighbourhood of the identity element $\mathrm{id} \in G$. Then for some D large enough we have that for infinitely many $\lambda \in \Lambda$

$$\mathrm{dist}(\mathfrak{q} \circ L_\lambda \circ \bar{\mathfrak{q}}(\mathrm{id}), \mathrm{id}) \leq D \Rightarrow \mathrm{dist}(L_\lambda \circ \bar{\mathfrak{q}}(\mathrm{id}), \bar{\mathfrak{q}}(\mathrm{id})) \leq D'\,,$$

where $D' = D'(L, C, D)$. This contradicts the fact that every ball in Λ is a finite set.

Also, one can argue that $G_1 \backslash X$ has finite volume roughly as follows. Consider a complementary set X_0 in X of a family of countably many pairwise disjoint open horoballs such that $G \backslash X_0$ is compact. Lemma 2.9 implies that Λ acts quasi-transitively by quasi-isometries on G, hence on X_0. It follows that G_1 acts "with compact quotient" on X_0. See [Sch$_1$, §10.4] for details.

3 Classes of groups complete with respect to quasi-isometries

Another way of interpreting Theorem 2.1, (2), is the following. Let $\mathcal{C}$ be the class of non-uniform lattices of isometries in $\mathbb{H}^n_{\mathbb{F}} \neq \mathbb{H}^2_{\mathbb{R}}$. Then every finitely generated group Λ quasi-isometric to a group $G \in \mathcal{C}$ is itself in $\mathcal{C}$, up to taking its quotient by a finite normal subgroup. One may ask what other classes of groups behave similarly. Possibly, to the operation of taking quotient by finite normal subgroup one has to add the other algebraic operation preserving the quasi-isometry class: taking a subgroup of finite index.

Definition 3.1. A class of finitely generated groups $\mathcal{C}$ with the property that if Λ is quasi-isometric to $G \in \mathcal{C}$ then $\Lambda_1 \in \mathcal{C}$, where Λ_1 is either a finite index subgroup of Λ or a quotient of Λ by a finite normal subgroup, is called *class of groups complete with respect to quasi-isometries* or *q.i. complete*.

The question of finding such classes has been asked for the first time by M. Gromov in [Gr_1].

3.1 List of classes of groups q.i. complete

We give a (non-exhaustive) list of classes of groups q.i. complete. We begin with the q.i. complete classes of lattices of isometries of symmetric spaces other than those discussed above. All the lattices that we consider are supposed to be *irreducible*.

1. uniform lattices of isometries of a symmetric space X for the list of spaces X below.
   - $X = \mathbb{H}^n_{\mathbb{R}}$, with $n \geq 3$, by the work of Sullivan and Tukia (see the lecture of Marc Bourdon and references therein);
   - $X = \mathbb{H}^n_{\mathbb{H}}$ and $X = \mathbb{H}^2_{Cay}$ by the work of P. Pansu [Pan_2];
   - $X = \mathbb{H}^n_{\mathbb{C}}, n \geq 2$, by the work of R. Chow [Chow];
   - $X = \mathbb{H}^2_{\mathbb{R}}$. In this case a proof of the q.i. completeness goes as follows:
     - A finitely generated group Λ quasi-isometric to $\mathbb{H}^2_{\mathbb{R}}$ is a hyperbolic group (see Example 6 below for a definition), with boundary at infinity homeomorphic to $\mathbb{S}^1$. Every hyperbolic group acts on its boundary as a convergence group [Tu_2]. For a definition of convergence groups see Section 8.
     - Every convergence group is conjugate to a Fuchsian group in $\text{Homeo}(\mathbb{S}^1)$. This follows from [$\text{Tu}_1$], [CJ] and [Ga].
   - X irreducible symmetric space of rank at least 2. This result is due to B. Kleiner and B. Leeb [KlL]. See also [EF] for another proof.
2. non-uniform lattices of isometries of a symmetric space X of rank at least 2. This is due to R. Schwartz for a family of $\mathbb{Q}$–rank one lattices containing the Hilbert modular groups [Sch_2] and to A. Eskin [E] in the general case, under the condition that X has no factors of rank 1. See also [Dr_2] for another proof of the general case.

   Moreover, in this case Statement (1), (b), of Theorem 2.1 holds as well, that is: two non-uniform lattices are quasi-isometric if and only if they are commensurable.

   Remark 3.2. In the cases of uniform lattices in $X = \mathbb{H}^n_{\mathbb{H}}$ and $X = \mathbb{H}^2_{Cay}$ or X of rank at least two, as well as in the case of non-uniform lattices of a symmetric space X of rank at least 2, the q.i. completeness result is obtained via a rigidity result similar to Theorem 2.3, that is : a quasi-isometry of the lattice is at bounded distance from an isometry. Moreover, in the case of non-uniform lattices, this isometry is in the commensurator of the lattice.

3. fundamental groups of non-geometric Haken manifolds with zero Euler characteristic [KaL$_2$] (see the lecture of M. Kapovich).

4. finitely presented groups [GH$_1$].

5. nilpotent groups. This follows from the Polynomial Growth Theorem of M. Gromov [Gr$_3$]. We recall that the growth function $B_S : \mathbb{N} \to \mathbb{N}$ of a group G with a finite set of generators S is defined by $B_S(n) =$ the cardinal of the ball $B_S(1, n)$ in the word metric dist_S. The theorem of M. Gromov states that the growth function with respect to some (hence any) finite set of generators is polynomial if and only if the group is virtually nilpotent.

   The subclass of Abelian groups is also q.i. complete, as follows from results in [Pan$_1$]. See the discussion in Section 4.3.

6. hyperbolic groups.

   We recall that a geodesic metric space is called *δ-hyperbolic* if in every geodesic triangle, each edge is contained in the δ-tubular neighbourhood of the union of the other two edges. If $\delta = 0$ the space is called *real tree* or *$\mathbb{R}$–tree*.

   A finitely generated group is called *hyperbolic* if its Cayley graph is hyperbolic. For instance, uniform lattices in rank one symmetric spaces are such.

   The q.i. completeness of the class of hyperbolic groups follows easily from the definition and from

   Lemma 3.3 (Morse lemma, see [GH$_2$]). *Every (L, C)–quasi-isometric segment in a δ-hyperbolic space is at Hausdorff distance at most D from the geodesic segment joining its endpoints, where $D = D(L, C, \delta)$.*

7. amenable groups [GH$_1$]. We recall that a discrete group G is *amenable* if for every finite subset K of G and every $\epsilon \in (0, 1)$ there exists a finite subset $F \subset G$ satisfying:

$$\mathrm{card}\, KF < (1 + \epsilon)\mathrm{card}\, F\,.$$

8. the whole class of solvable groups is not q.i. complete, as pointed out by the counter-example in [Dyu]. Still, there are some smaller classes of solvable groups that are q.i. complete. See for instance [FM$_1$], [FM$_2$].

3.2 Relatively hyperbolic groups: preliminaries

In the same way in which uniform lattices in rank one symmetric spaces inspired the notion of hyperbolic group, non-uniform lattices inspired the notion of *relatively hyperbolic group*. This notion was defined by M. Gromov in [Gr$_2$], then several equivalent definitions of the same notion were provided by B. Bowditch, B. Farb, D. Osin, A. Yaman.[2]

Here we recall the definition of B. Farb [Fa]. Let G be a finitely generated group and let $\{H_1, \ldots, H_m\}$ be a collection of subgroups of G. Let S be a finite generating set of G invariant with respect to inversion. The idea is to write down a list of properties which force G to behave with respect to $\{H_1, \ldots, H_m\}$ in the same way in which a non-uniform lattice Γ behaves with respect to its cusp subgroups $\{\Gamma_{\alpha_1}, \ldots, \Gamma_{\alpha_m}\}$ (see Figure 4).

[2] What is called in this paper relatively hyperbolic group is sometimes called in the litterature *strongly relatively hyperbolic group*, in contrast with *weakly relatively hyperbolic group*.

We denote by $\mathcal{H}$ the set $\bigsqcup_{i=1}^{m}(H_i \setminus \{1\})$. We can consider two Cayley graphs for the group G, Cayley(G, S) and Cayley$(G,\ S \cup \mathcal{H})$. We note that Cayley(G, S) is a subgraph of Cayley$(G,\ S \cup \mathcal{H})$, with the same set of vertices but a smaller set of edges and that Cayley$(G,\ S \cup \mathcal{H})$ is not locally finite. We have that $\mathrm{dist}_{S\cup\mathcal{H}}(u, v) \leq \mathrm{dist}_S(u, v)$, for every two vertices u, v.

Definition 3.4. Let $\mathfrak{p}$ be a path in Cayley$(G,\ S \cup \mathcal{H})$. An *$\mathcal{H}$–component* of $\mathfrak{p}$ is a maximal sub-path of $\mathfrak{p}$ contained in a left coset gH_i, $i \in \{1, 2, \ldots, m\}$, $g \in G$.

The path $\mathfrak{p}$ is said to be *without backtracking* if it does not have two distinct $\mathcal{H}$–components in the same left coset gH_i.

The notion of weak relative hyperbolicity has been introduced by B. Farb in [Fa]. We use a slightly different but equivalent definition. The proof of the equivalence can be found in [Os].

Definition 3.5. The group G is *weakly hyperbolic relative to* $\{H_1, \ldots, H_m\}$ if and only if the graph Cayley$(G,\ S \cup \mathcal{H})$ is hyperbolic.

This property is not enough to determine a picture as in Figure 4. For instance $G = \mathbb{Z}^2$ satisfies the previous property with respect to its subgroup $H = \mathbb{Z} \times \{0\}$. Or this case does not at all look as in Figure 4, in that the tubular neighbourhoods of left cosets of H do not, as in Figure 4, intersect in a finite set, but on the contrary the intersection may contain both left cosets. Vaguely speaking, in Figure 4 the left cosets stay close in the respective neighbourhoods of a pair of points realizing the minimal distance, and then diverge, while in the example above two left coset stay parallel.

