LECTURES ON NAKAJIMA'S QUIVER VARIETIES

VICTOR GINZBURG

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1. Outline

1.1. Introduction. Nakajima's quiver varieties are certain smooth (not necessarily affine) complex algebraic varieties associated with quivers. These varieties have been used by Nakajima to give a geometric construction of universal enveloping algebras of Kac-Moody Lie algebras (as well as a construction of quantized enveloping algebras for *affine* Lie algebras) and of all irreducible integrable (e.g., finite dimensional) representations of those algebras.
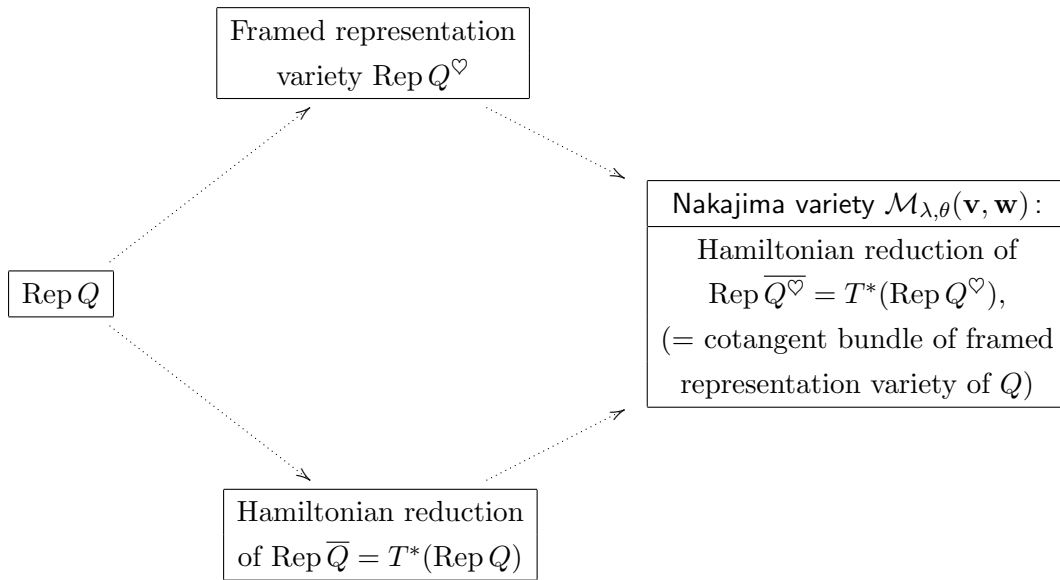
A connection between quiver representations and Kac-Moody Lie algebras has been first discovered by C. Ringel around 1990. Ringel produced a construction of $U_q(\mathfrak{n})$, the *positive part* of the quantized enveloping algebra $U_q(\mathfrak{g})$ of a Kac-Moody Lie algebra $\mathfrak{g}$, in terms of a *Hall algebra* associated with an appropriate quiver. Shortly afterwards, G. Lusztig combined Ringel's ideas with the powerful technique of perverse sheaves to construct a *canonical basis* of $U_q(\mathfrak{n})$, see [L2], [L3].

The main advantage of Nakajima's approach (as opposed to the earlier one by Ringel and Lusztig) is that it yields a geometric construction of the whole algebra $U(\mathfrak{g})$ rather than its positive part. At the same time, it also provides a geometric construction of simple integrable $U(\mathfrak{g})$-modules. Nakajima's approach also yields a similar construction of the algebra $U_q(L\mathfrak{g})$ and its simple integrable representations, where $L\mathfrak{g}$ denotes the loop Lie algebra associated to $\mathfrak{g}$. [1]

There are several steps involved in the definition of Nakajima's quiver varieties. Given a quiver Q, one associates to it three other quivers, $Q^{\heartsuit}$, $\overline{Q}$, and $\overline{Q^{\heartsuit}}$, respectively. In terms of these quivers,

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[1]Note however that, unlike the Ringel-Lusztig construction, the approach used by Nakajima does *not* provide a construction of the *quantized* enveloping algebra $U_q(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ itself. A similar situation holds in the case of Hecke algebras, where the *affine* Hecke algebra has a geometric interpretation in terms of equivariant K-theory, see [KL], [CG], while the Hecke algebra of a finite Weyl group does not seem to have such an interpretation.

various steps of the construction of Nakajima varieties may be illustrated schematically as follows

$$
\begin{array}{ccc}
& \boxed{\begin{array}{c}\text{Framed representation}\\ \text{variety } \operatorname{Rep} Q^{\heartsuit}\end{array}} & \\
\nearrow & & \searrow \\
\boxed{\operatorname{Rep} Q} & & \boxed{\begin{array}{c}\text{Nakajima variety } \mathcal{M}_{\lambda,\theta}(\mathbf{v},\mathbf{w}):\\ \hline \text{Hamiltonian reduction of}\\ \operatorname{Rep} \overline{Q^{\heartsuit}} = T^*(\operatorname{Rep} Q^{\heartsuit}),\\ (= \text{cotangent bundle of framed}\\ \text{representation variety of } Q)\end{array}} \\
\searrow & & \nearrow \\
& \boxed{\begin{array}{c}\text{Hamiltonian reduction}\\ \text{of } \operatorname{Rep} \overline{Q} = T^*(\operatorname{Rep} Q)\end{array}} &
\end{array}
$$

1.2. **Nakajima's varieties and symplectic algebraic geometry.** Nakajima's varieties also provide an important large class of examples of algebraic symplectic manifolds with extremely nice properties and rich structure, interesting in their own right. To explain this, it is instructive to consider a more general setting as follows.

Let X be a (possibly singular) affine variety equipped with an algebraic Poisson structure. In algebraic terms, this means that $\mathbb{C}[X]$, the coordinate ring of X, is equipped with a Poisson bracket $\{-,-\}$, that is, with a Lie bracket satisfying the Leibniz identity.

Recall that any smooth symplectic algebraic manifold carries a natural Poisson structure.

Definition 1.2.1. *Let X be an irreducible affine normal Poisson variety. A resolution of singularities $\pi : \widetilde{X} \to X$ is called a* symplectic resolution *of X provided $\widetilde{X}$ is a smooth complex algebraic symplectic manifold (with algebraic symplectic 2-form) such that the pull-back morphism $\pi^* : \mathbb{C}[X] \to \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is a Poisson algebra morphism.*

Below, we will be interested in the case where the variety X is equipped, in addition, with a $\mathbb{C}^{\times}$-action that rescales the Poisson bracket and contracts X to a (unique) fixed point $o \in X$. Equivalently, this means that the coordinate ring of X is equipped with a *nonnegative* grading $\mathbb{C}[X] = \bigoplus_{k \in \mathbb{Z}} \mathbb{C}^k[X]$ such that $\mathbb{C}^k[X] = 0 \, (\forall k < 0)$, and $\mathbb{C}^0[X] = \mathbb{C}$ and, in addition, there exists a (fixed) positive integer $m > 0$, such that one has

$$\{\mathbb{C}^i[X], \mathbb{C}^j[X]\} \subset \mathbb{C}^{i+j-m}[X], \quad \forall i, j \geq 0.$$

In this situation, given a symplectic resolution $\pi : \widetilde{X} \to X$, we call $\pi^{-1}(o)$, the fiber of π over the $\mathbb{C}^{\times}$-fixed point $o \in X$, the *central fiber.*

Symplectic resolutions of a Poisson variety with a contracting $\mathbb{C}^{\times}$-action as above enjoy a number of very favorable properties:
 (i) The map $\pi : \widetilde{X} \to X$ is automatically *semismall* in the sense of Goresky-MacPherson, i.e. one has $\dim(\widetilde{X} \times_X \widetilde{X}) = \dim X$, cf. [K1].
 (ii) We have a Poisson algebra *isomorphism* $\pi^* : \mathbb{C}[X] \xrightarrow{\sim} \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$, moreover, $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$ for all $i > 0$. The $\mathbb{C}^{\times}$-action on X admits a canonical lift to an algebraic $\mathbb{C}^{\times}$-action on $\widetilde{X}$, see [K1].

(iii) The Poisson variety X is a union of *finitely many* symplectic leaves $X = \sqcup X_\alpha$, [K4], and each symplectic leaf X_α is a locally closed smooth algebraic subvariety of X, [BG].

(iv) For any $x \in X$, we have $H^{\mathrm{odd}}(\pi^{-1}(x), \mathbb{C}) = 0$, moreover, the cohomology group $H^{2k}(\pi^{-1}(x), \mathbb{C})$ has pure Hodge structure of type (k, k), for any $k \geq 0$, cf. [EV] and [K3].

(v) Each fiber of π, equipped with reduced scheme structure, is an isotropic subvariety of $\widetilde{X}$. The central fiber $\pi^{-1}(o)$ is a homotopy retract of $\widetilde{X}$, in particular, we have $H^\bullet(\widetilde{X}, \mathbb{C}) \cong H^\bullet(\pi^{-1}(o), \mathbb{C})$.

The set $\widetilde{X} \times_X \widetilde{X}$ that appears in (i) may have several irreducible components and the semismallness condition says that the dimension of any such component is $\leq \dim X$; in particular, the diagonal $X \subset \widetilde{X} \times_X \widetilde{X}$ is one such component of maximal dimension. To prove (i), write ω for the symplectic 2-form on $\widetilde{X}$, and equip $\widetilde{X} \times \widetilde{X}$ with the 2-form $\Omega := p_1^*\omega + p_2^*(-\omega)$, where $p_i : \widetilde{X} \times \widetilde{X} \to \widetilde{X}$, $i = 1, 2$, denote the projections. Then, Ω is a symplectic form on $\widetilde{X} \times \widetilde{X}$ and it is not difficult to show that the restriction of Ω to the (regular locus of the) subvariety $\widetilde{X} \times_X \widetilde{X}$ vanishes. The inequality $\dim \widetilde{X} \times_X \widetilde{X} \leq \dim X$, hence the semismallness of π, follows from this.

Essential parts of statements (ii) and (iv) are special cases of the following more general result, to be proved in section 5.5 below.

Proposition 1.2.2. *Let $\pi : \widetilde{X} \to X$ be a proper morphism, where $\widetilde{X}$ is a smooth symplectic algebraic variety and X is an affine variety. Then, one has*

(i) $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$ *for all $i > 0$.*

(ii) *Any fiber of π is an isotropic subvariety.*

Example 1.2.3 (**Slodowy slices**). Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\langle e, h, f \rangle \subset \mathfrak{g}$ an $\mathfrak{sl}_2$-triple for a nilpotent element $e \in \mathfrak{g}$. Write $\mathfrak{z}_f$ for the centralizer of f in $\mathfrak{g}$, and $\mathcal{N}$ for the *nilcone*, the subvariety of all nilpotent elements of $\mathfrak{g}$. Slodowy has shown that the intersection $\mathcal{S}_e := \mathcal{N} \cap (e + \mathfrak{z}_f)$ is reduced, normal, and that there is a $\mathbb{C}^\times$-action on $\mathcal{S}_e$ that contracts $\mathcal{S}_e$ to e, cf. eg. [CG], §3.7 for an exposition.

The variety $\mathcal{S}_e$ is called the *Slodowy slice* for e.

Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by means of the Killing form, and view $\mathcal{S}_e$ as a subvariety in $\mathfrak{g}^*$. Then, the standard Kirillov-Kostant Poisson structure on $\mathfrak{g}^*$ induces a Poisson structure on $\mathcal{S}_e$. The symplectic leaves in $\mathcal{S}_e$ are obtained by intersecting $e + \mathfrak{z}_f$ with the various nilpotent conjugacy classes in $\mathfrak{g}$.

Let $\mathcal{B}$ denote the flag variety for $\mathfrak{g}$ and let $T^*\mathcal{B}$ be the cotangent bundle on $\mathcal{B}$. There is a standard resolution of singularities $\pi : T^*\mathcal{B} \to \mathcal{N}$, the *Springer resolution*, cf. eg. [CG, ch. 3]. It is known that $\widetilde{\mathcal{S}}_e := \pi^{-1}(\mathcal{S}_e)$ is a smooth submanifold in $T^*\mathcal{B}$ and the canonical symplectic 2-form on the cotangent bundle restricts to a nondegenerate, hence symplectic, 2-form on $\widetilde{\mathcal{S}}_e$. Moreover, restricting π to $\widetilde{\mathcal{S}}_e$ gives a symplectic resolution $\pi_e : \widetilde{\mathcal{S}}_e \to \mathcal{S}_e$, cf. [Gi2], Proposition 2.1.2. The central fiber of that resolution is $\pi_e^{-1}(e) = \mathcal{B}_e$, the fixed point set of the natural action of the element $e \in \mathfrak{g}$ on the flag variety $\mathcal{B}$.

In the (somewhat degenerate) case $e = 0$, we have $\mathcal{S}_e = \mathcal{N}$, and the corresponding symplectic resolution reduces to the Springer resolution itself.

Example 1.2.4 (**Symplectic orbifolds**). Let (V, ω) be a finite dimensional symplectic vector space and $\Gamma \subset Sp(V, \omega)$ a finite subgroup. The orbifold $X := V/\Gamma$ is an affine normal algebraic variety, and the symplectic structure on V induces a Poisson structure on X. Such a variety X may or may not have a symplectic resolution $\widetilde{X} \to X$, in general. This holds, for instance, in the case of *Kleinian singularities*, that is the case where $\Gamma \subset SL_2(\mathbb{C})$ and $X := \mathbb{C}^2/\Gamma$. Then, a symplectic resolution $\pi : \widetilde{X} \to X$ does exist. It is the canonical minimal resultion, see [Kro].