One has to add a second property in order to obtain the proper image, and thus to define (strong) relative hyperbolicity. Before formulating this property, we must mention another notable example of group weakly relatively hyperbolic and not strongly relatively hyperbolic. The Mapping Class Group of a hyperbolic surface Σ (also defined as $Out(\pi_1(\Sigma))$) is weakly hyperbolic (and *not* strongly hyperbolic) relative to finitely many stabilizers of closed geodesics on the surface Σ. The weak relative hyperbolicity follows from [MM] (see also [Bow$_3$]). The reason for which (strong) relative hyperbolicity is not satisfied is again that the intersection of two tubular neighbourhoods of left cosets is not finite. Indeed, two stabilizers of two closed geodesics intersect in the stabilizer of both, which can be itself infinite.

Notation: For every path $\mathfrak{p}$ in a metric space X, we denote the start of $\mathfrak{p}$ by $\mathfrak{p}_-$ and the end of $\mathfrak{p}$ by $\mathfrak{p}_+$.

Definition 3.6. The pair $(G,\ \{H_1, \ldots, H_m\})$ satisfies the *Bounded Coset Penetration (BCP) property* if for every $\lambda \geq 1$ there exists $a = a(\lambda)$ such that the following holds. Let $\mathfrak{p}$ and $\mathfrak{q}$ be two λ-bi-Lipschitz paths without backtracking in Cayley$(G,\ S \cup \mathcal{H})$ such that $\mathfrak{p}_- = \mathfrak{q}_-$ and $\mathrm{dist}_S(\mathfrak{p}_+, \mathfrak{q}_+) \leq 1$.

(1) Suppose that s is an $\mathcal{H}$–component of $\mathfrak{p}$ such that $\mathrm{dist}_S(s_-, s_+) \geq a$. Then $\mathfrak{q}$ has an $\mathcal{H}$–component contained in the same left coset as s;

(2) Suppose that s and t are two $\mathcal{H}$–components of $\mathfrak{p}$ and $\mathfrak{q}$, respectively, contained in the same left coset. Then $\mathrm{dist}_S(s_-, t_-) \leq a$ and $\mathrm{dist}_S(s_+, t_+) \leq a$.

In particular BCP property implies that if H_i is infinite two left cosets of H_i cannot be at finite Hausdorff distance one from the other. The proof is left as an exercise to the reader.

Definition 3.7. The group G is *(strongly) hyperbolic relative to* $\{H_1, \ldots, H_m\}$ if it is weakly hyperbolic relative to $\{H_1, \ldots, H_m\}$ and if $(G,\ \{H_1, \ldots, H_m\})$ satisfies the BCP property.

Both in the case of weak and strong relative hyperbolicity, the subgroups $H_1, \ldots, H_m$ are called *peripheral subgroups*. A subgroup conjugate to some H_i, $i \in \{1, ..., m\}$, is called a *maximal parabolic subgroup*. A subgroup contained in a maximal parabolic subgroup is called *parabolic*.

Other examples of relatively hyperbolic groups (besides non-uniform lattices):

1. $A *_F B$, where F is finite, is hyperbolic relative to A and B; more generally, fundamental groups of finite graphs of groups with finite edge groups are hyperbolic relative to the vertex groups [Bow$_1$].

2. a hyperbolic group Γ is hyperbolic relative to

   - $H = \{1\}$;
   - any class of quasi-convex subgroups $\{H_1, \ldots, H_k\}$ with the property that each H_i is its own normalizer in G and that $H_i^g \cap H_j$ is finite if $i \neq j$ or $g \notin H_i$.

     For instance let Γ be a uniform lattice of isometries of $\mathbb{H}^3_{\mathbb{R}}$ such that for some totally geodesic copy of the hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$ in $\mathbb{H}^3_{\mathbb{R}}$, $H = \Gamma \cap Isom(\mathbb{H}^2_{\mathbb{R}})$ is a uniform lattice of $\mathbb{H}^2_{\mathbb{R}}$. Then $\{H\}$ satisfies the previous properties.

3. fundamental groups of complete finite volume manifolds of pinched negative sectional curvature are hyperbolic relative to the fundamental groups of their cusps ([Bow$_1$], [Fa]);

4. fundamental groups of (non-geometric) Haken manifolds with at least one hyperbolic component are hyperbolic relative to fundamental groups of maximal graph-manifold components and to fundamental groups of tori and Klein bottles not contained in a graph-manifold component;

5. fully residually free groups, also known as limit groups, are hyperbolic relative to their maximal Abelian non-cyclic subgroups [Dah$_1$]. Moreover, according to [AB] these groups are known to be $CAT(0)$ with isolated flats, after the terminology in [Hr];

6. more generally, finitely generated groups acting freely on $\mathbb{R}^n$–trees are hyperbolic relative to their maximal Abelian non-cyclic subgroups [Gui].

Question 3.8. Is the class of relatively hyperbolic groups q.i. complete ?

Remark 3.9. Throughout the discussion of relatively hyperbolic groups we tacitly rule out the case of a group hyperbolic relative to itself, as well as the case of a finite group F_1 hyperbolic relative to any class of subgroups.

Thus, we are in the case of a group G and a finite collection (possibly reduced to one element) $\{H_1, \ldots, H_m\} \neq \{G\}$ of infinite subgroups of G. In this case it follows that each subgroup H_i has infinite index in G.

Before discussing Question 3.8, we define our main tools.

4 Asymptotic cones of a metric space

4.1 Definition, preliminaries

The notion of asymptotic cone was defined in an informal way in [Gr_3], and then rigorously in [VDW] and [Gr_4]. The idea is to construct, for a given metric space, an image of it seen from infinitely far away.

First one needs the notion of *non-principal ultrafilter*. This can be defined as a finitely additive measure ω defined on the set of all subsets of $\mathbb{N}$ (or, more generally, of a countable set) and taking only values zero and one, such that on all finite subsets it takes value zero. In particular if $\mathbb{N} = A_1 \sqcup \cdots \sqcup A_n$ and all A_i are infinite, then there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $\omega(A_{i_0}) = 1$ and $\omega(A_j) = 0$ for every $j \neq i_0$.

The fact that ω takes only values 0 and 1 immediately brings to one's mind the idea of a characteristic function. Indeed, ω satisfies the previous properties if and only if it is the characteristic function $\mathbf{1}_{\mathcal{U}}$ of a collection $\mathcal{U}$ of subsets of $\mathbb{N}$ which is

- an *ultrafilter*, that is, a maximal filter;
- moreover *nonprincipal*, that is containing *the Fréchet filter*.

For definitions of the notions above, that is, for a list of axioms, see Section 8. For more details see [Bou]. The main advantage of the second way of defining non-principal ultrafilters resides, besides the questionable pleasure of dealing with axioms, in the fact that it shows that such objects always exist. We also note that a functional analytic treatment of ultrafilters is possible. Thus, the notion of ω–limit can be seen as an application of the Hahn-Banach theorem to the space of relatively compact sequences, the subspace of convergent sequences and the limit map on it.

Since all ultrafilters in this paper are nonprincipal, we drop this adjective henceforth.

Given an ultrafilter ω and a sequence (x_n) in a topological space, one can define the *ω–limit* $\lim_\omega x_n$ of the sequence as an element x such that for every neighbourhood $\mathcal{N}$ of it,

$$\omega\left(\{n \in \mathbb{N} \mid x_n \in \mathcal{N}\}\right) = 1\,.$$

The following property emphasizes the main interest of (nonprincipal) ultrafilters.

Proposition 4.1. *[Bou] If (x_n) is contained in a compact, its ω–limit always exists.*

Note that, as soon as it exists, the ω–limit is also unique. Also, it is not difficult to see that it is a limit of a converging subsequence. Thus, an ultrafilter is a device to select a point of accumulation for any relatively compact sequence, in a coherent manner. In some sense, it is a systematic approach to the process of taking the diagonal subsequence, after selecting converging subsequences in countably many sequences.

With such a tool at hand, which makes almost any reasonable sequence converge, one can hope to define, for a given metric space (X, dist), an image of it seen from infinitely far away. More precisely, one has to take a sequence of positive numbers d_n diverging to infinity, and try to construct a limit of the sequence of metric spaces $\left(X, \frac{1}{d_n}\mathrm{dist}\right)$.

As in the formal construction of completion, one can simply take the set $\mathcal{S}(X)$ of all sequences (x_n) in X and try to define a metric on this space by

$$\mathrm{dist}_\omega\,(x, y) = \lim_\omega \frac{\mathrm{dist}(x_n, y_n)}{d_n}, \text{ for } x = (x_n),\, y = (y_n)\,.$$

The problem is that the latter limit can be $+\infty$, or it can be zero for two distinct sequences.

To avoid the situation $\mathrm{dist}_\omega(x,y) = +\infty$, one restricts to a subset of sequences defined as follows. For a fixed sequence $e = (e_n)$, consider

$$\mathcal{S}_e(X) = \left\{ (x_n) \in X^{\mathbb{N}} \,;\, \left(\frac{\mathrm{dist}(x_n, e_n)}{d_n} \right) \text{ is a bounded sequence} \right\} . \tag{5}$$

To deal with the situation when $\mathrm{dist}_\omega(x,y) = 0$ while $x \neq y$, one uses the classical trick of taking the quotient for the equivalence relation

$$x \sim y \Leftrightarrow \mathrm{dist}_\omega(x,y) = 0 .$$

The quotient space $\mathcal{S}_e(X)/\sim$ is denoted $\mathrm{Con}_\omega(X; e, d)$ and it is called *the asymptotic cone of X with respect to the ultrafilter ω, the scaling sequence $d = (d_n)$ and the sequence of observation centers e.*

A sequence of subsets (A_n) in X gives rise to a *limit subset* in the cone, defined by

$$\lim_\omega (A_n) = \{ \lim_\omega (a_n) \mid a_n \in A_n, \forall\, n \in \mathbb{N} \} .$$

If $\lim_\omega \frac{\mathrm{dist}(e_n, A_n)}{d_n} = +\infty$ then $\lim_\omega (A_n) = \emptyset$.