Recall that there is a correspondence, the McKay correspondence, between the (conjugacy classes of) finite subgroups $\Gamma \subset SL_2(\mathbb{C})$ and Dynkin graphs of $\mathbf{A}, \mathbf{D}, \mathbf{E}$ types, cf. [CS], and §4.6 below. It turns out that $\mathbb{C}^2/\Gamma$ is isomorphic, as a Poisson variety, to the Slodowy slice $\mathcal{S}_e$, where e is a *subregular* nilpotent in the simple Lie algebra $\mathfrak{g}$ associated with the Dynking diagram of the corresponding type.

Another important example is the case where $\Gamma \subset GL(\mathfrak{h})$ is a complex reflection group and $V := \mathfrak{h} \times \mathfrak{h}^* = T^*\mathfrak{h}$ is the cotangent bundle of the vector space $\mathfrak{h}$ equipped with the canonical symplectic structure of the cotangent bundle. We get a natural imbedding $\Gamma \subset Sp(V)$. One can show that, among all irreducible finite Weyl groups Γ, only those of types $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$, have the property that the orbifold $(\mathfrak{h} \times \mathfrak{h}^*)/\Gamma$ admits a symplectic resolution, see [GK], [Go].

In type $\mathbf{A}$, we have $\Gamma = S_n$, the Symmetric group acting diagonally on $\mathbb{C}^n \times \mathbb{C}^n$ (two copies of the permutation representation). Thus, $(\mathbb{C}^n \times \mathbb{C}^n)/S_n = (\mathbb{C}^2)^n/S_n$ is the n-th symmetric power of the plane $\mathbb{C}^2$. The orbifold $(\mathbb{C}^2)^n/S_n$ has a natural resolution of singularities $\pi : \mathrm{Hilb}^n(\mathbb{C}^2) \to (\mathbb{C}^2)^n/S_n$, where $\mathrm{Hilb}^n(\mathbb{C}^2)$ stands for the Hilbert scheme of n points in $\mathbb{C}^2$. The map π, called Hilbert-Chow morphism, turns out to be a symplectic resolution, cf. [Na3], §1.4.

Example 1.2.5 (**Quiver varieties**). Let Q be a finite quiver with vertex set I. Let $\mathbf{v}, \mathbf{w} \in \mathbb{Z}_{\geq 0}^I$ be a pair of dimension vectors. Nakajima varieties provide, in many cases, examples of a symplectic resolution of the form $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w}) \to \mathcal{M}_0(\mathbf{v}, \mathbf{w})$. Here, $\theta \in \mathbb{R}^I$ is a 'stability parameter', and we write $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$ for the Nakajima variety $\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w})$, as defined in Definition 5.1.10 of §5 below. For $\theta = 0$, the variety $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$ is known to be affine, see Theorem 4.5.6(i).

Assume now that θ is chosen to lie outside a certain collection $\{H_j\}$ of 'root hyperplanes' in $\mathbb{R}^I$. Then, under fairly mild conditions, the Nakajima variety $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$ turns out to be a smooth algebraic variety that comes equipped with a natural hyper-Kähler structure. The (algebraic) symplectic structure on $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$ is a part of that hyper-Kähler structure. This part is independent of the choice of the stability parameter θ as long as θ stays within a connected component of the set $\mathbb{R}^I \setminus (\cup_j H_j)$. In contrast, the Kähler structure on $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$ does depend on the choice of θ in an essential way.

Nakajima's varieties encorporate many of the examples described above. For a simple Lie algebra of type $\mathbf{A}$, for instance, *all* symplectic resolutions described in Example 1.2.3 come from appropriate quiver varieties, see [Ma].

Similarly, the minimal resolution of a Kleinian singularity and the resolution $\pi : \mathrm{Hilb}^n(\mathbb{C}^2) \to (\mathbb{C}^2)^n/S_n$, see Example 1.2.4, are also special cases of symplectic resolutions arising from quiver varieties. There are other important examples as well, eg. the ones where the group Γ is a wreath-product.

Quiver varieties provide a unifying framework for all these examples, from both conceptual and technical points of view. Here is an illustration of this.

Remark 1.2.6. The odd cohomology vanishing for the fibers of the Springer resolution, equivalently, for the e-fixed point varieties $\mathcal{B}_e \subset \mathcal{B}$, was standing as an open problem for quite a long time. This problem has been finally solved in [DCLP]. The argument in [DCLP] is quite complicated, in particular, it involves a case-by-case analysis.

The odd cohomology vanishing for the fibers of the map $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w}) \to \mathcal{M}_0(\mathbf{v}, \mathbf{w})$ was proved in [Na4]. Nakajima's proof is based on a standard result saying that rational homology groups of a complete variety that admits a 'resolution of diagonal' in K-theory, cf. [CG, Theorem 5.6.1], is spanned by the fundamental classes of algebraic cycles.[2]

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[2]It is not known whether it is true or not that, for any nilpotent element e in an arbitrary semisimple Lie algebra $\mathfrak{g}$, the variety $\widetilde{\mathcal{S}}_e$, cf. Example 1.2.3, admits a resolution of diagonal in K-theory.

Property (iv) of symplectic resolutions stated earlier in this subsection provides an alternative, more conceptual, unified approach to the odd cohomology vanishing of the fibers of the map π in the above examples.

1.3. **Reminder.** Throughout the paper, the ground field is the field $\mathbb{C}$ of complex numbers.

We fix a quiver Q, i.e., a finite oriented graph, with vertex set I and edge set E. We write Q^{op} for the opposite quiver obtained from Q by reversing the orientation of edges.

For any pair $i, j \in I$, let a_{ij} denote the number of edges of Q going from j to i. The matrix $A_Q := \|a_{ij}\|$ is called the *adjacency matrix* of Q.

On $\mathbb{C}^I$, one has the standard euclidean inner product $\alpha \cdot \beta := \sum_{i \in I} \alpha_i \beta_i$. Thus, the (non-symmetric) bilinear form associated with the adjacency matrix reads

$$A_Q \alpha \cdot \beta = \sum_{x \in E} \alpha_{\mathrm{tail}(x)} \beta_{\mathrm{head}(x)}, \qquad \alpha, \beta \in \mathbb{C}^I.$$

Let $\mathbb{C}I$ be the algebra of $\mathbb{C}$-valued functions on the set I, equipped with pointwise multiplication. This is a finite dimensional semisimple commutative algebra isomorphic to $\oplus_{i \in I} \mathbb{C}$. We write 1_i for the characteristic function of the one point set $\{i\} \subset I$.

Let $\mathbb{C}E$ be a $\mathbb{C}$-vector space with basis E. The vector space $\mathbb{C}E$ has a natural $\mathbb{C}I$-bimodule structure such that, for any edge $x \in E$, we have $1_j \cdot x \cdot 1_i = x$ if $j = \mathrm{tail}(x)$ and $i = \mathrm{head}(x)$, and $1_j \cdot x \cdot 1_i = 0$ otherwise.

One defines the *path algebra of Q* as $\mathbb{C}Q := T_{\mathbb{C}I}(\mathbb{C}E)$, the tensor algebra of the $\mathbb{C}I$-bimodule $\mathbb{C}E$. For each $i \in I$, the element $1_i \in \mathbb{C}I \subset \mathbb{C}Q$ may be identified with the trivial path at the vertex i.

Let B be an arbitrary $\mathbb{C}$-algebra equipped with an algebra map $\mathbb{C}I \to B$, eg. B is a quotient of the path algebra of a quiver. Abusing the notation, we also write 1_i for the image of the element $1_i \in \mathbb{C}I$ in B. Associated with any finite dimensional left B-module M, there is its *dimension vector* $\dim_I M \in \mathbb{Z}_{\geq 0}^I$, such that the ith coordinate of $\dim_I M$ equals $(\dim_I M)_i := \dim(1_i \cdot M)$, where we always write $\dim = \dim_{\mathbb{C}}$.

Note that a left $\mathbb{C}I$-module is the same thing as an I-graded vector space. Given an I-graded finite-dimensional vector space $V = \oplus_{i \in I} V_i$, we let $\mathrm{Rep}(B, V)$ denote the set of algebra homomorphisms $\rho : B \to \mathrm{End}_{\mathbb{C}} V$ such that $\rho|_{\mathbb{C}I}$, the pull-back of ρ to the subalgebra $\mathbb{C}I$, equals the homomorphism coming from the $\mathbb{C}I$-module structure on V. The group $\prod_{i \in I} GL(V_i)$ acts naturally on $\mathrm{Rep}(B, V)$ by 'base change' automorphisms.

Let $\mathbf{v} = (v_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I$ be an I-tuple, to be referred to as a 'dimension vector'. Given an I-graded vector space $V = \oplus_{i \in I} V_i$, such that $\dim V_i = v_i$ for all $i \in I$, we will often abuse the notation and write $GL(v_i)$ for $GL(V_i)$, resp. $\mathrm{Rep}(B, \mathbf{v})$ for $\mathrm{Rep}(B, V)$. In the special case $B = \mathbb{C}Q$, we simplify the notation further and write $\mathrm{Rep}(Q, \mathbf{v}) := \mathrm{Rep}(\mathbb{C}Q, \mathbf{v})$, the space of $\mathbf{v}$-dimensional representations of Q.

We put $G_{\mathbf{v}} := \prod_{i \in I} GL(v_i)$. Thus, $G_{\mathbf{v}}$ is a reductive group, and $\mathrm{Rep}(Q, \mathbf{v})$ is a vector space that comes equipped with a linear $G_{\mathbf{v}}$-action, by base change automorphisms. We have

$$\dim \mathrm{Rep}(Q, \mathbf{v}) = A_Q \mathbf{v} \cdot \mathbf{v}, \qquad \dim G_{\mathbf{v}} = \mathbf{v} \cdot \mathbf{v}. \tag{1.3.1}$$

Note the the subgroup $\mathbb{C}^{\times} \subset G_{\mathbf{v}}$, of *diagonally imbedded* invertible scalar matrices acts trivially on $\mathrm{Rep}(Q, \mathbf{v})$.

We will very often use the following elementary result

Lemma 1.3.2. *Let B be an algebra equipped with an algebra map $\mathbb{C}I \to B$. Then, the isotropy group of any point of $\mathrm{Rep}(B, \mathbf{v})$ is a connected subgroup of $G_{\mathbf{v}}$.*

Proof. Let M be a representation of B, and write $\mathrm{End}_B M$ for the algebra of B-module endomorphisms of M. It is known (and easy to see) that the isotropy group G^M of the point $M \in \mathrm{Rep}(B, \mathbf{v})$ may be identified with the group of invertible elements of the algebra $\mathrm{End}_B M$.

We claim that, more generally, the set $A^\times$ of of invertible elements of any finite dimensional $\mathbb{C}$-algebra A is connected. To see this, we observe that the set A^{sing}, of noninvertible elements of A, is a hypersurface in A given by the equation $\det m_a = 0$, where m_a denotes the operator of left multiplication by an element $a \in A$.

Such a hypersurface has real codimension ≥ 2 in A, hence cannot disconnect A, a real vector space. Therefore, the set $A^\times = A \smallsetminus A^{\mathrm{sing}}$, of invertible elements, must be connected. $\qquad\square$

2. Moduli of representations of quivers

2.1. **Categorical quotients.** Naively, one would like to consider a space of isomorphism classes of representations of Q of some fixed dimension $\mathbf{v}$. Geometrically, this amounts to considering the orbit space $\mathrm{Rep}(Q, \mathbf{v})/G_\mathbf{v}$. Such an orbit space is, however, rather 'badly behaved' in most cases. Usually, it does not have a reasonable Hausdorff topology, for instance.

One way to define a reasonable orbit space is to use a *categorical quotient*

$$\mathrm{Rep}(Q, \mathbf{v})//G_\mathbf{v} := \mathrm{Spec}\,\mathbb{C}[\mathrm{Rep}(Q, \mathbf{v})]^{G_\mathbf{v}},$$

the spectrum of the algebra of $G_\mathbf{v}$-invariant polynomials on the vector space $\mathrm{Rep}(Q, \mathbf{v})$. By definition, $\mathrm{Rep}(Q, \mathbf{v})//G_\mathbf{v}$ is an affine algebraic variety.

To understand the categorical quotient, we recall the following result of Le Bruyn and Procesi, [LBP],

Proposition 2.1.1. *The algebra $\mathbb{C}[\mathrm{Rep}(Q, \mathbf{v})]^{G_\mathbf{v}}$ is generated by the functions $\mathrm{Tr}(p, -) : V \mapsto \mathrm{Tr}(p, V)$, where p runs over the set of oriented cycles in Q of the form $p = p_{i_1, i_2} \cdot p_{i_2, i_3} \cdot \cdots \cdot p_{i_{s-1}, i_s} \cdot p_{i_s, i_1}$, $(p_{ij} \in E)$, and where $\mathrm{Tr}(p, V)$ denotes the trace of the operator corresponding to such a cycle in the representation $V \in \mathrm{Rep}(Q, \mathbf{v})$.*

The above proposition is a simple consequence of H. Weyl's '*first fundamental theorem of Invariant theory*', cf. [Kra]. The proposition yields

Corollary 2.1.2. *For a quiver Q without oriented cycles, one has $\mathbb{C}[\mathrm{Rep}(Q, \mathbf{v})]^{G_\mathbf{v}} = \mathbb{C}$, hence, we have $\mathrm{Rep}(Q, \mathbf{v})//G_\mathbf{v} = pt$.* $\qquad\square$

Combining Proposition 2.1.1 with standard results from invariant theory, cf. [Mu, Theorem 5.9], one obtains the following

Theorem 2.1.3. *Geometric (= closed) points of the scheme $\mathrm{Spec}\,\mathbb{C}[\mathrm{Rep}(Q, \mathbf{v})]^{G_\mathbf{v}}$ are in a natural bijection with $G_\mathbf{v}$-orbits of semisimple representations of Q.* $\qquad\square$

Corrolary 2.1.2 shows that the categorical quotient may often reduce to a point, so a lot of geometric information may be lost.

A better approach to the moduli problem is to use a stability condition and to replace the orbit space $\mathrm{Rep}(Q, \mathbf{v})/G_\mathbf{v}$, or the categorical quotient $\mathrm{Rep}(Q, \mathbf{v})//G_\mathbf{v}$, by an appropriate moduli space of (semi)-stable representations. There is a price to pay: moduli spaces arising in this way do depend on the choice of a stability condition, in general.

2.2. **Reminder on GIT.** The general theory of quotients by a reductive group action via stability conditions has been developed by D. Mumford, and is called Geometric Invariant Theory, cf. [GIT].

To fix ideas, let X be a not necessarily irreducible, affine algebraic G-variety, where G is a reductive algebraic group. Given a rational character (= algebraic group homomorphism) $\chi : G \to \mathbb{C}^\times$, Mumford defines a scheme $X//_\chi G$ in the following way. Let G act on the cartesian product $X \times \mathbb{C}$ by the formula $g : (x, z) \mapsto (gx, \chi(g)^{-1} \cdot z)$ (more generally, the cartesian product $X \times \mathbb{C}$ may be replaced here by the total space of any G-equivariant line bundle on X). The coordinate

ring of $X \times \mathbb{C}$ is the algebra $\mathbb{C}[X \times \mathbb{C}] = \mathbb{C}[X][z]$, of polynomials in the variable z with coefficients in the coordinate ring of X. This algebra has an obvious grading by degree of the polynomial.

Let $A_\chi := \mathbb{C}[X \times \mathbb{C}]^G$ be the subalgebra G-invariants. Clearly, this is a graded subalgebra which is, moreover, a finitely generated algebra by Hilbert's theorem on finite generation of algebras of invariants, cf. [Kra, ch. II, §3.1]. Explicitly, a polynomial $f(z) = \sum_{n=0}^{N} f_n \cdot z^n \in \mathbb{C}[X][z]$ is G-invariant if and only if, for each $n = 0, \ldots, N$, the function f_n is a χ^n-*semi-invariant*, i.e. if and only if one has

$$f_n(g^{-1}(x)) = \chi(g)^n \cdot f_n(x), \quad \forall g \in G, \ x \in X.$$

Write $\chi^n : \ g \mapsto \chi(g)^n$ for the n-th power of the character χ and let $\mathbb{C}[X]^{\chi^n} \subset \mathbb{C}[X]$ be the vector space of χ^n-semi-invariant functions. It is clear that we have

$$A_\chi := \mathbb{C}[X \times \mathbb{C}]^G = \bigoplus_{n \geq 0} \mathbb{C}[X]^{\chi^n},$$

and the direct sum decomposition on the right corresponds to the grading on the algebra A_χ.

Let $X /\!/_\chi G := \mathsf{Proj}\, A_\chi$ be the projective spectrum of the graded algebra A_χ. This is a quasi-projective scheme, called a *GIT quotient* of X by the G-action; the scheme $X /\!/_\chi G$ is reduced, resp. irreducible, whenever X is (since A_χ has no nilpotents, resp. no zero divisors, provided there are no nilpotents resp. no zero divisors, in $\mathbb{C}[X]$).

Put $A_\chi^{>0} := \bigoplus_{n > 0} \mathbb{C}[X]^{\chi^n}$. Let $\mathscr{I}$ be the set of homogeneous ideals $I \subset A_\chi$ such that one has $I \neq A_\chi$ and $A_\chi^{>0} \not\subset I$. An ideal $I \in \mathscr{I}$ is said to be a 'maximal homogeneous ideal' if it is not properly contained in any other ideal $I' \in \mathscr{I}$. Geometric points of the scheme $X /\!/_\chi G$ correspond to the maximal homogeneous ideals.

In general, for $n = 0$, we have $\mathbb{C}[X]^{\chi^n} = \mathbb{C}[X]^G$, is the algebra of G-invariants. Thus, we have a canonical algebra imbedding $\mathbb{C}[X]^G \hookrightarrow A_\chi$ as the degree zero subalgebra. Put another way, the algebra imbedding $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X \times \mathbb{C}]^G = A_\chi$ is induced by the first projection $X \times \mathbb{C} \to X$.

Standard results of algebraic geometry imply that the algebra imbedding $\mathbb{C}[X]^G \hookrightarrow A_\chi$ induces a *projective* morphism of schemes $\pi : \ \mathsf{Proj}\, A_\chi \to \mathsf{Spec}\, \mathbb{C}[X]^G = X /\!/ G$.

Remark 2.2.1. In the special case where $G = \mathbb{C}^\times$ and $A = \mathbb{C}[u_0, u_1, \ldots, u_m]$, is a polynomial algebra, we have $\mathsf{Proj}\, A = \mathbb{P}^m = (\mathbb{C}^{m+1} \smallsetminus \{0\}) / \mathbb{C}^\times$.

More generally, given a reductive group G and a *nontrivial* character $\chi : G \to \mathbb{C}^\times$, put $K := \mathsf{Ker}\, \chi$. Thus, K is a normal subgroup of G and one has $G/K = \mathbb{C}^\times$.

Now, let X be an affine G-variety such that $\mathbb{C}[X]^{\chi^n} = 0$ for any $n < 0$. Let $X /\!/ K = \mathsf{Spec}(\mathbb{C}[X]^K)$ be the categorical quotient of X by the K-action. There is a natural residual action of the group $G/K = \mathbb{C}^\times$ on $X /\!/ K$, equivalently, there is a natural nonnegative grading on the algebra $\mathbb{C}[X]^K$. Then, it is straightforward to show that $X /\!/_\chi G \cong \mathsf{Proj}(\mathbb{C}[X]^K)$. Furthermore, geometric points of the scheme $\mathsf{Proj}(\mathbb{C}[X]^K)$ correspond to $\mathbb{C}^\times$-orbits in $(X /\!/ K) \smallsetminus Y$, where Y denotes the set of $\mathbb{C}^\times$-fixed points in $X /\!/ K$. $\diamondsuit$

Remark 2.2.2. For any character $G \to \mathbb{C}^\times$ and any positive integer $m > 0$, one may view the algebra $A_{m\chi}$ as a graded subalgebra in A_χ via the natural imbedding $A_{m\chi} = \bigoplus_{n \geq 0, \ m|n} \mathbb{C}[X]^{\chi^n} \hookrightarrow A_\chi = \bigoplus_{n \geq 0} \mathbb{C}[X]^{\chi^n}$, called the *Veronese imbedding*. One can show that the Veronese imbedding induces an isomorphism $X /\!/_\chi G \xrightarrow{\sim} X /\!/_{m\chi} G$, of algebraic varieties. $\diamondsuit$

Given a nonzero homogeneous semi-invariant $f \in A_\chi$ we put $X_f := \{x \in X \mid f(x) \neq 0\}$. To get a better understanding of the GIT quotient $X /\!/_\chi G$, one introduces the following definition, see [GIT].

Definition 2.2.3. (i) *A point $x \in X$ is called χ-semistable if there exists $n \geq 1$ and a χ^n-semi-invariant $f \in \mathbb{C}[X]^{\chi^n}$ such that $x \in X_f$.*

(ii) *A point $x \in X$ is called χ-stable if there exists $n \geq 1$ and a χ^n-semi-invariant $f \in \mathbb{C}[X]^{\chi^n}$ such that $x \in X_f$ and, in addition, one has: (1) the action map $G \times X_f \to X_f$ is a closed morphism and (2) the isotropy group of the point x is finite.*

Write X_χ^{ss}, resp. X_χ^s, for the set of semistable, resp. stable, points. Thus, we have $X_\chi^s \subset X_\chi^{ss} \subset X$.

(iii) *Two χ-semistable points x, x' are called S-equivalent if and only if the orbit closures $\overline{G \cdot x}$ and $\overline{G \cdot x'}$ meet in X_χ^{ss}.*

Note that the G-orbit of a stable point is an orbit of maximal dimension, equal to $\dim G$, moreover, such a stable orbit is closed in X_χ^{ss}. Hence, two stable points are S-equivalent if and only if they belong to the same orbit.

By definition, we have that $X_\chi^{ss} = \cup_{\{f, \deg f > 0\}} X_f$ is a G-stable Zariski open subset of X. Furthermore, there is a well defined morphism $F : X_\chi^{ss} \to X//_\chi G$, of algebraic varieties, which is constant on G-orbits. The image of a G-orbit $\mathcal{O} \subset X_\chi^{ss}$ is a point corresponding to the maximal homogeneous ideal $\mathfrak{I}_\mathcal{O} \subset A_\chi$ formed by the functions $f \in A_\chi$ such that $f(\mathcal{O}) = 0$.

One of the basic results of GIT reads

Theorem 2.2.4. (i) *The morphism F induces a natural bijection between the set of S-equivalence classes of G-orbits in X_χ^{ss} and the set of geometric points of the scheme $X//_\chi G$.*

(ii) *The image of the set of stable points is a Zariski open (possibly empty) subset $F(X_\chi^s) \subset X//_\chi G$; moreover, the fibers of the restriction $F : X_\chi^s \to X//_\chi G$ are closed G-orbits of maximal dimension, equal to $\dim G$.*

Example 2.2.5. For the trivial character $\chi = 1$, we have $A_\chi = \mathbb{C}[X]^G[z]$. The regular function $z \in A_\chi$ is a homogeneous degree one regular function that does not vanish on X. Therefore, we have $X = X_z$ and any point $x \in X$ is χ-stable. Such a point is χ-stable if and only if the G-orbit of x is a closed orbit in X of dimension $\dim G$. Furthermore, one has

$$X//_\chi G = \mathsf{Proj}\, A_\chi = \mathsf{Proj}\left(\mathbb{C}[X]^G[z]\right) = \mathsf{Spec}\, \mathbb{C}[X]^G = X//G, \text{ for } \chi = 1.$$

In this case, the canonical map π becomes an isomorphism $X//_\chi G \xrightarrow{\sim} X//G$.

We will frequently use the following result which is, essentially, a consequence of definitions.

Corollary 2.2.6. (i) *Let X be a smooth G-variety such that the isotropy group of any point of X is connected. Then the set $F(X^s)$ is contained in the smooth locus of the scheme $X//_\chi G$.*

(ii) *Assume, in addition, that X is affine and that the G-action on X^{ss} is free. Then any semistable point is stable, the scheme $X//_\chi G$ is smooth. Furthermore, the morphism $F : X^{ss} \to X//_\chi G$ is a principal G-bundle (in ètale topology).* $\qquad\square$

In the situation of (ii) above, one often calls the map F (or the variety $X//_\chi G$) a *universal geometric quotient*.

2.3. **Stability conditions for quivers.** A. King introduced a totally different, purely algebraic, notion of stability for representations of algebras. He then showed that, in the case of quiver representations, his definition of stability is actually equivalent to Mumford's Definition 2.2.3.

To explain King's approach, fix a quiver Q and fix $\theta \in \mathbb{R}^I$. It will become clear shortly that the parameter θ is an analogue of the group character $\chi : G \to \mathbb{C}^\times$, in Mumford's theory.

Let $V = \oplus_{i \in I} V_i$ be a finite dimensional nonzero representation of Q with dimension vector $\dim_I V \in \mathbb{Z}^I$. One defines the *slope* of V by the formula $slope_\theta(V) := (\theta \cdot \dim_I V)/\dim_\mathbb{C} V$, where $\dim_\mathbb{C} V := \sum_{i \in I} \dim V_i$. Using the vector $\theta^+ := (1, 1, \ldots, 1) \in \mathbb{Z}^I$, one can alternatively write $\dim_\mathbb{C} V = \theta^+ \cdot \dim_I V$.

Definition 2.3.1. *A nonzero representation V of Q is said to be θ-semistable if, for any subrepresentation $N \subset V$, we have $slope_\theta(N) \leq slope_\theta(V)$.*

A nonzero representation is called θ-stable if the strict inequality holds for any nonzero proper subrepresentation $N \subset V$.

Example 2.3.2. Let $\theta = 0$. Then, any representation is θ-semistable. Such a representation is θ-stable if and only if it is simple as an $\mathbb{C}Q$-module.

Remark 2.3.3. (i) Our definition of semistability in terms of slopes follows the approach of Rudakov [Ru, §3]. In the case where $\theta \cdot \dim_I V = 0$ the inequality of slopes in Definition 2.3.1 reduces to the condition that $\theta \cdot N \leq 0$. The latter condition, combined with the requirement that $\theta \cdot \dim_I V = 0$, is the original definition of semistability used by King [Ki]. Rudakov's approach is more flexible since it works well without the assumptions that $\theta \cdot \dim_I V = 0$.

(ii) Let $\theta \in \mathbb{R}^I$ and put $\theta' = \theta - c \cdot \theta^+$, where $c \in \mathbb{R}$ is an arbitrary constant. It is easy to see that a representation V is θ-semistable in the sense of Definition 2.3.1 if and only if it is θ'-semistable. On the other hand, given V, one can always find a constant $c \in \mathbb{R}$ such that one has $\theta' \cdot \dim_I V = 0$, see [Ru], Lemma 3.4. $\diamondsuit$

The definition of (semi)stability given above is a special case of a more general approach due to A. King [Ki], who considers the case of an arbitrary associative $\mathbb{C}$-algebra A.

Given such an algebra A, let $K_{\mathrm{fin}}(A)$ denote the Grothendieck group of the category of *finite dimensional* A-modules. This is a free abelian group with the basis formed by the classes of simple finite dimensional A-modules. Note that the assignment $V \mapsto \dim_\mathbb{C} V$ extends to an additive group homomorphism $K_{\mathrm{fin}}(A) \to \mathbb{R}$.