Properties of asymptotic cones:

1. $\mathrm{Con}_\omega(X; e, d)$ is a complete metric space;
2. every limit subset $\lim_\omega(A_n)$, if non-empty, is closed;
3. if X is geodesic then every asymptotic cone $\mathrm{Con}_\omega(X; e, d)$ is geodesic;
4. an (L, C)–quasi-isometry between two metric spaces $\mathfrak{q} : X \to Y$ gives rise to a bi-Lipschitz map between asymptotic cones

$$\begin{array}{rcl} \mathfrak{q}_\omega : \mathrm{Con}_\omega(X; e, d) & \to & \mathrm{Con}_\omega(Y; \mathfrak{q}(e), d) \\ \lim_\omega(x_n) & \to & \lim_\omega(\mathfrak{q}(x_n)) \, ; \end{array}$$

5. If G is a group then every $\mathrm{Con}_\omega(G; e, d)$ is isometric to $\mathrm{Con}_\omega(G; 1, d)$, where 1 denotes here the constant sequence equal to 1;
6. $\mathrm{Con}_\omega(G; 1, d)$ is a homogeneous space.

Proofs of the previous properties can be found in [Gr$_4$], [KlL], [KaL$_1$]. None of them is difficult; they are good exercises in order to get familiar with the notion. We shall take a closer look only at the last property, namely we shall exhibit the group acting transitively by isometries on $\mathrm{Con}_\omega(G; 1, d)$. Let $G^{\mathbb{N}}$ be the set of all sequences in G and let $\mathcal{S}_{(1)}(G)$ be the subset of sequences defined as in (5). We consider the equivalence relation

$$(g_n) \approx (g'_n) \Leftrightarrow \omega(\{n \in \mathbb{N} \mid g_n = g'_n\}) = 1 \, .$$

The quotient space $\Pi_\omega G = G^{\mathbb{N}} / \approx$ is a group, called the *ω–ultrapower of G*. The subgroup $G^\omega = \mathcal{S}_{(1)}(G) / \approx$ acts transitively by isometries on $\mathrm{Con}_\omega(G; 1, d)$ by:

$$(g_n)^\omega \lim_\omega(x_n) = \lim_\omega(g_n x_n) \, .$$

One can put a condition in order to restrict the growth of the scaling sequence with respect to the ultrafilter. The idea is to choose a sequence d and an ultrafilter ω such that there is no set E with $\omega(E) = 1$ and such that $(d_n)_{n \in E}$ grows faster than exponentially. The definition is as follows:

Definition 4.2. The pair (ω, d) is *non-sparse* if:

(i) For every $a > 1$ we have $d_n \leq a^n$ ω—almost surely.

(ii) For every ordered infinite subset $E = \{i_1, i_2, \dots, i_n, \dots\}$ such that there exists $a > 1$ satisfying $\lim_{n \to \infty} \frac{d_{i_n}}{a^n} = +\infty$, we have that $\omega(E) = 0$.

In one of the properties (i) and (ii) is not satisfied, we say that the pair (ω, d) is *sparse*.

Both sparse and non-sparse pairs exist. In order to construct a sparse pair, it suffices to take an ultrafilter ω, and via the injection $n \mapsto 3^n$ to identify it to an ultrafilter supported by the set $\{3^n \, ; \, n \in \mathbb{N}\}$.

To construct an non-sparse pair, take for instance $d_n = n$ and all the sets E described in (ii), for this choice. The collection of subsets of $\mathbb{N}$ having complementaries in $\mathbb{N}$ either finite or contained in a set of type E is a filter. An ultrafilter $\mathcal{U}$ containing it is non-principal and the pair $(\mathbf{1}_\mathcal{U} \, , \, (n))$ is non-sparse.

4.2 A sample of what one can do using asymptotic cones

Proposition 4.3 ([Gr$_4$]). *Let Γ be a discrete group endowed with a metric* dist*, left invariant with respect to the action of the group on itself, such that all balls are finite.*

(1) If all the asymptotic cones of (Γ, dist) are path-connected then Γ is finitely generated.

(2) If moreover all the asymptotic cones of (Γ, dist) are simply connected then Γ is finitely presented.

Remark 4.4. (a) A metric as in Proposition 4.3, (1), can be obtained for instance if the group acts properly discontinuously and freely by isometries on a proper metric space X. Given $x \in X$ we identify Γ with the orbit Γx and we take the induced metric.

A particular case of the previous situation is when Γ is a subgroup of a finitely generated group G. Then we can take as X the Cayley graph of G. If we choose $x = 1$, the induced metric dist is the word metric of G restricted to Γ.

(b) The reciprocal of Proposition 4.3, (2), is not true. This can be seen for instance in the case of Baumslag-Solitar groups $BS(p, q)$ or in the case of uniform lattices in the solvable group Sol. These groups are finitely presented, nevertheless their asymptotic cones have uncountable fundamental group [Bu].

Proof. **(1)** Step 1. We first prove that between every pair of elements x, y in Γ there exists a discrete path composed of at most N steps of length at most $\frac{\mathrm{dist}(x,y)}{M}$, where both M and N are fixed.

Here is the precise statement with quantifiers: for every $M > 1$ there exist $N \in \mathbb{N}$, $N \geq 2$, and $D > 0$ such that for every $x, y \in \Gamma$ with $\mathrm{dist}(x, y) \geq D$, there exists a finite sequence of points $t_0 = x, t_1, \ldots, t_m = y$ with $m \leq N$ and $\mathrm{dist}(t_i, t_{i+1}) \leq \frac{\mathrm{dist}(x,y)}{M}$ for every $i \in \{0, 1, \ldots, m-1\}$.

We argue by contradiction and suppose it is not the case. Then there exists M and a sequence of pairs of points $(x_n, y_n) \in \Gamma \times \Gamma$ with $d_n = \mathrm{dist}(x_n, y_n) \geq n$ and such that every discrete path of at most n steps between x_n and y_n has at least one step of length $\geq \frac{\mathrm{dist}(x_n,y_n)}{M}$. In the asymptotic cone $\mathrm{Con}_\omega(\Gamma; x_n, d_n)$ the two sequences (x_n) and (y_n) give two points x_ω and y_ω at distance 1 such that for every n, every discrete path joining x_ω and y_ω and having n steps has at least one step of length $\geq \frac{1}{M}$. On the other hand, since $\mathrm{Con}_\omega(\Gamma; x_n, d_n)$ is path-connected, x_ω and y_ω can be joined by a path. On this path can be chosen a finite discrete path of steps at most $\frac{1}{M}$ between x_ω and y_ω. Thus we obtain a contradiction.

Step 2. By iterating the result obtained in Step 1, one can deduce that every pair of points in Γ can be joined by a discrete path of step at most D. Now it suffices to take the finite set of all the elements in Γ at distance at most D from 1. By the previous statement, this is a set of generators in Γ.

(2) The proof is in the same spirit as for (1). In (1) we dealt with pairs of points and we had to "fill the space between them" with a discrete path composed of steps of bounded length. Then all the possibilities for the steps gave the finite set of generators.

We can see a pair of points as an image of the zero dimensional sphere $\mathbb{S}^0$. If we go one dimension up, instead of pairs of points we shall have loops in the Cayley graph. These are nothing else than all relations in the group. To show that Γ is finitely presented means to show that an arbitrary such loop/relation can be "filled" with loops of uniformly bounded length. "Filled" means here that by putting one next to the other a set of loops of uniformly bounded length one obtains a diagram having as boundary the initial arbitrary loop. Then the (finite) set of loops of bounded length gives the set of relations in the finite presentation.

The proof is just an appropriate transcription of the proof of (1). See [Gr_4] and [Dr_1] for details. □

Remark 4.5. Simple connectedness of asymptotic cones implies much more than the conclusion of Proposition 4.3: it implies that the group Γ has polynomial Dehn function and linear filling radius. See [Dr_1] and references therein.

4.3 Examples of asymptotic cones of groups

All the groups considered below are finitely generated.

(1) A group is virtually nilpotent if and only if all its asymptotic cones are locally compact ([Gr_3], [Dr_1]).

Morality: Outside the class of virtually nilpotent groups one should not expect the asymptotic cones to be locally compact.

Moreover, in this case it was proved by P. Pansu in [Pan_1] that all asymptotic cones are isometric to a graded Lie group canonically associated to G, as follows. Let $\mathrm{tor}(G)$ be the finite normal subgroup of G generated by elements of finite order. The nilpotent group $\bar{G} = G/\mathrm{tor}(G)$ is without torsion, hence it can be embedded, according to [Mal], as a uniform lattice in a nilpotent Lie group. To this Lie group one canonically associates a graded Lie group, and it is this graded Lie group endowed with a Carnot-Caratheodory metric that is isometric to all asymptotic cones.

If two virtually nilpotent groups are quasi-isometric, the graded Lie groups associated to them are not only bi-Lipschitz equivalent as usually for asymptotic cones, but moreover isomorphic. This points out new quasi-isometry invariants: the degree of nilpotency of $\bar{G} = G/\mathrm{tor}(G)$ and the rank of each of the Abelian groups $\bar{G}^i/\bar{G}^{i+1}$, where $\bar{G}^i$ is the i-th group in the lower central series of $\bar{G}$.

In particular, if a group G is quasi-isometric to an Abelian group, then G itself is virtually Abelian.

(2) A group is hyperbolic if and only if all its asymptotic cones are real trees ([Gr_3], [Dr_1]).

Moreover, all asymptotic cones are isometric to a $2^{\aleph_0}$–universal real tree [DP].

Remarks 4.6.

- In the "if" part of the previous two statements as well as in every similar statement in this paper, it is enough to take all asymptotic cones for a *fixed* ultrafilter.
- Note that Proposition 4.3, (2), implies that hyperbolic groups are finitely presented. This can also be obtained directly from the definition, and in fact much more is known: the Dehn function of hyperbolic groups is linear [Gr_2].

(3) Let G be a uniform lattice of isometries of a symmetric space or Euclidean building of rank at least 2. Every asymptotic cone $\mathrm{Con}_\omega(G; 1, d)$ is a (non-discrete) Euclidean building [KlL].

As for the question whether in this case all asymptotic cones are isometric or not, it turns out to be related to the Continuum Hypothesis (the hypothesis stating that there is no cardinal number between $\aleph_0$ and $2^{\aleph_0}$).