Given any other additive group homomorphism $\phi : K_{\mathrm{fin}}(A) \to \mathbb{R}$ and a nonzero finite dimensional A-module V, one puts $slope_\phi(V) := \phi([V])/\dim_\mathbb{C} V$, where $[V]$ stands for the class of V in $K_{\mathrm{fin}}(A)$. Following King and Rudakov, one says that a finite dimensional A-module V is ϕ-semistable, if for any nonzero A-submodule $N \subset V$, we have $slope_\phi(N) \leq slope_\phi(V)$.

This definition specializes to Definition 2.3.1 as follows. One takes $A := \mathbb{C}Q$. Then, the assignment $[V] \mapsto \dim_I V$ yields a well defined group homomorphism $\dim_I : K_{\mathrm{fin}}(\mathbb{C}Q) \to \mathbb{Z}^I$. Now, for any $\theta \in \mathbb{R}^I$, define a group homomorphism $\phi_\theta : \mathbb{Z}^I \to \mathbb{R}, \ x \mapsto \sum_i \theta_i \cdot x_i$. This yields an obvious isomorphism $\mathbb{R}^I \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Z}^I, \mathbb{R}), \ \theta \mapsto \phi_\theta$. Thus, given $\theta \in \mathbb{R}^I$, one may form a composite homomorphism $K_{\mathrm{fin}}(\mathbb{C}Q) \to \mathbb{Z}^I \to \mathbb{R}, \ [V] \mapsto \theta \cdot \dim_I V$.

For this last homomorphism, the general definition of semistability for A-modules reduces to Definition 2.3.1.

Remark 2.3.4. Assume that the quiver Q has no oriented cycles. Then, it is easy to show that any simple representation V of Q is 1-dimensional, i.e., there exists a vertex $i \in I$ such that $V_i = \mathbb{C}$ and $V_j = 0$ for any $j \neq i$. It follows that the map $\dim_I : K_{\mathrm{fin}}(\mathbb{C}Q) \to \mathbb{Z}^I, \ [V] \mapsto \dim_I V$ is in this case a group isomorphism. $\diamondsuit$

Proposition 2.3.5. *Fix an additive group homomorphism $\phi : K_{fin}(A) \to \mathbb{R}$. Then, finite dimensional ϕ-semistable A-modules form an abelian category. An A-module is ϕ-stable if and only if it is a simple object of this category.*

Proof. The proposition states that, for any pair M, N, of ϕ-semistable A-modules, the kernel, resp. cokernel, of an A-module map $f : M \to N$ is ϕ-semistable again.

To prove this, put $K := \mathrm{Ker}(f)$ and write $slope(-)$ for $slope_\theta(-)$. Then, the imbedding $M/K \hookrightarrow N$ yields $slope(M/K) \leq slope(N) = slope(M)$. Hence, alying [Ru, Lemma 3.2 and Definition 1.1] to the short exact sequence $K \to M \to M/K$, we get $slope(K) \geq slope(M)$. On the other hand, since K is a submodule of M, a semistable module, we have $slope(K) \leq slope(M)$. Thus, we

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obtain that $slope(K) = slope(M)$. It follows that, for any submodule $E \subset K$, we have $slope(E) \leq slope(M) = slope(K)$, since $E \subset M$. Thus, we have proved that K is ϕ-semistable.

Next, write C for the cokernel of the map f. Then, one proves that $slope(C) = slope(N)$ and, moreover, $slope(C) \leq slope(F)$ for any quotient F of C. This implies that C is a ϕ-semistable A-module, see [Ru], Definition 1.6 and the discussion after it.

We leave details to the reader. $\qquad\square$

We return to the quiver setting, and fix a quiver Q with vertex set I. As a corollary of Proposition 2.3.5, we deduce that any θ-semistable representation V, of Q, has a Jordan-Hölder filtration $0 = V_0 \subset V_1 \subset \ldots \subset V_m = V$, by subrepresentations, such that V_k/V_{k-1} is a θ-stable representation for any $k = 1, \ldots, m$. The associated graded representation $\mathrm{gr}^s V := \oplus_k V_k/V_{k-1}$ does not depend, up to isomorphism, on the choice of such a filtration.

To relate Mumford's and King's notions of stability, we associate with an integral vector $\theta = (\theta_i)_{i \in I} \in \mathbb{Z}^I$, a rational character

$$\chi_\theta : \ G_\mathbf{v} \to \mathbb{C}^\times, \quad g = (g_i)_{i \in I} \ \mapsto \ \prod_{i \in I} \det(g_i)^{-\theta_i}.$$

Remark 2.3.6. Fix a representation V, of Q, and put $\mathbf{v} = \dim_I V$. It is clear that the character χ_θ vanishes on the subgroup $\mathbb{C}^\times \subset G_\mathbf{v}$ if and only if we have $\theta \cdot \mathbf{v} = 0$. $\qquad\diamond$

The main result of King relating the two notions of stability reads

Theorem 2.3.7. *For any dimension vector $\mathbf{v}$ and any $\theta \in \mathbb{Z}^I$ such that $\theta \cdot \mathbf{v} = 0$, we have*

(i) *A representation $V \in \mathrm{Rep}(Q, \mathbf{v})$ is χ_θ-semistable, resp. χ_θ-stable, in the sense of Definition 2.2.3 if and only if it is θ-semistable, resp. θ-stable, in the sense of Definition 2.3.1.*

(ii) *A pair V, V', of χ_θ-semistable representations, are S-equivalent in the sense of Definition 2.2.3 if and only if one has $\mathrm{gr}^s V \cong \mathrm{gr}^s V'$.* $\qquad\square$

Let $\mathrm{Rep}^s_\theta(Q, \mathbf{v})$ denote the set of stable, resp. $\mathrm{Rep}^{ss}(Q, \mathbf{v})$ denote the set of semistable, representations of dimension $\mathbf{v}$. We write $\mathcal{R}_\theta(\mathbf{v}) = \mathcal{R}_\theta(Q, \mathbf{v}) := \mathrm{Rep}^{ss}_\theta(Q, \mathbf{v}) /\!/_{\chi_\theta} G_\mathbf{v}$. By Theorem 2.2.4, this is a quasi-projective variety.

Corollary 2.3.8 (A. King). (i) *The group $G_\mathbf{v}/\mathbb{C}^\times$ acts freely on the set $\mathrm{Rep}^s_\theta(Q, \mathbf{v})$, of θ-stable representations. The orbit set $\mathcal{R}^s_\theta(\mathbf{v}) := \mathrm{Rep}^s_\theta(Q, \mathbf{v})/G_\mathbf{v}$ is contained in $\mathcal{R}_\theta(\mathbf{v})$ as a Zariski open (possibly empty) subset.*

(ii) *Assume that Q has no edge loops. Then, the vector $\mathbf{v} \in \mathbb{Z}^I_{\geq 0}$ is a* Schur vector *(i.e. there exists a simple representation of Q of dimension $\mathbf{v}$) for Q if and only if there exists $\theta \in \mathbb{Z}^I$ such that*

$$\theta \cdot \mathbf{v} = 0 \quad \text{and} \quad \mathrm{Rep}^s_\theta(Q, \mathbf{v}) \neq \emptyset.$$

For such a θ, we have $\dim \mathcal{R}^s_\theta(\mathbf{v}) = 1 + \langle A_Q \mathbf{v}, \mathbf{v} \rangle - \mathbf{v} \cdot \mathbf{v}$.

Proof of (i). Let g be an element of the isotropy group of V, such that $g \notin \mathbb{C}^\times$, and let $c \in \mathbb{C}$ be an eigenvalue of g. Then $N := \mathrm{Ker}(g - c\mathrm{Id})$ is a nontrivial subrepresentation of V. Clearly, the group $\mathbb{C}^\times$ acts trivially on N. Hence, we have $\dim_I N \cdot \mathbf{v} = 0$, contradicting the definition of stability. It follows that the group $G_\mathbf{v}/\mathbb{C}^\times$ acts freely on $\mathrm{Rep}^s_\theta(Q, \mathbf{v})$. $\qquad\square$

According to Example 2.2.5, we get

Corollary 2.3.9. *In the special case $\theta = 0$, one has*

$$\mathcal{R}_0(\mathbf{v}) \ = \ \mathrm{Rep}(Q, \mathbf{v}) /\!/ G_\mathbf{v} \ = \ \mathrm{Spec}\, \mathbb{C}[\mathrm{Rep}(Q, \mathbf{v})]^{G_\mathbf{v}}.$$

For any $\theta \in \mathbb{Z}^I$, there is a canonical projective morphism $\pi : \ \mathcal{R}_\theta(Q, \mathbf{v}) \to \mathrm{Rep}(Q, \mathbf{v}) /\!/ G_\mathbf{v}$. $\qquad\square$

Remark 2.3.10. Note that a representation V of Q is θ-semistable if and only if V^*, the dual representation of Q^{op}, is $(-\theta)$-semistable. Thus, taking the dual representation yields canonical isomorphisms

$$\mathrm{Rep}_\theta^{ss}(Q, \mathbf{v}) \xrightarrow{\sim} \mathrm{Rep}_{-\theta}^{ss}(Q^{\mathrm{op}}, \mathbf{v}), \quad \text{resp.} \quad \mathcal{R}_\theta(Q, \mathbf{v}) \xrightarrow{\sim} \mathcal{R}_{-\theta}(Q^{\mathrm{op}}, \mathbf{v}).$$

3. Framings

Our exposition in this section will be close to the one given by Nakajima in [Na5].

3.1. The set $\mathcal{R}_\theta(Q, \mathbf{v})$ is often empty in various interesting cases of quivers Q and dimension vectors $\mathbf{v}$. Introducing a framing is a way to remedy the situation.

To explain this, fix a quiver Q with vertex set I. We introduce another quiver $Q^\heartsuit$, called the *framing of,* Q as follows. The set of vertices of $Q^\heartsuit$ is defined to be $I \sqcup I'$, where I' is another copy of the set I, equipped with the bijection $I \xrightarrow{\sim} I'$, $i \mapsto i'$. The set of edges of $Q^\heartsuit$ is, by definition, a disjoint union of the set of edges of Q and a set of additional edges $\mathbf{j}_i : i \to i'$, from the vertex i to the corresponding vertex i', one for each vertex $i \in I$.

Thus, giving a representation of $Q^\heartsuit$ amounts to giving a representation $\mathbf{x}$, of the original quiver Q, in a vector space $V = \oplus_{i \in I} V_i$ together with a collection of linear maps $V_i \to W_i$, $i \in I$, where $W = \oplus_{i \in I} W_i$ is an additional collection of finite dimensional vector spaces, where W_i is 'placed' at the vertex $i' \in I'$. We let $\mathbf{w} := \dim_I W \in \mathbb{Z}_{\geq 0}^I$ denote the corresponding dimension vector, and write $\mathbf{j} : V \to W$ to denote a collection of linear maps $\mathbf{j}_i : V_i \to W_i$, $i \in I$, as above.

With this notation, a representation of $Q^\heartsuit$ is a pair $(\mathbf{x}, \mathbf{j})$, where $\mathbf{x}$ is a representation of Q in $V = \oplus_i V_i$, and $\mathbf{j} : V \to W$ is arbitrary additional collection of linear maps. Accordingly, dimension vectors for the quiver $Q^\heartsuit$ are elements $\mathbf{v} \times \mathbf{w} \in \mathbb{Z}^I \times \mathbb{Z}^I = \mathbb{Z}^{I \sqcup I'}$. We write $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w}) := \mathrm{Rep}(Q^\heartsuit, \mathbf{v} \times \mathbf{w})$ for the space of representations $(\mathbf{x}, \mathbf{j})$, of $Q^\heartsuit$, of dimension $\dim_I V = \mathbf{v}$, $\dim_I W = \mathbf{w}$.

We define a $G_{\mathbf{v}}$-action on $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ by $g : (\mathbf{x}, \mathbf{j}) \mapsto (g\mathbf{x}g^{-1}, \mathbf{j} \circ g^{-1})$, where we write $\mathbf{j} \circ g^{-1}$ for the collection of maps $V_i \xrightarrow{(g_i)^{-1}} V_i \xrightarrow{\mathbf{j}_i} W_i$.

Remark 3.1.1. The group $G_{\mathbf{v}} = \prod_{i \in I} GL(V_i)$ may be viewed as a subgroup in $G_{\mathbf{v}} \times G_{\mathbf{w}} = \prod_{i \in I} GL(W_i) \times \prod_{i \in I} GL(W_i)$. The later group acts on $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ according to the general rule of §1.3 applied in the case of the quiver $Q^\heartsuit$. The $G_{\mathbf{v}}$-action defined above is nothing but the restriction of the $G_{\mathbf{v}} \times G_{\mathbf{w}}$-action to the subgroup $G_{\mathbf{v}}$.

From now on, we will view $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ as a $G_{\mathbf{v}}$-variety, and will ignore the action of the other factor, the group $G_{\mathbf{w}}$. $\diamond$

There is a slightly different but equivalent point of view on framings, discovered by Crawley-Boevey [CB1], p.261. Given a quiver Q, with vertex set I, and a dimension vector $\mathbf{w} = (w_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I$, Crawley-Boevey considers a quiver $Q^{\mathbf{w}}$ with the vertex set equal to $I \sqcup \{\infty\}$, where ∞ is a new additional vertex. The set of edges of the quiver $Q^{\mathbf{w}}$ is obtained from the set of edges of Q by adjoing w_i additional edges $i \to \infty$, for each vertex $i \in I$.