Using a description of asymptotic cones in terms of fields and valuations (a similar description has been obtained independently by B. Leeb and A. Parreau [Par]), Kramer, Shelah, Tent and Thomas have shown in [KSTT] that:

- if the Continuum Hypothesis (CH) is not true then any uniform lattice in $SL(n, \mathbb{R})$, $n \geq 3$, has $2^{2^{\aleph_0}}$ non-isometric asymptotic cones;

- if the CH is true then all asymptotic cones of a uniform lattice in $SL(n,\mathbb{R})$, $n \geq 3$, are isometric. Moreover, a finitely generated group has at most a continuum of non-isometric asymptotic cones.

(4) In [DS$_1$] can be found an example of two-generated (and recursively presented - but not finitely presented) group with continuously many non-homeomorphic asymptotic cones. The construction is independent of CH.

Question 4.7. Can one characterize relatively hyperbolic groups also in terms of asymptotic cones ?

An answer to this question would give a better idea of how relatively hyperbolic groups look like and also it might serve to prove some rigidity result about relatively hyperbolic groups. For instance, in the rigidity result of B. Kleiner and B.Leeb [KlL] the main ingredient is the description of asymptotic cones of uniform lattices.

5 Relatively hyperbolic groups: image from infinitely far away and rigidity

5.1 Tree-graded spaces and cut-points

Definition 5.1 (tree-graded spaces). Let $\mathbb{F}$ be a complete geodesic metric space and let $\mathcal{P}$ be a collection of closed geodesic subsets of $\mathbb{F}$ (called *pieces*) such that the following two properties are satisfied:

(T_1) Every two different pieces have at most one common point.

(T_2) Every simple geodesic triangle (a simple loop composed of three geodesics) in $\mathbb{F}$ is contained in one piece.

Then we say that the space $\mathbb{F}$ is *tree-graded with respect to* $\mathcal{P}$.

Property (T_2) can be replaced by one of the following properties:

(T_2') For every topological arc $\mathfrak{c} : [0,d] \to \mathbb{F}$ and $t \in [0,d]$, let $\mathfrak{c}[t-a, t+b]$ be a maximal sub-arc of $\mathfrak{c}$ containing $\mathfrak{c}(t)$ and contained in one piece. Then every other topological arc with the same endpoints as $\mathfrak{c}$ must contain the points $\mathfrak{c}(t-a)$ and $\mathfrak{c}(t+b)$.

(T_2'') Every simple loop in $\mathbb{F}$ is contained in one piece.

Remark 5.2 ([DS$_1$]). If one replaces property (T_2) by the stronger property (T_2'') in the definition of a tree-graded space then one can weaken the condition on $\mathcal{P}$ and ask only that each set in $\mathcal{P}$ is path-connected.

The structure of tree-graded space appears naturally as soon as a space has a cut-point, as shown by the following result.

Proposition 5.3 ([DS$_1$], Section §2.4). *Let X be a complete geodesic metric space containing at least two points and let $\mathcal{C}$ be a non-empty set of global cut-points in X.*

(a) There exists a largest (in an appropriate sense) collection $\mathcal{P}$ of subsets of X such that X is tree-graded with respect to $\mathcal{P}$ and such that any piece in $\mathcal{P}$ is either a singleton or a set with no global cut-point in $\mathcal{C}$.

Moreover the intersection of any two distinct pieces from $\mathcal{P}$ is either empty or a point in $\mathcal{C}$.

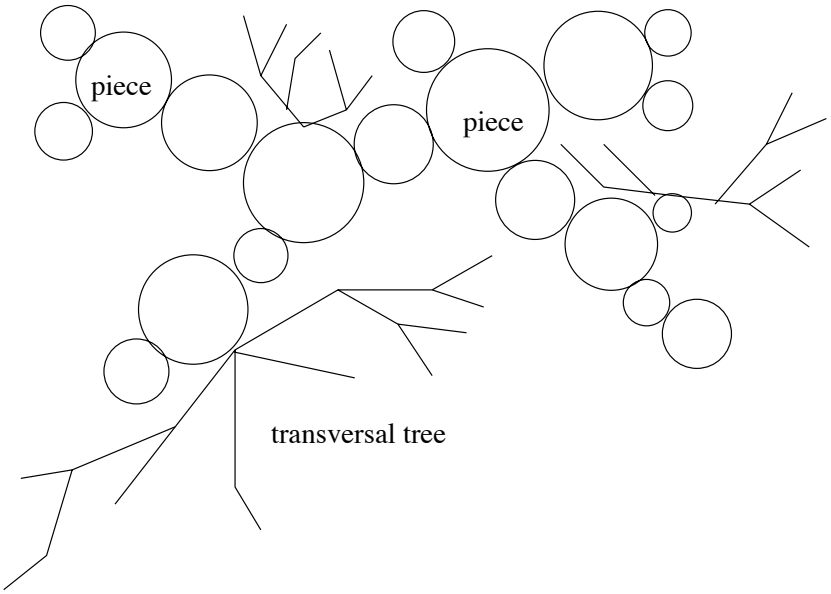

Figure 6: A tree-graded space.

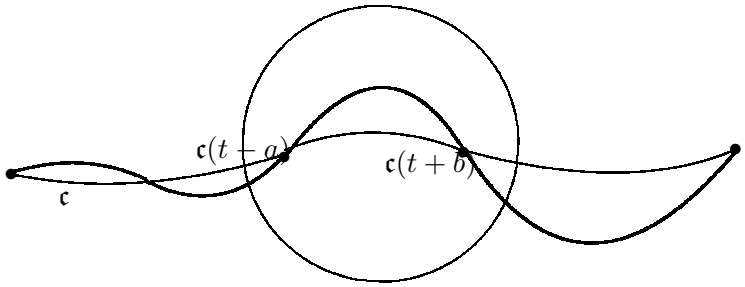

Figure 7: Property (T_2').

(b) If $\mathcal{C} = X$ then all pieces in $\mathcal{P}$ are either singletons or sets without cut-point. In particular this is true if X is a homogeneous space with a cut-point.

We should point out here that a systematic approach to homogeneous spaces having cut-points had already been done by B. Bowditch when working at the Bestvina-Mess conjecture (stating that the boundary at infinity of a one-ended hyperbolic group has no global cut-point). Thus, he considered a topological space with a cut-point and such that the group of its homeomorphisms acts transitively on it (having in mind the boundary at infinity of a group) and he showed that it projects on a tree. For details see [Bow$_2$].

Properties of tree-graded spaces:

1. If all the pieces are real trees then $\mathbb{F}$ is a real tree.

2. For every $x \in \mathbb{F}$ we define the set $\mathcal{T}_x$ to be the set of points $y \in \mathbb{F}$ which can be joined to x by a topological arc intersecting each piece in at most a point. For every x the set $\mathcal{T}_x$ is a real tree and a closed subset in $\mathbb{F}$. For every $y \in \mathcal{T}_x$, $\mathcal{T}_y = \mathcal{T}_x$. We call such a tree *transversal tree*.

3. Every point of intersection of two distinct pieces as well as every point in a non-trivial transversal tree is a cut-point for $\mathbb{F}$.

4. Every path-connected subset *without cut-points* is contained in a piece.

5. For every point outside a piece M there exists a unique point on the piece minimizing the distance. This allows to define a projection map from $\mathbb{F}$ to M.

6. Every path-connected subset intersecting a piece M in at most one point projects onto the piece M in a unique point.

7. Suppose that there exists $\epsilon > 0$ such that every loop of length at most ϵ and contained in a piece is contractible. Then $\pi_1(\mathbb{F})$ coincides with the free product of the fundamental groups of the pieces, $*_{M\in\mathcal{P}}\pi_1(M)$.

   We note that the hypothesis that short loops are contractible in any piece is a necessary condition, as shown by the example of the Hawaiian earring.

8. Let $(\mathbb{F}, \mathcal{P})$ be a tree-graded space. If ϕ is a homeomorphism from $\mathbb{F}$ to another geodesic metric space X, then X is tree-graded with respect to the collection of pieces $\{\phi(M) \mid M \in \mathcal{P}\}$.

9. Let $(\mathbb{F}, \mathcal{P})$ and $(\mathbb{F}', \mathcal{P}')$ be two tree-graded spaces with all pieces without cut-points. Every homeomorphism $\phi : \mathbb{F} \to \mathbb{F}'$ sends each piece onto a piece.

For proofs of these properties and other properties of tree-graded spaces see [DS$_1$].

5.2 The characterization of relatively hyperbolic groups in terms of asymptotic cones

Theorem 5.4 (Druţu-Osin-Sapir[DS$_1$]). *A finitely generated group G is hyperbolic relatively to a finite family $\{H_1, ..., H_n\}$ of finitely generated subgroups if and only if every asymptotic cone $\mathrm{Con}_\omega(G; 1, d)$ is tree-graded with respect to the collection of pieces*

$$\mathcal{P} = \left\{ \lim_\omega (g_n H_i) \mid (g_n) \text{ sequence in } G, i \in \{1, 2, \dots, n\} \right\} .$$

The "only if" part is proven by D. Osin and M. Sapir in the Appendix of [DS$_1$]. The "if" part is proven in [DS$_1$]. Note that for the "if" part one does not need to ask that the peripheral subgroups are finitely generated. It follows immediately from the fact that the limit sets of their left cosets, which are isometric to their asymptotic cones with the metric induced from G, are geodesic, since they are pieces in a tree-graded space. It remains to apply Proposition 4.3, (1).

Also, Proposition 4.3, (2), and property 7 of tree-graded spaces implies that if H_i all have simply connected asymptotic cones then G is finitely presented. On the other hand, from the equivalent definition of relative hyperbolicity given by D. Osin in [Os] it follows that the same is true if all H_i are finitely presented (which is a weaker hypothesis than the previous).

In particular Theorem 5.4 is true for $G = \Gamma$ a non-uniform lattice in rank one and $\{\Gamma_{\alpha_1}, \dots, \Gamma_{\alpha_m}\}$ its cusp subgroups. Thus the image of the space X_0 in Figure 4 seen from infinitely far away is a homogeneous version of the Figure 6. It is not difficult to show that for this particular tree-graded space each transversal tree is in fact a $2^{\aleph_0}$–universal real tree.