Next, associated with any dimension vector $\mathbf{v} = (v_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^I$, introduce a dimension vector $\widehat{\mathbf{v}} \in \mathbb{Z}_{\geq 0}^{I \sqcup \{\infty\}}$ such that $\widehat{v}_i := v_i$ for any $i \in I$, and $v_\infty := 1$. We have a natural group imbedding $G_{\mathbf{v}} \hookrightarrow G_{\widehat{\mathbf{v}}}$ that sends an element $g = (g_i) \in \prod_{i \in I} GL(v_i)$ to the element $\widehat{g} = (\widehat{g}_i)_{i \in I \sqcup \{\infty\}} \in \prod_{i \in I \sqcup \{\infty\}} GL(\widehat{v}_i)$, where $\widehat{g}_i := g_i$ for any $i \in I$ and $g_\infty := \mathrm{Id}$. Note that this imbedding induces an isomorphism $G_{\mathbf{v}} \xrightarrow{\sim} G_{\widehat{\mathbf{v}}}/\mathbb{C}^\times$. Thus, we may (and will) view $\mathrm{Rep}(Q^{\mathbf{w}}, \widehat{\mathbf{v}})$ as a $G_{\mathbf{v}}$-variety via that isomorphism.

Let $\widehat{V} = \oplus_{\{I \sqcup \{\infty\}\}} V_i$, resp. $W = \oplus_{i \in I} W_i$, be a vector space such that $\dim_I \widehat{V} = \widehat{\mathbf{v}}$, resp. $\dim_I W = \mathbf{w}$. We identify V_∞ with $\mathbb{C}$, a 1-dimensional vector space with a fixed base vector.

For each $i \in I$, we choose a basis of the vector space W_i. Then, given any collection of w_i linear maps $V_i \to V_\infty$ (corresponding to the w_i edges $i \to \infty$ of the quiver $Q^{\mathbf{w}}$), one can use the basis in W_i to assemble these maps into a single linear map $\mathbf{j}_i : V_i \to W_i$. This way, we see that any representation of the quiver $Q^{\mathbf{w}}$ in the vector space $\widehat{V}$ gives rise to a representation of the quiver $Q^{\heartsuit}$, that is, to a point in $\mathrm{Rep}(Q^{\heartsuit}, V, W)$. The resulting map $\mathrm{Rep}(Q^{\mathbf{w}}, \widehat{\mathbf{v}}) \to \mathrm{Rep}(Q^{\heartsuit}, \mathbf{v}, \mathbf{w})$ is a $G_{\mathbf{v}}$-equivariant vector space isomorphism that depends on the choice of basis of the vector space W. However, the morphisms corresponding to different choices of basis are obtained from each other by composing with an invertible linear map $g : \mathrm{Rep}(Q^{\heartsuit}, \mathbf{v}, \mathbf{w}) \to \mathrm{Rep}(Q^{\heartsuit}, \mathbf{v}, \mathbf{w})$ that comes from the action on $\mathrm{Rep}(Q^{\heartsuit}, \mathbf{v}, \mathbf{w})$ of an element g of the group $G_{\mathbf{w}}$.

3.2. **Stability for framed representations.** We may apply the general notion of stability in GIT, cf. Definition 2.2.3, in the special case of the $G_{\mathbf{v}}$-action on the variety $\mathrm{Rep}(Q^{\heartsuit}, \mathbf{v}, \mathbf{w})$ and a character $\chi_\theta : G_{\mathbf{v}} \to \mathbb{C}^\times$.

The notion of (semi)stability for *framed* representations of the quiver $Q^{\heartsuit}$ in the sense of Definition 2.2.3 may *not* agree with the notion of (semi)stability for representations of $Q^{\heartsuit}$ in the sense of Definition 2.3.1. This is because the general Definition 2.2.3 refers to a choice of group action. Considering a representation of $Q^{\heartsuit}$ as an ordinary representation without framing refers implicitly to the action of the group $G = G_{\mathbf{v}} \times G_{\mathbf{w}}$, while considering the same representation as a framed representation refers to the action of the group $G = G_{\mathbf{v}}$.

Let $\theta \in \mathbb{R}^I$. A convenient way to relate the θ-stability of framed representations to King's results is to use the $G_{\mathbf{v}}$-equivariant isomorphism $\mathrm{Rep}(Q^{\mathbf{w}}, \widehat{\mathbf{v}}) \to \mathrm{Rep}(Q^{\heartsuit}, \mathbf{v}, \mathbf{w})$ described at the end of the previous subsection. Recall that we have $G_{\mathbf{v}} \cong G_{\widehat{\mathbf{v}}}/\mathbb{C}^\times$, where the subgroup $\mathbb{C}^\times \subset G_{\widehat{\mathbf{v}}}$, of scalar matrices, acts trivially on $\mathrm{Rep}(Q^{\mathbf{w}}, \widehat{\mathbf{v}})$. Further, define a vector $\widehat{\theta} \in \mathbb{R}^{I \sqcup \{\infty\}}$ by $\widehat{\theta}_i := \theta_i$ for any $i \in I$ and $\widehat{\theta}_\infty := -\sum_{i \in I} \theta_i \cdot v_i$. In this way, all the results of §2.3 concerning $\widehat{\theta}$-stability for the $G_{\widehat{\mathbf{v}}}$-action on $\mathrm{Rep}(Q^{\mathbf{w}}, \widehat{\mathbf{v}})$ may be transferred into corresponding results concerning θ-stability for the $G_{\mathbf{v}}$-action on $\mathrm{Rep}(Q^{\heartsuit}, \mathbf{v}, \mathbf{w})$.

Remark 3.2.1. Note that the $G_{\mathbf{v}}$-action on $\mathrm{Rep}(Q^{\heartsuit}, \mathbf{v}, \mathbf{w})$ does *not* factor through the quotient $G_{\mathbf{v}}/\mathbb{C}^\times$.

Observe also that our definition of the vector $\widehat{\theta}$ insures that one has $\widehat{\mathbf{v}} \cdot \widehat{\theta} = 0$. $\Diamond$

Below, we restrict ourselves to the special case of the vector

$$\theta^+ := (1, 1, \ldots, 1) \in \mathbb{Z}^I_{>0}. \tag{3.2.2}$$

We write 'semistable' for 'θ^+-semistable', and let $\mathrm{Rep}^{ss}(Q^{\heartsuit}, \mathbf{v}, \mathbf{w})$ denote the set of semistable representations of $Q^{\heartsuit}$ of dimension $(\mathbf{v}, \mathbf{w})$. Further let $\mathcal{R}(\mathbf{v}, \mathbf{w}) := \mathrm{Rep}^{ss}(Q^{\heartsuit}, \mathbf{v}, \mathbf{w}) /\!/_{\chi_{\theta^+}} G_{\mathbf{v}}$ be the corresponding GIT quotient.

We have the following result.

Lemma 3.2.3. (i) *A representation* $(\mathbf{x}, \mathbf{j}) \in \mathrm{Rep}(Q^{\heartsuit}, \mathbf{v}, \mathbf{w})$, *in vector spaces* (V, W), *is semistable (with respect to the $G_{\mathbf{v}}$-action on $\mathrm{Rep}(Q^{\heartsuit}, \mathbf{v}, \mathbf{w})$) if and only if there is no nontrivial subrepresentation* $V' \subset V$, *of the quiver Q, contained in* $\mathrm{Ker}\,\mathbf{j}$.

(ii) *The group $G_{\mathbf{v}}$ acts freely on the set* $\mathrm{Rep}^{ss}(Q^{\heartsuit}, \mathbf{v}, \mathbf{w})$, *moreover, any semistable representation is automatically stable.*

(iii) $\mathcal{R}(\mathbf{v}, \mathbf{w})$ *is a smooth quasi-projective variety and the canonical map* $\mathrm{Rep}^{ss}(Q^{\heartsuit}, \mathbf{v}, \mathbf{w})/G_{\mathbf{v}} \to \mathcal{R}(\mathbf{v}, \mathbf{w})$ *is a bijection of sets.*

Proof. Part (i) follows by directly applying Theorem 2.3.7 to the $G_{\widehat{\mathbf{v}}}$-action on $\mathrm{Rep}(Q^{\mathbf{w}}, \widehat{\mathbf{v}})$. To prove (ii), let $g \neq \mathrm{Id}$ be an element of the isotropy group of a representation $V \in \mathrm{Rep}^{ss}(Q^{\heartsuit}, \mathbf{v}, \mathbf{w})$. Then, $V' := \mathrm{Ker}(g - \mathrm{Id})$ is a subrepresentation of V that violates the condition of part (i). Part (ii) follows from this. Part (iii) follows from (ii) by Corollary 2.2.6. $\square$

Proposition 3.2.4 (King). (i) *Assume that Q has no edge loops and the set of θ-stable $(\mathbf{v}, \mathbf{w})$-dimensional framed representations of Q is nonempty. Then, we have*

$$\dim \mathcal{R}_\theta(\mathbf{v}, \mathbf{w}) = \mathbf{v}\cdot\mathbf{w} + A_Q\mathbf{v}, \mathbf{v} - \mathbf{v}\cdot\mathbf{v}, \tag{3.2.5}$$

(ii) *If Q has no oriented cycles then the scheme $\mathcal{R}_\theta(\mathbf{v}, \mathbf{w})$ is a (smooth) projective variety.*

Sketch of proof of formula (3.2.5). Observe first that we have

$$\dim \operatorname{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w}) = \mathbf{w} \cdot \mathbf{v} + A_Q\mathbf{v} \cdot \mathbf{v}.$$

Furthermore, one shows that, for θ as in the statement of the proposition. the set $\operatorname{Rep}_\theta^s(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ is Zariski open in $\operatorname{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$. The $G_\mathbf{v}$-action on $\operatorname{Rep}_\theta^s(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ being free, we compute

$$\begin{aligned}
\dim \mathcal{R}_\theta(\mathbf{v}, \mathbf{w}) &= \dim \big(\operatorname{Rep}^s(Q^\heartsuit, \mathbf{v}, \mathbf{w})/G_\mathbf{v} \big) \\
&= \dim \operatorname{Rep}^s(Q^\heartsuit, \mathbf{v}, \mathbf{w}) - \dim G_\mathbf{v} = \dim \operatorname{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w}) - \dim G_\mathbf{v} \\
&= \mathbf{w} \cdot \mathbf{v} + A_Q\mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{v}. \quad \square
\end{aligned}$$

Example 3.2.6 (Jordan quiver). Let Q be a quiver with a single vertex and a single edge-loop at this vertex. For any positive integers $n, m \in \mathbb{Z}^I = \mathbb{Z}$, we have $\operatorname{Rep}(Q, n) = \operatorname{End}\mathbb{C}^n$. Further, we have

$$Q^\heartsuit : \quad \circlearrowleft \bullet \xrightarrow{\ \mathbf{j}\ } \bullet.$$

Hence, we get $\operatorname{Rep}(Q^\heartsuit, n, m) = \operatorname{End}\mathbb{C}^n \times \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^m)$.

First, let $m = 0$, so we are considering representations of Q, not of $Q^\heartsuit$. It is clear that, for $\theta = \theta^+ = 1$, any n-dimensional representation of Q is θ-semistable. There are no stable representations unless $n \leq 1$..

Let $\mathbb{S}_n$ denote the Symmetric group and let $\mathbb{C}^n/\mathbb{S}_n$ be the set of unordered n-tuples of complex numbers viewed as an affine variety. The map sending an $n \times n$-matrix to the (unordered) n-tuple of its eigenvalues yields an isomorphism $\mathcal{R}(n) = \operatorname{Rep}(Q, n)//GL_n \xrightarrow{\sim} \mathbb{C}^n/\mathbb{S}_n$.

Now, take $m = 1$, so we get $\operatorname{Rep}(Q^\heartsuit, n, m) = \operatorname{End}\mathbb{C}^n \times \operatorname{Hom}(\mathbb{C}^n, \mathbb{C})$. A pair $(\mathbf{x}, \mathbf{j}) \in \operatorname{End}\mathbb{C}^n \times (\mathbb{C}^n)^*$ is semistable if and only if the linear function $\mathbf{j} : \mathbb{C}^n \to \mathbb{C}$ is a *cyclic vector* for $\mathbf{x}^* : (\mathbb{C}^n)^* \to (\mathbb{C}^n)^*$, the dual operator.

It is known that the GL_n-action on the set $\operatorname{Rep}^s(Q^\heartsuit, n, 1)$, of such pairs $(\mathbf{x}, \mathbf{j})$, is free. Moreover, sending $(\mathbf{x}, \mathbf{j})$ to the unordered n-tuple of the eigenvalues of $\mathbf{x}$ yields a bijection between the set of GL_n-orbits in $\operatorname{Rep}^s(Q^\heartsuit, n, 1)$ and $\mathbb{C}^n/\mathbb{S}_n$. Thus, in this case, we have isomorphisms $\mathcal{R}(Q^\heartsuit, n, 1) \cong \mathcal{R}^s(n) \cong \mathbb{C}^n/\mathbb{S}_n$.

Example 3.2.7 (Type **A** Dynkin quiver).

$$Q : \quad \overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longleftarrow \cdots \longleftarrow \overset{n-2}{\bullet} \longleftarrow \overset{n-1}{\bullet} \longleftarrow \overset{n}{\bullet}$$

In this case, we have $I = \{1, 2, \ldots, n\}$ and $\operatorname{Rep}(Q, \mathbf{v})//GL_\mathbf{v} = pt$, since Q has no oriented cycles.

We let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (r, 0, 0, \ldots, 0)$, where $r > v_1 > v_2 > \ldots > v_n > 0$, is a strictly decreasing sequence of positive integers. An element of $\operatorname{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ has the form $(\mathbf{x}, \mathbf{j})$, where $\mathbf{x} = (x_{i-1,i} : \mathbb{C}^{v_i} \to \mathbb{C}^{v_{i-1}})_{i=2,\ldots,n}$, and the only nontrivial component of $\mathbf{j}$ is a linear map $j := \mathbf{j}_1 : \mathbb{C}^{v_1} \to \mathbb{C}^r$.