A straightforward consequence of Theorem 5.4 is the following.

Corollary 5.5. *If a group G is hyperbolic relative to $\{H_1, \dots, H_m\}$ and if each H_i is hyperbolic relative to a collection of subgroups $\{H_i^1, \dots, H_i^{n_i}\}$ then G is hyperbolic relative to $\{H_i^j \mid i \in \{1, \dots, m\}, j \in \{1, \dots, n_i\}\}$.*

Remark 5.6. This process may not terminate: for instance if G is a free group and $H = \langle h \rangle$ is a cyclic subgroup, one can consider $H_n = \langle h^{2^n} \rangle$, G is hyperbolic relative to $\{H_n\}$ and H_n is hyperbolic relative to $\{H_{n+1}\}$. Still in this situation there is a terminal point: G hyperbolic relative to $\{1\}$.

In general a terminal point would be a family $\{H_1, \dots, H_m\}$ of subgroups relative to which the ambient group G is hyperbolic and such that no H_i is relatively hyperbolic. Such a family might not exist for an arbitrary relatively hyperbolic group. The example of inaccessible group constructed by Dunwoody in [Du_2] is an example of such a group (the argument showing it can be found in [BDM]). See Question 7.5 in the Open questions section.

Theorem 5.4 and properties (2) and (9) of tree-graded spaces suggest that the "good objects" for a rigidity theory for relatively hyperbolic groups are the finitely generated groups such that all their asymptotic cones are without cut-points. We call such groups *asymptotically without cut-points*. To avoid trivial cases and different technical complications we also assume that finite groups are not asymptotically without cut-points.

Remark 5.7. A group asymptotically without cut-points is one-ended. This follows from Stallings' Ends Theorem stating that a finitely generated group splits as a free product or HNN-extension with finite amalgamation if and only if it has more than one end [Sta]. The converse is not true: the asymptotic cones of any hyperbolic group are $\mathbb{R}$–trees, and there are one-ended hyperbolic groups (uniform lattices in $\mathbb{H}^3_{\mathbb{R}}$ for instance).

Examples of groups asymptotically without cut-points:

1. uniform lattices in symmetric spaces/Euclidean buildings of rank at least two; this is because their asymptotic cones are non-discrete Euclidean buildings of rank at least two, and these do not have cut-points [KlL];

2. groups with elements of infinite order in the center, not virtually cyclic [DS_1];

3. groups satisfying an identity (a law), not virtually cyclic [DS_1].

   We recall what *satisfying an identity (a law)* means for a group. Let $w(x_1, \dots, x_n)$ be a non-trivial reduced word in the n letters $x_1, \dots, x_n$ and their inverses. *Reduced* means that all sequences of type $x\, x^{-1}$ are deleted. The group G satisfies the identity $w(x_1, \dots, x_n) = 1$ if the equality is satisfied in G whenever replacing $x_1, \dots, x_n$ with arbitrary elements in G.

   Examples of such groups:

   - Abelian groups: here $w = x_1 x_2 x_1^{-1} x_2^{-1}$;
   - more generally solvable groups of class at most $m \in \mathbb{N}$;
   - free Burnside groups. We recall that the free Burnside group $B(n, m)$ is the group with n generators satisfying the identity $x^m = 1$ and all the relations that can be obtained from this identity (and no other). A rigorous way to define it is to say that it is the quotient of $\mathbb{F}_n$ by its normal subgroup generated by all elements of the form f^m, $f \in \mathbb{F}_m$. It is known that these groups are infinite for m large enough (see [Ad_1], [Ly], [DG] and references therein).

- uniformly amenable groups, not virtually cyclic.

  A discrete group G is *uniformly amenable* if there exists a function $\mathfrak{C} : (0,1) \times \mathbb{N} \to \mathbb{N}$ such that for every finite subset K of G and every $\epsilon \in (0,1)$ there exists a finite subset $F \subset G$ satisfying:

  (i) $\operatorname{card} F \leq \mathfrak{C}(\epsilon, \operatorname{card} K)$;

  (ii) $\operatorname{card} KF < (1+\epsilon)\operatorname{card} F$.

  For details on this notion see [Kel], [Boż] and [Wys]. In [DS$_1$] it is shown that a uniformly amenable group always satisfies a law.

5.3 Rigidity of relatively hyperbolic groups

Theorem 5.8 ([DS$_1$]). *Let G be a finitely generated group that is hyperbolic relative to its subgroups $H_1, ..., H_m$, and let S be a group asymptotically without cut-points. Then the image of S under any (L,C)–quasi-isometric embedding $S \to G$ is in the M–tubular neighbourhood of a coset gH_i, $g \in G, i = 1, ..., m$, where M depends on L, C, G and S.*

We shall not discuss the proof here, but only mention that it follows from Property (4) of tree-graded spaces.

Remarks 5.9. (a) Theorem 5.8 is not quite surprising for S Abelian or more generally nilpotent, since the picture of the Cayley graph of a relatively hyperbolic group, as given in a particular case by Figure 4, shows that it has the peripheral left cosets gH_i, $g \in G, i = 1, ..., m$, and "outside them" the graph is hyperbolic. Thus, the quasi-isometric image of an Euclidean plane, for instance, cannot wander in the hyperbolic geometry which is in between the left cosets.

On the other hand, Theorem 5.8 becomes more surprising when one takes S a solvable group or Burnside group. The asymptotic cones of such groups contain many copies of the Hawaiian earring (see [Gr$_4$], [Bu]). Therefore it is *a priori* not clear why they should display such a rigid behaviour when q.i. embedded in a relatively hyperbolic group.

(b) In the particular case when S is a non-virtually cyclic group satisfying an identity, the constant M in Theorem 5.8 depends on the identity only.

(c) The hypothesis of Theorem 5.8 that the group S is asymptotically without cut-points can be weakened. See [DS$_1$] and [BDM].

(d) In [PW, §3] Theorem 5.8 is proven for G a fundamental group of a graph of groups with finite edge groups and S a one-ended group.

One cannot hope however to weaken the hypothesis of Theorem 5.8 to "S is a one ended group". For instance non-uniform lattices in $\mathbb{H}^3_{\mathbb{R}}$ are one-ended groups on one hand and hyperbolic relative to their cusp subgroups on the other. Thus, they are quasi-isometrically embedded into themselves and not uniformly near a left coset of a cusp subgroup.

Corollary 5.10. *Let G be a finitely generated group that is hyperbolic relative to its subgroups $H_1, ..., H_m$, and let S be an undistorted subgroup of G asymptotically without cut-points. Then S is contained in H_i^g for some $g \in G, i \in \{1, ..., m\}$.*

As the proofs of the Theorems of R. Schwartz show, a rigidity result such as Theorem 5.8 can be used to get a result on q.i. completeness. Indeed, this can also be done in this case.

Theorem 5.11 ([DS$_1$]). *Let G be a finitely generated group hyperbolic relative to $\{H_1, ..., H_m\}$. Suppose that all the subgroups H_i, $i = 1, ..., m$, are asymptotically without cut-points.*

Let Λ be a finitely generated group that is quasi-isometric to G. Then Λ is hyperbolic relative to a finite collection of subgroups $S_1, ..., S_n$ each of which is quasi-isometric to one of the subgroups $H_1, ..., H_m$.

Remarks 5.12. (a) The number of subgroups relative to which the group is hyperbolic is *not* a quasi-isometry invariant. This can be seen in the example of a finite covering $M \to N$ of a finite volume non-compact hyperbolic 3-manifold by another. The group $\Gamma_M = \pi_1(M)$ is a finite index subgroup of $\Gamma_N = \pi_1(N)$, so they are quasi-isometric. On the other hand, the number of cusp subgroups of Γ_M can be larger than the number of cusp subgroups of Γ_N.

(b) Particular cases of this Theorem follow from the results in [Sch$_1$], [KaL$_1$], [KaL$_2$], [PW].

(c) A more general version of Theorem 5.11 can be found in [BDM].

Outline of proof of Theorem 5.11.

I. Let $\mathfrak{q} : \Lambda \to G$ and $\bar{\mathfrak{q}} : G \to \Lambda$ be two (L, C)–quasi-isometries quasi-converse to each other. Lemma 2.9 implies that using them one can construct a quasi-action quasi-transitive and of finite kernel of Λ on the Cayley graph of G. For simplicity we denote by $\mathcal{A}$ the set $\{gH_i \mid g \in G, i = 1, 2, \ldots, m\}$. By quasi-transitivity, for every left coset $A \in \mathcal{A}$ and every point g in it, there exists $\lambda \in \Lambda$ such that $\mathfrak{q}_\lambda(g) \in B(1, C_1)$, where $C_1 = C_1(L, C)$. On the other hand, by Theorem 5.8, $\mathfrak{q}_\lambda(A)$ is contained in the M–tubular neighbourhood of another left coset $A' \in \mathcal{A}$, where $M = M(L, C, G)$. It follows that A' intersects $B(1, C_1 + M)$. We conclude that the finite set of left cosets intersecting $B(1, C_1 + M)$, $\{A_1, \ldots, A_k\}$, is in some sense a set of representatives for $A \in \mathcal{A}$. It is not minimal though, therefore we diminish it.

II. First we note that if for some λ in Λ, A and B in $\mathcal{A}$, and $M > 0$ we have that $\mathfrak{q}_\lambda(A) \subset \mathcal{N}_M(B)$, then $\mathrm{dist}_H(\mathfrak{q}_\lambda(A), B) \leq M'$ for some $M' = M'(L, C)$. This follows by applying the same result to $\mathfrak{q}_{\lambda^{-1}}(B)$ and from the fact that two left cosets cannot be at finite Hausdorff distance one from the other unless they coincide.