Observe that the collection of vector spaces

$$F_i := \operatorname{Image}(j \circ \mathbf{x}_{21} \circ \ldots \circ \mathbf{x}_{i-1,i}) \subset \mathbb{C}^r, \qquad i = 1, \ldots, n,$$

form an n-step partial flag, $F_1 \subset F_2 \subset \ldots \subset F_n = \mathbb{C}^r$, in $\mathbb{C}^r$. Now, the stability condition amounts, in this case, to the *injectivity* of each of the maps j, $\mathbf{x}_{12}, \ldots, \mathbf{x}_{n-1,n}$. It follows that we have

$$\dim F_i = \dim \operatorname{Image}(j \circ \mathbf{x}_{21} \circ \ldots \circ \mathbf{x}_{i-1,i}) = \dim \mathbb{C}^{v_i} = v_i.$$

Let $\mathcal{F}(n, W)$ be the variety formed by n-step partial flags $F = (F_1 \subset F_2 \subset \ldots \subset F_n = W)$, such that $\dim F_i = v_i, \forall i \in I$. In this way, for the corresponding moduli space, one obtains an isomorphism $\mathcal{R}(\mathbf{v}, \mathbf{w}) \cong \mathcal{F}(n, W)$. In particular, $\mathcal{R}(\mathbf{v}, \mathbf{w})$ is a smooth projective variety, in accordance with Proposition 3.2.4(ii).

4. Hamiltonian reduction for representations of quivers

4.1. **Symplectic geometry.** To motivate later constructions, we first remind a few basic definitions.

Let X be a smooth manifold, write $T^*X \to X$ for the the cotangent bundle on X. The total space T^*X, of the cotangent bundle, comes equipped with a canonical symplectic structure, i.e. there is a canonically defined nondegenerated closed 2-form ω on T^*X.

In the case where X is a vector space, the only case we will use below, we have $T^*X = X \times X^*$, where X^* denotes the vector space dual to X. The canonical symplectic structure on $X \times X^*$ is given, in this special case, by a constant 2-form defined by the formula

$$\omega(x \times x^*, \ y \times y^*) := \langle y^*, x \rangle - \langle x^*, y \rangle, \quad \forall x, y \in X, \ x^*, y^* \in X^*, \tag{4.1.1}$$

where $\langle -, - \rangle$ stands for the canonical pairing between a vector space and the dual vector space.

Now, let a Lie group G act on an arbitrary smooth manifold X. Let $\mathfrak{g}$ be the Lie algebra of G. Given $u \in \mathfrak{g}$, write $\vec{u}$ for the vector field on X corresponding to the 'infinitesimal u-action' on X, and let $\vec{u}_x$ be the value of that vector field at a point $x \in X$.

Associated with the G-action on X, there is a natural G-action on T^*X and a canonical *moment map*

$$\mu : \ T^*X \to \mathfrak{g}^*, \quad \alpha_x \mapsto \mu(\alpha), \quad \text{defined by } \mathfrak{g}^* \ni \mu(\alpha_x) : u \mapsto \langle \alpha, \vec{u}_x \rangle, \tag{4.1.2}$$

where $\alpha_x \in T_x^*X$ stands for a covector at a point $x \in X$.

The following properties of the map (4.1.2) are straightforward consequences of the definitions.

Proposition 4.1.3. (i) *If the group G is* connected *then the moment map is G-equivariant, i.e. it intertwines the G-action on T^*X and the coadjoint G-action on $\mathfrak{g}^*$.*

(ii) *Writing T_Y^*X for the conormal bundle of a submanifold $Y \subset X$, one has*

$$\mu^{-1}(0) = \bigcup_{Y \in X/G} T_Y^*(X). \tag{4.1.4}$$

Here, X/G stands for the set of G-orbits on X.

From the last formula one easily derives the following result.

Corollary 4.1.5. *Assume that the Lie group G acts freely on X, and that the orbit space X/G is a well defined smooth manifold. Then,*
- *The G-action on T^*X is free, and the moment map (4.1.2) is a submersion.*
- *For any coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$, the orbit space $\mu^{-1}(\mathcal{O})/G$ has a natural structure of smooth symplectic manifold.*
- *For $\mathcal{O} = \{0\}$, there is, in addition, a canonical symplectomorphism*

$$T^*(X/G) \cong \mu^{-1}(0)/G. \tag{4.1.6}$$

Formula (4.1.6) explains the importance of the zero fiber of the moment map. Later on, we will consider quotients of $\mu^{-1}(0)$ by the group action in situations where the group action on X is no longer free, so the naive orbit set X/G can not be equipped with a reasonable structure of a manifold. In those cases, various quotiets of $\mu^{-1}(0)$ by G involving stability conditions serve as substitutes for the contangent bundle on a nonexisting space X/G.

The above discussion was in the framework of differential geometry, where 'manifold' means a C^∞-manifold. There are similar constructions and results in the algebraic geometric framework where G now stands for an affine algebraic group and X stands for a G-variety.

For any affine algebraic group G, the differential of a rational group homomorphism $G \to \mathbb{C}^\times$ gives a linear function $\mathfrak{g} \to \mathbb{C}$, i.e. a point $\lambda \in \mathfrak{g}^*$. The points of $\mathfrak{g}^*$ arising in this way are automatically fixed by the coadjoint action of G on $\mathfrak{g}^*$. If the group G is connected, then the corresponding fiber $\mu^{-1}(\lambda)$ is necessarily a G-stable subvariety, by Proposition 4.1.3(i). The varieties of that form play the role of 'twisted cotangent bundles' on X/G, cf. [CG], Proposition 1.4.14 and discussion after it. These varieties share many features of the zero fiber of the moment map.

The following elementary result will be quite useful in applications to quiver varieties.

Lemma 4.1.7. Let $\lambda \in \mathfrak{g}^*$ be a fixed point of the coadjoint action of a connected group G, and let G act on a manifold X with an associated moment map μ as in (4.1.2). Then, the following holds:

A geometric point $\alpha \in \mu^{-1}(\lambda)$ is a smooth point of the scheme theoretic fiber $\mu^{-1}(\lambda)$ if and only if α has finite isotropy in G. In such a case, the symplectic form on T^*X induces a nondegenerate bilinear form on the vector space $T_\alpha(T^*X)/\operatorname{Lie} G^\alpha$.

Proof. Put $M := T^*X$, for short, let $\alpha \in M$, and write $G^\alpha \subset G$ for the isotropy group of the point α. Further, let $d_\alpha\mu : T_\alpha M \to \mathfrak{g}^*$ stand for the differential of the moment map μ at the point α.

Now let $u \in \mathfrak{g}$ and write $\vec{u}_\alpha \in T_\alpha M$ for the tangent vector corresponding to the infinitesimal u-action on M. Also, one may view $u \in \mathfrak{g}$ as a linear function on $\mathfrak{g}^*$. The crucial observation, that follows directly from the definition of the moment map, cf. (4.1.2), is that one has

$$\langle d_\alpha\mu(v),\ u\rangle = d_\alpha(u \circ \mu)(v) = \omega(\vec{u}_\alpha, v), \quad \forall u \in \mathfrak{g},\ v \in T_\alpha^* M. \tag{4.1.8}$$

Using this, we deduce

$$
\begin{aligned}
G^\alpha \text{ is finite} \quad &\Longleftrightarrow \quad \operatorname{Lie} G^\alpha = 0 \\
&\Longleftrightarrow \quad \vec{u}_\alpha \neq 0 \text{ for any } u \neq 0 \\
&\Longleftrightarrow \quad \text{There is no } u \in \mathfrak{g},\ u \neq 0, \text{ such that } \langle d_\alpha\mu(v),\ u\rangle = \omega(\vec{u}_\alpha, v) = 0 \\
&\Longleftrightarrow \quad d_\alpha\mu \text{ is surjective} \\
&\Longleftrightarrow \quad \alpha \text{ is a smooth point.}
\end{aligned}
$$

This proves the first statement of the lemma. The second statement easily follows from (4.1.8) by similar arguments. We leave details to the reader. $\qquad\square$

4.2. Fix a finite set I and a dimension vector $\mathbf{v} = (v_i)_{i \in I} \in \mathbb{Z}^I$.

From now on, we specialize to the case where the algebraic group G is a product of general linear groups, ie. is a group of the form $G_\mathbf{v} = \prod_{i \in I} GL(v_i)$. Thus, we have $\mathfrak{g}_\mathbf{v} := \operatorname{Lie} G_\mathbf{v} = \oplus_{i \in I} \mathfrak{gl}(v_i)$. The center of each summand $\mathfrak{gl}(v_i)$ is a 1-dimensional Lie algebra of scalar matrices. Therefore, the center of $\mathfrak{g}_\mathbf{v}$ may be identified with the vector space $\mathbb{C}^I$.

Observe further that any Lie algebra homomorphism $\mathfrak{g}_\mathbf{v} \to \mathbb{C}$ has the form $\mathbf{x} = (x_i)_{i \in I} \mapsto \sum_{i \in I} \lambda_i \cdot \operatorname{Tr} x_i$. We deduce that the fixed point set of the coadjoint $G_\mathbf{v}$-action on $\mathfrak{g}_\mathbf{v}^*$ is a vector space $\mathbb{C}^I \subset \mathfrak{g}_\mathbf{v}^*$. Explicitly, an element $\lambda = (\lambda_i)_{i \in I} \in \mathbb{C}^I$ corresponds to the point in $\mathfrak{g}_\mathbf{v}^*$ given by the linear function $\mathbf{x} \mapsto \lambda \cdot \mathbf{x} = \sum_{i \in I} \lambda_i \cdot \operatorname{Tr} x_i$, on $\mathfrak{g}_\mathbf{v}$.

4.3. **The double $\overline{Q}$.** Given a quiver Q, let $\overline{Q} = Q \sqcup Q^{\mathrm{op}}$ be the *double* of Q, the quiver that has the same vertex set as Q and whose set of edges is a disjoint union of the sets of edges of Q and of Q^{op}, an opposite quiver. Thus, for any edge $x \in Q$, there is a reverse edge $x^* \in Q^{\mathrm{op}} \subset \overline{Q}$.

Definition 4.3.1. *For any $\lambda = (\lambda_i)_{i \in I} \in \mathbb{C}^I$, let $\Pi_\lambda = \Pi_\lambda(\overline{Q})$ be a quotient of the path algebra $\mathbb{C}\overline{Q}$, of the double quiver $\overline{Q}$, by the two-sided ideal generated by the following element*

$$\sum_{x \in Q} (xx^* - x^*x) - \sum_{i \in I} \lambda_i \cdot 1_i \ \in \mathbb{C}\overline{Q}.$$

Thus, $\Pi_\lambda(\overline{Q})$ is an associative algebra called preprojective algebra *of Q with parameter λ.*

The defining relation for the preprojective algebra may be rewritten more explicitly as a collection of relations, one for each vertex $i \in I$, as follows:

$$\sum_{\{x \in Q:\ \mathrm{head}(x)=i\}} xx^* \ - \ \sum_{\{x \in Q:\ \mathrm{tail}(x)=i\}} x^*x \ = \ \lambda_i \cdot 1_i, \qquad i \in I.$$

Clearly, one has $\mathrm{Rep}(\overline{Q}, \mathbf{v}) \cong \mathrm{Rep}(Q, \mathbf{v}) \times \mathrm{Rep}(Q^{\mathrm{op}}, \mathbf{v})$. We will write a point of $\mathrm{Rep}(\overline{Q}, \mathbf{v})$ as a pair $(\mathbf{x}, \mathbf{y}) \in \mathrm{Rep}(Q, \mathbf{v}) \times \mathrm{Rep}(Q^{\mathrm{op}}, \mathbf{v})$.

Recall that, for any pair, E, F, of finite dimensional vector spaces, there is a canonical perfect pairing

$$\mathrm{Hom}(E, F) \times \mathrm{Hom}(F, E) \to \mathbb{C}, \quad f \times f' \mapsto \mathrm{Tr}(f \circ f') = \mathrm{Tr}(f' \circ f).$$

Using this pairing, one obtains canonical isomorphisms of vector spaces

$$\mathrm{Rep}(Q^{\mathrm{op}}, \mathbf{v}) \cong \mathrm{Rep}(Q, \mathbf{v})^*, \quad \text{resp.} \quad \mathfrak{g}_{\mathbf{v}} \cong \mathfrak{g}_{\mathbf{v}}^*. \tag{4.3.2}$$

We deduce the following isomorphisms

$$\mathrm{Rep}(\overline{Q}, \mathbf{v}) \cong \mathrm{Rep}(Q, \mathbf{v}) \times \mathrm{Rep}(Q, \mathbf{v})^* \cong T^*\big(\mathrm{Rep}(Q, \mathbf{v})\big). \tag{4.3.3}$$

The natural $G_{\mathbf{v}}$-action on $\mathrm{Rep}(\overline{Q}, \mathbf{v})$ corresponds, via the isomorphisms above, to the $G_{\mathbf{v}}$-action on the cotangent bundle induced by the $G_{\mathbf{v}}$-action on $\mathrm{Rep}(Q, \mathbf{v})$. Associated with the latter action, there is a moment map μ. It is given by the following explicit formula, a special case of formula (4.1.2):

$$\mu: \ \mathrm{Rep}(\overline{Q}, \mathbf{v}) = T^*\big(\mathrm{Rep}(Q, \mathbf{v})\big) \ \longrightarrow \ \mathfrak{g}_{\mathbf{v}}^* = \mathfrak{g}_{\mathbf{v}}, \quad (\mathbf{x}, \mathbf{y}) \mapsto \sum (x \circ y - y \circ x) \ \in \ \mathfrak{g}_{\mathbf{v}}.$$

We explain the above formula in the simplest case of the Jordan quiver.

Example 4.3.4. Let Q be a quiver with one vertex and one edge-loop. Then, $\overline{Q}$ is a quiver with a single vertex and two edge-loops at that vertex. Thus, given a positive integer $\mathbf{v} \in \mathbb{Z}^I = \mathbb{Z}$, we have $\mathrm{Rep}(\overline{Q}, \mathbf{v}) = \mathfrak{gl}_{\mathbf{v}} \times \mathfrak{gl}_{\mathbf{v}}$. The action of the group $G_{\mathbf{v}}$ on the space $\mathrm{Rep}(\overline{Q}, \mathbf{v})$ becomes, in this case, the $\mathrm{Ad}\, GL_{\mathbf{v}}$-diagonal action on pairs of $(\mathbf{v} \times \mathbf{v})$-matrices.

Further, the isomorphism $\mathfrak{g}_{\mathbf{v}} \xrightarrow{\sim} \mathfrak{g}_{\mathbf{v}}^*$, resp. $\mathrm{Rep}(Q^{\mathrm{op}}, \mathbf{v}) \xrightarrow{\sim} \mathrm{Rep}(Q, \mathbf{v})^*$, sends a matrix $\mathbf{x} \in \mathfrak{g}_{\mathbf{v}}$ to a linear function $\mathbf{y} \mapsto \mathrm{Tr}(\mathbf{x} \cdot \mathbf{y})$. Hence, in the notation of §4, for any $u \in \mathfrak{gl}_{\mathbf{v}}$, we have $\vec{u} = \mathrm{ad}\, u$.

Now, according to definitions, see formula (4.1.2), the moment map sends a point $(\mathbf{x}, \mathbf{y}) \in T^*(\mathfrak{gl}_{\mathbf{v}}) = \mathfrak{gl}_{\mathbf{v}} \times \mathfrak{gl}_{\mathbf{v}}$ to a linear function

$$\mu(\mathbf{x}, \mathbf{y}): \ \mathfrak{gl}_{\mathbf{v}} \to \mathbb{C}, \quad u \mapsto \langle \mathbf{y}, \ \vec{u}_{\mathbf{x}} \rangle = \langle \mathbf{y}, \ \mathrm{ad}\, u(\mathbf{x}) \rangle = \mathrm{Tr}\big(\mathbf{y} \cdot [u, \mathbf{x}]\big) = \mathrm{Tr}\big([\mathbf{x}, \mathbf{y}] \cdot u\big).$$

We see that the linear function $\mu(\mathbf{x}, \mathbf{y}) \in \mathfrak{gl}_{\mathbf{v}}^*$ corresponds, under the isomorphism $\mathfrak{gl}_{\mathbf{v}}^* \xrightarrow{\sim} \mathfrak{gl}_{\mathbf{v}}$, to the matrix $[\mathbf{x}, \mathbf{y}]$. We conclude that the moment map for the $\mathrm{Ad}\, GL_{\mathbf{v}}$-diagonal action on $T^*(\mathfrak{gl}_{\mathbf{v}}) = \mathfrak{gl}_{\mathbf{v}} \times \mathfrak{gl}_{\mathbf{v}}$ has the following final form

$$\mu: \ \mathfrak{gl}_{\mathbf{v}} \times \mathfrak{gl}_{\mathbf{v}} \ \longrightarrow \ \mathfrak{gl}_{\mathbf{v}}, \quad \mathbf{x} \times \mathbf{y} \mapsto [\mathbf{x}, \mathbf{y}].$$

This is nothing but the general formula (4.3.3) in a special case of the Jordan quiver Q. $\qquad \diamondsuit$

In general, it is clear from Definition 4.3.1 that, inside $\mathrm{Rep}(\mathbf{v}, \overline{Q})$, one has an equality:

$$\mathrm{Rep}(\Pi_\lambda, \mathbf{v}) = \mu^{-1}(\lambda) := \{(\mathbf{x}, \mathbf{y}) \in \mathrm{Rep}(\overline{Q}, \mathbf{v}) \mid [\mathbf{x}, \mathbf{y}] = \lambda\}, \qquad \lambda \in \mathbb{C}^I. \tag{4.3.5}$$

This is, in fact, an isomorphism of schemes.

Remark 4.3.6. Observe that, for any Π_λ-representation V of dimension $\mathbf{v}$, in view of the defining relation for the preprojective algebra, one must have

$$\lambda \cdot \mathbf{v} = \sum_{i \in I} \lambda_i \cdot \mathrm{Tr}_V(1_i) = \mathrm{Tr}_V \left(\sum_{i \in I} \lambda_i 1_i \right) = \mathrm{Tr}_V \left(\sum_{x \in Q} (xx^* - x^*x) \right) = 0,$$

where in the last equation we have used that the trace of any commutator vanishes. We deduce that the algebra Π_λ has no $\mathbf{v}$-dimensional representations unless $\lambda \cdot \mathbf{v} = 0$.

This is consistent with (4.3.5). Indeed, the group $\mathbb{C}^\times \subset G_\mathbf{v}$ acts trivially on $\mathrm{Rep}(\overline{Q}, \mathbf{v})$, hence the image of the moment map μ is contained in the hyperplane $(\mathrm{Lie}\, \mathbb{C}^\times)^\perp \subset \mathfrak{g}_\mathbf{v}^*$. Therefore, the fiber $\mu^{-1}(\lambda)$ over a point $\lambda \in \mathbb{C}^I \subset \mathfrak{g}_\mathbf{v}^*$ is empty unless we have $\lambda \cdot \mathbf{v} = 0$. $\diamond$

Remark 4.3.7. It is important to emphasize that, up to a relabelling $\lambda \mapsto \lambda'$ of parameters, one has:

The quiver $\overline{Q}$, hence also the scheme $\mu^{-1}(\lambda)$ and the algebra $\Pi_\lambda(\overline{Q})$, depend only on the underlying graph of Q, and do not depend on the orientation of the quiver Q.

4.4. The cotangent bundle projection $p : T^*\big(\mathrm{Rep}(Q, \mathbf{v})\big) \to \mathrm{Rep}(Q, \mathbf{v})$ may be clearly identified with the natural projection to $\mathrm{Rep}(\overline{Q}, \mathbf{v}) \to \mathrm{Rep}(Q, \mathbf{v})$, $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x}$, cf. (4.3.3). Restricting the latter projection to a fiber of the moment map one obtains a map $p_\lambda : \mathrm{Rep}(\Pi_\lambda, \mathbf{v}) = \mu^{-1}(\lambda) \to \mathrm{Rep}(Q, \mathbf{v})$.

Observe further that the composite $\mathbb{C}Q \hookrightarrow \mathbb{C}\overline{Q} \twoheadrightarrow \Pi_\lambda$ yields an algebra imbedding $\mathbb{C}Q \hookrightarrow \Pi_\lambda$. In terms of the latter imbedding, the map p_λ amounts to restricting representations of the algebra Π_λ to the subalgebra $\mathbb{C}Q$. Thus, we obtain, cf. [CBH, Lemma 4.2],

Proposition 4.4.1. *For any $\mathbf{x} \in \mathrm{Rep}(Q, \mathbf{v})$, the set $p_\lambda^{-1}(X)$ is canonically identified with the set of extensions of $\mathbf{x}$ to a Π_λ-module $(\mathbf{x}, \mathbf{y}) \in \mathrm{Rep}(\Pi_\lambda, \mathbf{v})$.* $\square$

In some important cases, one can say quite a bit about the structure of the variety (4.3.5). To explain this, we need to introduce some notation.

Let $R_Q \subset \mathbb{Z}^I$ be the set of roots for Q, as defined eg. in [CB1], p. 262. Given $\lambda \in \mathbb{C}^I$, we put $R_\lambda^+ := \{\alpha \in R_Q \mid \alpha \geq 0 \ \& \ \lambda \cdot \alpha = 0\}$ where, in general, we write $\mathbf{v} \geq \mathbf{v}'$ whenever $\mathbf{v} - \mathbf{v}' \in \mathbb{Z}_{\geq 0}^I$. Further, for any $\mathbf{v} \in \mathbb{Z}^I$, we define

$$\mathsf{p}(\mathbf{v}) := 1 + A_Q \mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{v}.$$

Recall the hyperplane $(\mathrm{Lie}\, \mathbb{C}^\times)^\perp \subset \mathfrak{g}_\mathbf{v}^*$ that corresponds to the diagonal imbedding $\mathbb{C}^\times \subset G_\mathbf{v}$. One has the the following result.

Theorem 4.4.2. *Fix $\lambda \in \mathbb{C}^I$. Let $\mathbf{v}$ be a dimension vector such that $\lambda \cdot \mathbf{v} = 0$ and, for any decomposition $\mathbf{v} = \alpha_1 + \ldots + \alpha_r$, $\alpha_j \in R_\lambda^+$, the following inequality holds*

$$\mathsf{p}(\mathbf{v}) \geq \mathsf{p}(\alpha_1) + \ldots + \mathsf{p}(\alpha_r). \tag{4.4.3}$$

Then, we have

(i) The moment map $\mu : \mathrm{Rep}(\overline{Q}, \mathbf{v}) \to (\mathrm{Lie}\, \mathbb{C}^\times)^\perp$ is flat and the scheme $\mathrm{Rep}(\Pi_\lambda, \mathbf{v})$, in (4.3.5), is a complete intersection in $\mathrm{Rep}(\overline{Q}, \mathbf{v})$.

(ii) *The irrducible components of* $\mathrm{Rep}(\Pi_\lambda, \mathbf{v})$ *are in one-to-one correspondence with decomposi-tions* $\mathbf{v} = \alpha_1 + \ldots + \alpha_r, \ \alpha_j \in R_\lambda^+$, *such that the corresponding inequality* (4.4.3) *is an equality. Each irreducible component has dimension* $1 + 2A_Q\mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{v}$.

(iii) *If the inequality* (4.4.3) *is strict for any* $\mathbf{v} = \alpha_1 + \ldots + \alpha_r, \ \alpha_j \in R_\lambda^+$, *with* $r > 1$, *then the scheme* $\mathrm{Rep}(\Pi_\lambda, \mathbf{v})$ *is reduced and irreducible, moreover, the general point in this scheme is a simple representation of the algebra* Π_λ.

Here, parts (i) and (iii) are due to Crawley-Boevey, [CB1], Theorems 1.1 and 1.2. Part (ii) is [GG], Theorem 3.1.

4.5. **Hamiltonian reduction.** For any $\lambda \in \mathbb{C}^I$ such that $\lambda \cdot \mathbf{v} = 0$, the fiber $\mu^{-1}(\lambda)$ is a nonempty closed $G_\mathbf{v}$-stable subscheme of $\mathrm{Rep}(\overline{Q}, \mathbf{v})$, not necessarily reduced, in general. Thus, given $\theta \in \mathbb{R}^I$ such that $\theta \cdot \mathbf{v} = 0$, one may consider the following GIT quotient

$$\mathcal{M}_{\lambda,\theta}(\mathbf{v}) := \mu^{-1}(\lambda)/\!\!/_{\chi_\theta} G_\mathbf{v} = \mathrm{Rep}^{ss}(\Pi_\lambda, \mathbf{v})/S\text{-equivalence}, \qquad \forall\, \lambda \cdot \mathbf{v} = \theta \cdot \mathbf{v} = 0. \qquad (4.5.1)$$

Remark 4.5.2. One may identify $\mathbb{C}^I \times \mathbb{R}^I = \mathbb{R}^3 \otimes \mathbb{R}^I$ and view a pair $(\lambda, \theta) \in \mathbb{C}^I \times \mathbb{R}^I$ as a point in $\mathbb{R}^3 \otimes \mathbb{R}^I$. Further, given $\mathbf{v} = (v_i)_{i \in I}$, view $\mathbb{C}^{v_i}$ as a hermitian vector space with respect to the standard euclidean (hermitian) inner product. These inner products induce hermitian inner products on the spaces $\mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j})$. The resulting hermitian inner product on $\mathrm{Rep}(\overline{Q}, \mathbf{v})$ combined with the ($\mathbb{C}$-bilinear) symplectic 2-form, see (4.1.1), give $\mathrm{Rep}(\overline{Q}, \mathbf{v})$ the structure of a hyper-Kähler vector space.

One can show, cf. [Kro] for a special case, that the Hamiltonian reduction $\mu^{-1}(\lambda)/\!\!/_{\chi_\theta} G_\mathbf{v}$ may be identified with a *hyper-Kähler reduction* of $\mathrm{Rep}(\overline{Q}, \mathbf{v})$ with respect to the maximal compact subgroup of the complex algebraic group $G_\mathbf{v}$ formed by the elements which preserve the metric.

To proceed further, we need to introduce the *Cartan matrix* of the underlying graph of Q defined as follows $C_Q := 2\mathrm{Id} - A_{\overline{Q}}$. This is a symmetric Cartan matrix in the sense of Kac, [Ka], provided Q has no edge loops.

Corollary 4.5.3. (i) *Any simple* Π_λ-*module of dimension* $\mathbf{v}$ *corresponds to a point in* $\mu^{-1}(\lambda)^{\mathrm{reg}}$, *the smooth locus of the scheme* (4.3.5)

(ii) *The group* $G_\mathbf{v}/\mathbb{C}^\times$ *acts freely on* $\mu^{-1}(\lambda)^{\mathrm{reg}}$.

(iii) *Let* $T_{G_\mathbf{v}\alpha}(\mu^{-1}(\lambda))$ *be the normal space, at* $\alpha \in \mu^{-1}(\lambda)^{\mathrm{reg}}$, *to the orbit* $G_\mathbf{v}\alpha \subset \mu^{-1}(\lambda)^{\mathrm{reg}}$. *Then, the vector space* $T_{G_\mathbf{v}\alpha}(\mu^{-1}(\lambda))$ *has a canonical symplectic structure and, we have*

$$\dim T_{G_\mathbf{v}\alpha}(\mu^{-1}(\lambda)) = 2 - C_Q\mathbf{v} \cdot \mathbf{v}.$$

Proof. Part (i) follows, thanks to Schur's lemma, from Lemma 1.3.2 and Lemma 4.1.7. The last lemma also yields part (ii).