Now we consider the equivalence relation in $\mathcal{A}$

$$A \sim B \Leftrightarrow \exists \lambda \in \Lambda,\ \exists M > 0 \text{ such that } \mathfrak{q}_\lambda(A) \subset \mathcal{N}_M(B)\,.$$

In the set $\{A_1, \ldots, A_k\}$ we select one representative in each equivalence class and obtain thus a possibly smaller set $\{B_1, \ldots, B_n\}$. Also, for every $A_i \sim A_j$ we fix λ_{ij} such that $\mathfrak{q}_{\lambda_{ij}}(A_i) \subset \mathcal{N}_M(A_j)$, and we consider $K_0 = \max_{i,j} \mathrm{dist}(\mathfrak{q}_{\lambda_{ij}}(1), 1)$.

III. We define for each $A \in \mathcal{A}$ the subgroup in Λ

$$\mathrm{Stab}_{M'}(A) = \{\lambda \in \Lambda\,;\ \mathrm{dist}_H(\mathfrak{q}_\lambda(A), A) \leq M'\}\,.$$

Using the arguments in **I** and the choice made in **II** it is not difficult to show that for every $A \in \mathcal{A}$, $\mathrm{Stab}_{M'}(A)$ acts C_2–quasi-transitively on A, in the sense that every orbit of a point under the action of the group contains A in its C_2–tubular neighbourhood. Here C_2 is a constant which is computed by means of K_0.

This in particular implies that $\mathrm{dist}_H(\bar{\mathfrak{q}}(B_i), \mathrm{Stab}_{M'}(B_i)) \leq \kappa$ for some constant $\kappa = \kappa(L, C, C_2)$. The last statement together with the argument in **I** imply that for every $A \in \mathcal{A}$ there exists $\lambda \in \Lambda$ and $i \in \{1, 2, \ldots, n\}$ such that

$$\mathrm{dist}_H(\bar{\mathfrak{q}}(A), \lambda \mathrm{Stab}_{M'}(B_i)) \leq \chi\,,$$

for some $\chi = \chi(L, C, \kappa)$.

IV. Now we have the following sequence of implications. G is hyperbolic relative to $\{H_1, \dots, H_m\}$ $\Rightarrow$ (by Theorem 5.4) every asymptotic cone $\mathrm{Con}_\omega(G; 1, d)$ is tree-graded with set of pieces $\{\lim_\omega(A_n)\}$; $A_n \in \mathcal{A}\}$ $\Rightarrow$ (by Property 8 of tree-graded spaces and Property 4 of asymptotic cones) every asymptotic cone $\mathrm{Con}_\omega(\Lambda; 1, d)$ is tree-graded with set of pieces $\{\lim_\omega(\bar{\mathfrak{q}}(A_n))\}$; $A_n \in \mathcal{A}\}$ $\Rightarrow$ (by the last statement in **III**) every asymptotic cone $\mathrm{Con}_\omega(\Lambda; 1, d)$ is tree-graded with set of pieces $\{\lim_\omega(\lambda_n \mathrm{Stab}_{M'}(B_i))$; $\lambda_n \in \Lambda$, $i \in \{1, 2, \dots, n\}\}$ $\Rightarrow$ (again by Theorem 5.4) Λ is hyperbolic relative to $\{\mathrm{Stab}_{M'}(B_1), \dots, \mathrm{Stab}_{M'}(B_n)\}$.

5.4 More rigidity of relatively hyperbolic groups: outer automorphisms

The group of outer automorphisms of a group G is the quotient group $Out(G) = Aut(G)/Inn(G)$, where $Inn(G)$ is the normal subgroup of automorphisms c_g given by the conjugacy with an element $g \in G$. The group $Inn(G)$ is called *the group of inner automorphisms*.

We recall that in the case of hyperbolic groups the following result is known.

Theorem 5.13 ([Pau]). *(1) Let G be a hyperbolic group. If $Out(G)$ is infinite then G acts isometrically on an $\mathbb{R}$–tree with virtually cyclic edge stabilizers and without global fixed point.*

(2) Let G be a finitely generated hyperbolic group with Kazhdan property (T). Then $Out(G)$ is finite.

Statement (2) follows immediately from (1) because property (T) implies that every action by isometries on a real tree has a global fixed point.

Remark 5.14. Theorem 5.13, (1), together with [BF, Theorem 9.5] imply that if G is a hyperbolic group and if $Out(G)$ is infinite then either G splits as an amalgamated product or as an HNN extension over a virtually cyclic subgroup, or G is itself virtually cyclic.

Examples of hyperbolic groups with property (T):

- uniform lattices of isometries of $\mathbb{H}^n_{\mathbb{H}}$;
- all their hyperbolic quotients. The quotient of a group G with property (T) also has property (T). But hyperbolicity is not automatically inherited by a quotient. Nevertheless, it appears that "almost every" quotient of a hyperbolic group is hyperbolic, in the following sense. A quotient of the group G means the prescription of new relations. A different way of saying it is that, given some finite generating set S of the group G, one chooses a set of reduced words in the alphabet S and puts the condition that they become equal to 1. Since we want a hyperbolic quotient we prescribe finitely many new relations, that is we pick finitely many reduced words in S. There are several ways to introduce the probabilistic language into the picture. One of them is as follows. Choose randomly $e^{\beta\ell}$ new relations among the reduced words of length ℓ in S. Given a certain property (*), count the number $N_{\beta,\ell}$ of choices that give a quotient with property (*). The *probability that the quotient has property (*)* is the ratio of $N_{\beta,\ell}$ over the number of all possible choices of relations under the parameters given above. According to the results in [Gr_4], [Gr_5] and [Oll], for every non-elementary hyperbolic group G and finite generating set S of it, there exists $\alpha = \alpha(G, S) > 0$ such that the following holds:
  - for every $\beta < \alpha$, the probability that the quotient is non-elementary hyperbolic goes to 1 as ℓ goes to infinity;

– if $\beta > \alpha$ then with probability going to 1 the quotient is either trivial or $\mathbb{Z}/2\mathbb{Z}$.

In the particular case when G is the free group of rank m, $\mathbb{F}_m$, and S is the set of $2m$ generators, $\alpha = \frac{\ln(2m-1)}{2}$.

This gives a large choice of relations and of hyperbolic quotients for any hyperbolic group G. It allows in particular for the possibility of constructing approximate copies of very complicated graphs in the Cayley graph of a quotient of G, without loosing the property of hyperbolicity.

- In [$\dot{\text{Z}}$], a slightly different notion of random group is considered. Given $\mathbb{F}_m$ the free group of rank m, one chooses randomly relations of length 3 - on the whole there are $2m(2m-1)^2$ candidates. Suppose that one chooses randomly $(2m-1)^{3\beta}$ relations. In[$\dot{\text{Z}}$] it is proved that:
  - if $\beta > \frac{1}{3}$ then the probability that the quotient of $\mathbb{F}_m$ thus obtained has property (T) goes to 1 as m goes to infinity;
  - if $\beta < \frac{1}{2}$ then with probability going to 1 as m goes to infinity, the quotient of $\mathbb{F}_m$ is non-elementary hyperbolic;
  - consequently for $\beta \in \left(\frac{1}{3}, \frac{1}{2}\right)$, with probability going to 1 the quotient of $\mathbb{F}_m$ is both hyperbolic and with property (T).

In the case of relatively hyperbolic groups, the following two theorems provide a sample of results on the group of outer automorphisms.

Theorem 5.15 ([DS$_1$]). *Let G be a group relatively hyperbolic with respect to $\{H_1, \ldots, H_n\}$ and suppose that all H_i are asymptotically without cut-points. Then for every $i \in \{1, \ldots, n\}$, there exists a homomorphism from a subgroup of index at most $n!$ in $Out(G)$ to $Out(H_i)$.*

A result more in the line of Theorem 5.13 is the following.

Theorem 5.16 (Theorem 1.7 in [DS$_2$]). *Let G be a group relatively hyperbolic with respect to $\{H_1, \ldots, H_n\}$ and suppose that no H_i is relatively hyperbolic with respect to proper subgroups. If $Out(G)$ is infinite then one of the followings cases occurs:*

(a) G splits as an amalgamated product or HNN extension over a virtually cyclic subgroup;

(b) G splits as an amalgamated product or HNN extension over a parabolic subgroup.

Remark 5.17. In particular a group G as in Theorem 5.16 which moreover has property (T) has finite $Out(G)$.

We finish the exposition of the rigidity results for relatively hyperbolic groups by making the following important remark.

Remark 5.18. In all the rigidity results in Sections 5.3 and 5.4, the hypothesis "asymptotically without cut-points" can be replaced by the weaker hypothesis "one asymptotic cone has no cut-points". Further generalizations can be found in [BDM].

6 Groups asymptotically with(out) cut-points

First we come back to the list of examples of groups asymptotically without cut-points and discuss the last two.

6.1 Groups with elements of infinite order in the center, not virtually cyclic

Let us see what happens if the asymptotic cone $\mathrm{Con}_\omega(G; 1, d)$ of an infinite group G has a cut-point. Proposition 5.3 implies that it is a tree-graded space with respect to a set of pieces $\mathcal{P}$ such that each piece is either a point or a geodesic subset without cut-point. In particular, if all pieces are points the cone is a tree. Note that by homogeneity in this case it can be either a line or a tree in which every point is a branching point.

The case when one asymptotic cone is a line turns out to be quite particular.

Proposition 6.1 (Corollary 6.2 in [DS$_1$]). *A finitely generated group such that one asymptotic cone is a point or a line is virtually cyclic.*

Let now G be a non-virtually cyclic group with a central infinite cyclic subgroup $\langle h \rangle$. We have to show that G cannot have cut-points in any asymptotic cone. Suppose that one of its asymptotic cones $\mathrm{Con}_\omega(G; 1, d)$ has cut-points. It follows that it is tree-graded and that it is not a line.

Every element ζ in the center of G is an isometry with the property that every $g \in G$ is translated by ζ at a fixed distance, as $\mathrm{dist}(\zeta g, g) = \mathrm{dist}(\zeta, 1)$. It is not difficult to deduce from this the following. For every $\epsilon > 0$ there exists an isometry h_ω of $\mathrm{Con}_\omega(G; 1, d)$, $h_\omega \in G^\omega$, such that for every $x \in \mathrm{Con}_\omega(G; 1, d)$, $\mathrm{dist}(h_\omega(x)\,,\, x) = \epsilon$.

On the other hand no tree-graded space different from a line admits such set of isometries. This is clearly seen for instance in the particular case of a real tree with at least one branching point. There is no way in which to translate a small tripod in this tree such that all its points move at the same distance.

6.2 Groups satisfying an identity, not virtually cyclic

Again we argue by contradiction and suppose that such a group G has an asymptotic cone $\mathrm{Con}_\omega(G; 1, d)$ with cut-points, consequently an asymptotic cone which is tree-graded and different from a line. By the argument in the end of Section 4.1, the group G^ω acts transitively on $\mathrm{Con}_\omega(G; 1, d)$. Note that the ω–ultrapower $\Pi_\omega G$ and its subgroup G^ω satisfy the same identity as G. Even more can be said:

Lemma 6.2 (Lemma 6.15, [DS$_1$]). *Let ω be any ultrafilter. The group G satisfies a law if and only if its ω–ultrapower $\Pi_\omega G$ does not contain free non-Abelian subgroups.*

If $\mathrm{Con}_\omega(G; 1, d)$ is a tree then G^ω cannot act on it by fixing a point in the boundary of the tree [DS$_1$, §6]. This fact and [Ch, Proposition 3.7, page 111] imply that G^ω contains a free non-Abelian subgroup. This contradicts Lemma 6.2.

So in what follows we may assume that $\mathrm{Con}_\omega(G; 1, d)$ is not a tree, consequently that it contains at least one (hence by homogeneity continuously many) pieces without cut-points which are not singletons.

To conclude we need the following result.

Proposition 6.3 ([DS$_1$]). *Let $\mathbb{F}$ be a tree-graded space with at least one non-singleton piece, and let $\mathcal{G}$ be a group acting transitively on $\mathbb{F}$ and permuting pieces. The group $\mathcal{G}$ contains a non-Abelian free subgroup.*

Outline of proof. Let $\mathcal{P}$ be the set of pieces of $\mathbb{F}$, containing at least one piece P different from a point. It follows that P has cardinality $2^{\aleph_0}$. Then it can be shown that for every pair of distinct points a, b in P there exists an isometry $g \in \mathcal{G}$ such that $g(P) \cap P = \emptyset$, $g(P)$ projects onto P

in a and P projects onto $g(P)$ in $g(b)$. We denote by Π_x the set of all points projecting in P in the point x. Property 6 of tree-graded spaces implies that for every $x \neq b$, $g(\Pi_x) \subset \Pi_a$.

From this one can easily deduce that $g^{-1}(P) \cap P = \emptyset$, $g^{-1}(P)$ projects onto P in b and P projects onto $g^{-1}(P)$ in $g^{-1}(a)$. Moreover, for every $y \neq a$, $g^{-1}(\Pi_y) \subset \Pi_b$.

Now we choose a second pair of distinct points c, d in $P \setminus \{a, b\}$. We choose for this pair a second isometry h with the same properties as g for a, b.

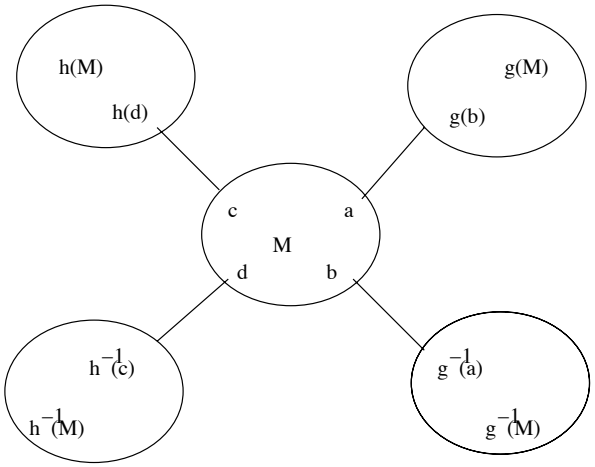

Figure 8: The ping-pong argument.

It is not difficult to show by a ping-pong argument that g and h generate a free group. □

As already mentioned, it turns out that uniformly amenable groups are a particular case of groups satisfying a law. This observation is maybe worth some explanations. Recall the following results.

Theorem 6.4 ([Wys]). *Let G be a countable discrete group.*

(1) If G is uniformly amenable then for any ultrafilter ω the ultrapower $\Pi_\omega G$ is uniformly amenable.

(2) If there exists an ultrafilter ω such that the ultrapower $\Pi_\omega G$ is amenable then G is uniformly amenable.

In particular, if the ultrapower of a discrete countable group is amenable then it is uniformly amenable.

Now we recall some classical results.

Proposition 6.5. *A subgroup S of an amenable group G is amenable.*

Remark 6.6. Note that no other assumption is made on S or G - except the amenability for G, as defined page 15 - not discreteness, nor local compactness nor anything.

Proof. Take $\epsilon > 0$ arbitrary small. Take K a finite subset in S. There exists a subset F in G such that $\operatorname{card} KF < (1 + \epsilon)\operatorname{card} F$. Consider a graph whose vertices are the elements of the set F, and whose edges correspond to the pairs of points $(f_1, f_2) \in F \times F$ such that $f_2 = kf_1$, where $k \in K$. Let C be a connected component of this graph with set of vertices $\mathcal{V}_C$. Then $K\mathcal{V}_C$ does not intersect the sets of vertices of other connected components. Hence there exists a connected component C such that $\operatorname{card} K\mathcal{V}_C < (1+\epsilon)\operatorname{card} \mathcal{V}_C$ (otherwise if all these inequalities

have to be reversed, the sum of them gives a contradiction with the choice of F). Without loss of generality, we can assume that $\mathcal{V}_C$ contains 1. Otherwise we can shift it to 1 by multiplying on the right by c^{-1} for some $c \in \mathcal{V}_C$. Then $\mathcal{V}_C$ can be identified with a finite subset of S. Therefore S contains a subset $\mathcal{V}_C$ such that $\operatorname{card} K\mathcal{V}_C < (1+\epsilon)\operatorname{card} \mathcal{V}_C$. □

Proposition 6.7. *A non-Abelian free group is not amenable.*

A nice proof of this can be found in [GLP, §6.C].

Corollary 6.8. *A group having a non-Abelian free subgroup is not amenable.*

Remark 6.9. The remark that the existence of a free subgroup excludes amenability was first made by J. von Neumann in [vN], the very paper in which he introduced the notion of amenable group, under the name of measurable group. It is this remark that raised the question known later as *the von Neumann problem*: whether a non-amenable group always contains a free non-Abelian subgroup. In [Ti] it was shown that for linear groups the von Neumann problem has an affirmative answer, moreover a linear group without any free non-Abelian subgroup is solvable-by-finite. The first examples of non-amenable groups with no (non-Abelian) free subgroups were given in [Olsh]. In [Ad$_2$] it was shown that the free Burnside groups $B(n,m)$ with $n \geq 2$ and $m \geq 665$, m odd, are also non-amenable. The first finitely presented examples of non-amenable groups with no (non-Abelian) free subgroups were given in [OlS].

Lemma 6.2 and Corollary 6.8 imply the following.

Corollary 6.10 (Corollary 5.9 in [Kel], Corollary 6.16 in [DS$_1$]). *A finitely generated group which is uniformly amenable satisfies a law. In particular no asymptotic cone of it has a cut-point.*

6.3 Existence of cut-points in asymptotic cones and relative hyperbolicity

A natural question to ask is the following.

Question 6.11. Can one improve the characterization of relatively hyperbolic groups by their asymptotic cones to: a group G is relatively hyperbolic if and only if all its asymptotic cones have cut-points ?

The"only if" part is already proved. Concerning the "if" part let us note that if an arbitrary asymptotic cone $\mathcal{C}$ of G has a cut-point then Proposition 5.3 implies that it is tree-graded with respect to some collection of pieces $\mathcal{P}$ (which are either points or without cut-point). Still it is not granted that there exists a finite set of subgroups of G such that all pieces are limit sets of left cosets of these subgroups.

It turns out that the answer to Question 6.11 is negative, and that the property of having cut-points in every asymptotic cone appears oftener than relative hyperbolicity. Here are some examples of non-relatively hyperbolic groups that have cut-points in every asymptotic cone:

1. The mapping class group of an orientable finite type surface S with

$$3 \cdot \operatorname{genus}(S) + \#\,\text{punctures} \geq 5;$$

   The fact that it has cut-points in any asymptotic cone is proved in [B], while its non-relative hyperbolicity can be deduced from arguments in [Bow$_4$] and [KN], and it is explicitely proved in [AAS] and [BDM].

2. Many right angled Artin groups [BDM].

3. Fundamental groups of graph manifolds. They are not relatively hyperbolic according to [BDM], while they have cut-points in any asymptotic cone by arguments in [KaL$_3$] and [KKL].

There also exists a metric example of the same sort, in which the relative hyperbolicity is to be taken in its purely metric sense given in [DS$_1$]. More precisely, for any surface S with $3 \cdot \text{genus}(S) + \#\,\text{punctures} \geq 9$, the Teichmüller space with the Weil-Petersson metric is not relatively hyperbolic [BDM], while it has cut-points in any asymptotic cone [B].

It follows from arguments in [KKL] that the property of having cut-points in all asymptotic cones is common to many fundamental groups of non-positively curved manifolds.

Proposition 6.12 ([KKL]). *If M is a compact non-positively curved manifold then either the universal cover of M is a symmetric space or $\pi_1(M)$ has cut-points in any asymptotic cone of it.*

7 Open questions

Question 7.1. Is it true that an amenable group has at least one asymptotic cone without cut-points ? Is it true that all its asymptotic cones are without cut-points ?

The interest in such a question comes from Remark 5.18. An affirmative answer to it would generalize Corollary 6.10.

Question 7.2. Do relatively hyperbolic groups have uniform exponential growth ?[3]

A finitely generated group is said to have *exponential growth* if for some set of generators S (hence for every S), the growth function $B_S(n) = \text{card}\, B(1,n)$ is exponential. One can define $\alpha_S = \lim_{n\to\infty} \frac{\ln B_S(n)}{\ln n}$ and then exponential growth means that $\alpha_S > 0$. One can also define $\alpha = \inf_S \alpha_S$. If $\alpha > 0$ then the group is said to have *uniform exponential growth*. Hyperbolic groups for instance have uniform exponential growth [Kou]. There are also examples of groups having exponential growth but not uniform exponential growth [Wi]. For a survey of the subject see [dH].

The usual way in which uniform exponential growth is proved is to show that there exists some n_0 such that $B_S(1,n_0)$ contains two elements generating a free subgroup (or a free subsemigroup), for every generating set S.

Question 7.3. Can one drop the assumption that the groups $H_1, \ldots, H_m$ are asymptotically without cut-points from the rigidity theorem 5.11 ?

Obviously the methods used in the proof of Theorem 5.11 no longer work. More precisely, the main tool Theorem 5.8 no longer holds without the hypothesis "asymptotically without cut-points". Nevertheless the rigidity result might still be true.

Note that versions of Theorem 5.11 with weaker hypotheses on the groups $H_1, \ldots, H_m$ can be found in [DS$_1$] and [BDM].

Question 7.4. How does weak relative hyperbolicity behave with respect to quasi-isometries ?

Again the methods used for (strong) relative hyperbolicity no longer work. Theorem 5.8 again does not hold as can be easily seen by taking $G = \mathbb{Z}^n$, $H = \mathbb{Z}^{n-1} \times \{0\}$ and $S = \mathbb{Z}^{n-1}$. A quasi-isometric embedding of S has no reason to stay close to a left coset of H, as illustrated

[3] A positive answer to this question has been announced in [Xie].

by many examples: it can be transversal to all left cosets of H or it can be composed of many horizontal and vertical pieces etc.

Up to now there is no general result on the behavior up to quasi-isometry of weakly relatively hyperbolic groups. In [KaL$_2$], [Pap], [DS], [MSW$_1$] and [MSW$_2$] strong quasi-isometric rigidity results are proved for some particular cases of weakly relatively hyperbolic groups—in fact all of them fundamental groups of some graphs of groups. The notion of thick group introduced in [BDM] can be seen as a first attempt towards a study of weakly relatively hyperbolic groups from the quasi-isometry rigidity viewpoint.

Question 7.5 ("accessibility" for relatively hyperbolic groups). Under which conditions does the process described in Corollary 5.5 have a terminal point, that is: when does a relatively hyperbolic group G have a list of peripheral subgroups that are non-relatively hyperbolic ? Does this hold when G is torsion-free, when it is finitely presented ? Note that both conditions are not satisfied by the inaccessible groups of Dunwoody (see Remark 5.6).

We recall the standard theory of accessibility of groups, to which this question relates. By Stalling's Ends Theorem [Sta], a finitely generated group with more than one end splits as a free product or HNN-extension with finite amalgamation. The question is whether in an arbitrary finitely generated group one can keep on doing this splitting until no more splitting is possible, that is until all the factor groups are finite or one-ended. The answer is positive for finitely generated torsion-free groups (the Grushko-Neumann theorem) and for finitely presented groups [Du$_1$]. But it is not true for all finitely generated groups [Du$_2$].

Question 7.6. Given a group G hyperbolic relative to the subgroups $H_1, \ldots, H_m$ can one say that the group G has all asymptotic cones isometric to each other, under the obvious necessary condition that each H_i has all asymptotic cones isometric to each other ?

This would generalize the result of [DP] from hyperbolic groups to relatively hyperbolic groups.

8 Dictionary

- **Boundary at infinity.** Given X either a simply connected Riemannian manifold of non-positive curvature (or more generally a $CAT(0)$–space) or an infinite graph, its boundary at infinity $\partial_\infty X$ is the quotient $\mathcal{R}/\sim$ of the set $\mathcal{R}$ of geodesic rays in X with respect to the equivalence relation $r_1 \sim r_2 \Leftrightarrow \mathrm{dist}_H(r_1, r_2) < +\infty$.

- **(Abstractly) commensurable groups.** Two discrete groups G_1 and G_2 are called *abstractly commensurable* if they have finite index subgroups that are isomorphic.

- **Commensurable groups in an ambient larger group.** When both G_1 and G_2 are subgroups in a group G, we say that G_1 and G_2 are *commensurable (in G)* if there exists $g \in G$ such that $G_1^g \cap G_2$ has finite index both in G_1^g and in G_2.

- **Commensurator.** In the above case the set of $g \in G$ such that $G_1^g \cap G_2$ has finite index both in G_1^g and in G_2 is called *the commensurator of G_1 to G_2 in G*, and it is denoted $Comm_G(G_1, G_2)$. When $G_1 = G_2$ we simply write $Comm_G(G_1)$. Also, when there is no possibility of confusion, we drop the index G.

- **Convergence group.** It is a subgroup G of Homeo($\mathbb{S}^1$) such that every sequence of distinct elements in G contains a subsequence (g_n) for which there exist $x, y \in \mathbb{S}^1$ with

the property that on $\mathbb{S}^1 \setminus \{x, y\}$, g_n converges to x and g_n^{-1} converges to y uniformly on compact subsets.

- **Filter.** A *filter* $\mathcal{F}$ over a set I is a collection of subsets of I satisfying the following conditions:

  (F_1) If $A \in \mathcal{F}$, $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$;

  (F_2) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;

  (F_3) $\emptyset \notin \mathcal{F}$.

  For instance, if $I = \mathbb{N}$, the collection of all complementaries of finite sets is a filter over $\mathbb{N}$, called *the Fréchet filter*.

- **Fuchsian group.** It is a discrete subgroup of $PSL(2, \mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2_{\mathbb{R}})$.

- **Fully residually * group (also called ω–residually * group).** Here * represents a family of groups (finite groups, free groups etc.) A group G is fully residually * if for every finite subset F in G there exists a homomorphism from G onto a * group which is injective on F.

- **(Global) cut-point.** A point p in a topological space X such that $X \setminus \{p\}$ has several connected components.

- **Geodesic metric space.** It is a length metric space such that for every pair of points, the shortest path joining them exists. By Hopf-Rinow Theorem [GLP] a complete locally compact length metric space is geodesic.

- **Hausdorff distance.** If A and B are two subsets in a metric space X, then the Hausdorff distance $\mathrm{dist}_H(A, B)$ between A and B is the minimum of all $\delta > 0$ such that A is contained in the δ-tubular neighbourhood of B and B is contained in the δ-tubular neighbourhood of A. If no such finite δ exists, one puts $\mathrm{dist}_H(A, B) = +\infty$.

- **Hawaiian earring**. It is the topological space $\bigcup_{n\in\mathbb{N}} C\left(\left(0, \frac{1}{n}\right), \frac{1}{n}\right)$ with the topology induced from $\mathbb{R}^2$, where $C\left(\left(0, \frac{1}{n}\right), \frac{1}{n}\right)$ denotes the circle of center $\left(0, \frac{1}{n}\right)$ and of radius $\frac{1}{n}$. Its fundamental group is uncountable and non-free [DES].

- **Horoball, horosphere**. Let ϱ be a geodesic ray in a simply connected Riemannian manifold of non-positive curvature (more generally in a $CAT(0)$–space) X. It defines a point at infinity $\alpha \in \partial_\infty X$. The *open horoball $Hbo(\varrho)$ determined by ϱ* is the union of open balls $\bigcup_{t>0} B(\varrho(t), t)$. Its closure $Hb(\varrho)$ is the *closed horoball determined by ϱ*, and its boundary $H(\varrho)$ is the *horosphere determined by ϱ*.

  Note that if ϱ_1 , ϱ_2 are asymptotic rays then there exists $\kappa > 0$ such that $\mathcal{N}_\kappa(Hbo(\varrho_i)) = Hbo(\varrho_j)$, where $\{i, j\} = \{1, 2\}$. Thus, one horoball defines all the other horoballs determined by rays in the same asymptotic class. Therefore it makes sense to no longer specify the ray, but only the point at infinity α corresponding to it, and to speak about all horoballs corresponding to rays with the same point at infinity as *horoballs of basepoint α*. For details on this notion see [BrH].

- **Length (or path) metric space.** A metric space (X, dist_ℓ) such that for every pair x, y in X, $\mathrm{dist}_\ell(x, y) =$ the infimum of the lengths of the paths joining x and y. A priori the path realizing the infimum might not exist.

Note that given a metric space (X, dist), one can define the length of curves in it. Consequently one can define a "length metric" dist_ℓ on X. The problem is that in this case dist_ℓ might take the value $+\infty$, because in case x and y are not joined by any path of finite length, or simply by any path, one puts $\mathrm{dist}_\ell(x, y) = +\infty$.

- **Net.** A *net* in a metric space X is a subset N of X which is
  - δ-*separated* for some $\delta > 0$: for every $n_1, n_2 \in N$, $\mathrm{dist}(n_1, n_2) \geq \delta$;
  - ϵ-*covering* for some $\epsilon > 0$: $X \subset \mathcal{N}_\epsilon(N)$.

  When more precision is needed, N is also called (δ, ϵ)–net.

- **Proper metric space**. A metric space with the property that all its closed balls are compact. Note that by the Hopf-Rinow Theorem [GLP] every complete, locally compact length metric space is proper.

- **Reduced word.** Given an alphabet $S = \{a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\}$, a word in the alphabet S is called *reduced* if it does nor contain subwords of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$.

- **Tubular neighbourhood**. For a set A in a metric space X and for $\delta > 0$ we define the *δ-tubular neighbourhood $\mathcal{N}_\delta(A)$ of A* as the set

$$\{x \mid \mathrm{dist}(x, A) < \delta\}.$$

- **Ultrafilter.** An *ultrafilter* over a set I is a filter $\mathcal{U}$ over I which is a maximal element in the ordered set of all filters over I with respect to the inclusion. An ultrafilter can also be defined as a collection of subsets of I satisfying the conditions (F_1), (F_2), (F_3) defining a filter and the additional condition:

$$(F_4) \text{ For every } A \subseteq I \text{ either } A \in \mathcal{U} \text{ or } I \setminus A \in \mathcal{U}.$$

  A *non-principal ultrafilter* is an ultrafilter containing the Fréchet filter.

- **Virtually *.** A group is said to have property * *virtually* if a finite index subgroup has the property *.

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