To prove (iii), put $U := \mu^{-1}(\lambda)^{\mathrm{reg}}$, let $G := G_\mathbf{v}/\mathbb{C}^\times$ and $\mathfrak{g} := \mathrm{Lie}\, G$. Thus, we have $\dim \mathfrak{g} = \dim G_\mathbf{v} - 1$.

For any $\alpha \in U$, the tangent space to U/G at the point corresponding to the image of α equals $(T_\alpha U)/\mathfrak{g}$, where we identify the Lie algebra $\mathfrak{g}$ with its image under the action map $\mathfrak{g} \to T_\alpha U, \ u \mapsto \vec{u}_\alpha$. Furthermore, the (proof of) Lemma 4.1.7 implies that this last map is injective. Also, the symplectic structure on $(T_\alpha U)/\mathfrak{g}$ is provided by the last statement of Lemma 4.1.7.

Now, using the surjectivity of the differential of the moment map $d_\alpha\mu : T_\alpha \operatorname{Rep}(\overline{Q}, \mathbf{v}) \to \mathfrak{g}^*$ is surjective by Lemma 4.1.7, we compute

$$
\begin{aligned}
\dim U/G &= \dim \left((T_\alpha U)/\mathfrak{g}\right) \\
&= \dim \operatorname{Ker}(d_\alpha\mu) - \dim \mathfrak{g} \qquad\qquad\qquad\qquad (4.5.4) \\
&= \left[\dim \operatorname{Rep}(\overline{Q}, \mathbf{v}) - \dim \mathfrak{g}^*\right] - \dim \mathfrak{g} = \dim \operatorname{Rep}(\overline{Q}, \mathbf{v}) - 2\dim \mathfrak{g} \\
&= \dim \operatorname{Rep}(\overline{Q}, \mathbf{v}) - 2(\dim G_\mathbf{v} - 1).
\end{aligned}
$$

Finally, from formula (1.3.1) applied to the quiver $\overline{Q}$, we find

$$
2\dim G_\mathbf{v} - \dim \operatorname{Rep}(\overline{Q}, \mathbf{v}) = 2\mathbf{v}\cdot\mathbf{v} - A_{\overline{Q}}\mathbf{v}\cdot\mathbf{v} = C_Q\mathbf{v}\cdot\mathbf{v}. \qquad (4.5.5)
$$

The last formula of Corollary 4.5.3 now follows by combining (4.5.4) with (4.5.5). $\qquad\square$

Many of the results concerning stability of quiver representations carry over in a straightforward way to Π_λ-modules. In particular, we have

Theorem 4.5.6. (i) *For $\theta = 0$, the scheme $\mathcal{M}_{\lambda,0}(\mathbf{v}) = \operatorname{Spec}\mathbb{C}[\mu^{-1}(\lambda)]^{G_\mathbf{v}}$ is a normal affine variety, cf.* [CB3, Theorem 1.1]; *geometric points of this scheme correspond to semisimple Π_λ-modules.*

(ii) *Geometric points of the scheme $\mathcal{M}_{\lambda,\theta}(\mathbf{v})$ correspond to S-equivalence classes of θ-semistable Π_λ-modules.*

(iii) *The group $G_\mathbf{v}$ acts freely on the set $\mu^{-1}(\lambda)^s$, of θ-stable points; isomorphism classes of θ-stable Π_λ-modules form a Zariski open subset $\mathcal{M}_{\lambda,\theta}^s(\mathbf{v}) \subset \mathcal{M}_{\lambda,\theta}(\mathbf{v})$, of dimension $2 - (\mathbf{v}, C_Q\mathbf{v})$.*

(iv) *The canonical map $\pi : \mathcal{M}_{\lambda,\theta}(\mathbf{v}) \to \mathcal{M}_{\lambda,0}(\mathbf{v})$ is a projective morphism that takes a Π_λ-module V to its semi-simplification.*

Sketch of Proof. Part (i) of the theorem is a consequence of Corollary 2.3.9.

To prove (iii), let $V \in \mu^{-1}(\lambda)^s$ be a stable $\Pi_\lambda(\overline{Q})$-module. A version of Corollary 2.3.8(ii) implies that the isotropy group of V is equal to $\mathbb{C}^\times$. It follows that V gives a smooth point of the fiber $\mu^{-1}(\lambda)$, by Lemma 4.1.7. Furthermore, Corollary 4.5.3 applies and we find

$$
\dim \mathcal{M}_{\lambda,\theta}^s(\mathbf{v}) = 2(\dim \operatorname{Rep}(Q, \mathbf{v}) - \dim(G_\mathbf{v}/\mathbb{C}^\times)) = 2 - (\mathbf{v}, C_Q\mathbf{v}).
$$

Other statements of the theorem are obtained by applying Theorem 2.3.7 to the quiver $\overline{Q}$. $\qquad\square$

Corollary 4.5.7. *If the set $\mathcal{M}_{\lambda,\theta}^s(\mathbf{v})$ is nonempty then, we have $C_Q\mathbf{v}\cdot\mathbf{v} \leq 2$.* $\qquad\square$

In the special case $\lambda = 0$, using isomorphism (4.1.4), we deduce

Proposition 4.5.8. *The variety $\mathcal{M}_{0,\theta}(\mathbf{v})$ contains $T^*\mathcal{R}_\theta^s(\mathbf{v})$, the cotangent space to the moduli space $\mathcal{R}_\theta^s(\mathbf{v})$, as an open (possibly empty) subset of the smooth locus of $\mathcal{M}_{0,\theta}(\mathbf{v})$.*

Example 4.5.9 (Dynkin quivers). Let Q be an ADE quiver, and fix a dimension vector $\mathbf{v}$.

The number of $G_\mathbf{v}$-orbits in $\operatorname{Rep}(Q, \mathbf{v})$ is finite by the Gabriel theorem. Thus, we see from (4.1.4) that $\mu^{-1}(0)$ is in this case a finite union of conormal bundles, hence a Lagrangian subvariety in $T^* \operatorname{Rep}(Q, \mathbf{v})$.

We claim next that the zero representation $0 \in \operatorname{Rep}(\overline{Q}, \mathbf{v})$ is contained in the closure of any $G_\mathbf{v}$-orbit. This is clear for the $G_\mathbf{v}$-orbit of the point corresponding to an indecomposable representation (if such a represntation of dimension $\mathbf{v}$ exists), since the corresponding orbit is Zariski dense by the Gabriel theorem. From this, one deduces easily that our claim must hold for the orbit of a point corresponding to a direct sum of indecomposable representations as well.

Our claim implies that the conormal bundle on any $G_\mathbf{v}$-orbit is a subset of $\operatorname{Rep}(\overline{Q}, \mathbf{v})$ which is stable under the $\mathbb{C}^\times$-action on the vector space $\operatorname{Rep}(\overline{Q}, \mathbf{v})$ by dilations (this is not the action

obtained by restricting the natural $G_{\mathbf{v}}$-action on $\mathrm{Rep}(\overline{Q}, \mathbf{v})$ to the subgroup $\mathbb{C}^\times \subset G_{\mathbf{v}}$; the latter $\mathbb{C}^\times$-action is trivial). It follows that the set $\mu^{-1}(0)$, the union of conormal bundles, is also $\mathbb{C}^\times$-stable under the dilation action.

We conclude that any homogeneous $G_{\mathbf{v}}$-invariant polynomial on $\mu^{-1}(0)$ of positive degree vanishes. Thus, in the Dynkin case, we have $\mathcal{M}_0(\mathbf{v}) = pt$.

4.6. The McKay correspondence.
Associated to any finite group Γ and a finite dimensional representation $\Gamma \to GL(E)$, there is a quiver Q_Γ, called the *McKay quiver* for Γ (that depends on the representation E as well). The vertex set of this quiver is defined to be the set I of equivalence classes of irreducible representations of Γ. We write L_i for the irreducible representation corresponding to a vertex $i \in I$. In particular, there is a distinguished vertex $o \in I$ corresponding to the trivial representation.

Further, the adjacency matrix $A_{Q_\Gamma} = \|a_{ij}\|$, of the McKay quiver, is defined by the formula

$$a_{ij} := \dim \mathrm{Hom}_\Gamma(L_i, L_j \otimes E). \tag{4.6.1}$$

The matrix A_Q is symmetric if and only if E is a self-dual representation of Γ. In such a case, one can write $Q_\Gamma = \overline{Q}$, for some quiver Q. Note also that the quiver Q_Γ has no edge-loops if and only if E does not contain the trivial representation of Γ as a direct summand.

Now, fix a 2-dimensional symplectic vector space (E, ω), and a finite subgroup $\Gamma \subset Sp(E, \omega) = SL_2(\mathbb{C})$. The imbedding $\Gamma \hookrightarrow GL(E)$ gives a self-dual representation of Γ.

It follows from the well known classification of platonic solids that conjugacy classes of finite subgroups of the group $SL_2(\mathbb{C})$ are in one-to-one correspondence with Dynkin graphs of A, D, E types. McKay observed that this correspondence may be obtained by assigning to $\Gamma \subset Sp(E, \omega)$ its McKay quiver Q_Γ (so that Q_Γ then becomes the double of the corresponding extended Dynkin graph of type $\widetilde{A}, \widetilde{D}, \widetilde{E}$, equipped with any choice of orientation).

Associated with the extended Dynkin diagram, there is a root system $R \subset \mathbb{Z}^I$. The vector $\delta = (\delta_i)_{i \in I} \in \mathbb{Z}^I$, where $\delta_i := \dim L_i$, turns out to be equal to the minimal imaginary root of that root system.

Let $\mathbb{C}\Gamma$ be the group algebra of Γ and, for each $i \in I$, choose a minimal idempotent $e_i \in \mathbb{C}\Gamma$ such that $\mathbb{C}\Gamma \cdot e_i \cong L_i$. Put $e = \sum_{i \in I} e_i$, an idempotent in $\mathbb{C}\Gamma$. The Γ-action on E induces one on $\mathrm{Sym}\, E$, the symmetric algebra of E. We write $(\mathrm{Sym}\, E) \rtimes \Gamma$ for the corresponding cross-product algebra, resp. $(\mathrm{Sym}\, E)^\Gamma \subset \mathrm{Sym}\, E$ for the subalgebra of Γ-invariants. Note that the self-duality of E implies that one has algebra isomorphisms $(\mathrm{Sym}\, E)^\Gamma = \mathbb{C}[E]^\Gamma = \mathbb{C}[E/\Gamma]$.

One way of stating the McKay correspondence is as follows, cf. [CBH], Theorem 0.1.

Theorem 4.6.2. (i) *There is an algebra isomorphism $\Pi_0(Q_\Gamma) \cong e\big[(\mathrm{Sym}\, E) \rtimes \Gamma\big]e$. In particular, the algebras $\Pi_0(Q_\Gamma)$ and $(\mathrm{Sym}\, E) \rtimes \Gamma$ are Morita equivalent.*

(ii) *There is a canonical algebra isomorphism $1_o \cdot \Pi_0(Q_\Gamma) \cdot 1_o \cong (\mathrm{Sym}\, E)^\Gamma$.*

Outline of Proof. Let $T_\mathbb{C}V$ denote the tensor algebra of the vector space E. An elementary argument based on formula (4.6.1) yields an algebra isomorphism $\phi: T_{\mathbb{C}I}(\mathbb{C}E_{Q_\Gamma}) \xrightarrow{\sim} e\big[(T_\mathbb{C}E) \rtimes \Gamma\big]e$, see [CBH, §2]. Recall that, for the path algebra of any quiver Q, we have $\mathbb{C}Q = T_{\mathbb{C}I}(\mathbb{C}E_Q)$, see §1.3. Thus, we may identify the algebra on the left hand side of the isomorphism ϕ with the path algebra $\mathbb{C}Q_\Gamma$.

Next, one verifies that the two-sided ideal of $\mathbb{C}Q_\Gamma$ generated by the element $\sum_{x \in Q_\Gamma}(xx^* - x^*x)$, cf. Definition 4.3.1, goes under the isomorphism ϕ to the two-sided ideal J generated by the elements $e_1 \otimes e_2 - e_2 \otimes e_1 \in T_\mathbb{C}^2 E$, $e_1, e_2 \in E$. The isomorphism of part (i) of the theorem is now induced by the isomorphism ϕ using that one has

$$\mathbb{C}Q_\Gamma \Big/ \Big(\sum_{x \in Q_\Gamma}(xx^* - x^*x) \Big) = \Pi_0(Q_\Gamma), \quad \text{and} \quad [(T_\mathbb{C}E) \rtimes \Gamma]/J = (\mathrm{Sym}\, E) \rtimes \Gamma.$$

To complete the proof, we observe that the isomorphism ϕ constructed above sends the idempotent $1_o \in \mathbb{C}I$ to $\mathsf{p} := \frac{1}{|\Gamma|} \sum_{g \in \Gamma} g \in \mathbb{C}\Gamma$, the averaging idempotent. Furthermore, it is easy to show that the natural imbedding $(\operatorname{Sym} E)^\Gamma \hookrightarrow \operatorname{Sym} E$ induces an algebra isomorphism $(\operatorname{Sym} E)^\Gamma \xrightarrow{\sim} \mathsf{p}[(\operatorname{Sym} E) \rtimes \Gamma]\mathsf{p}$. Part (ii) of the theorem follows from these observations using the isomorphism of part (i). $\qquad \square$

The orbit space $\mathbb{C}^2/\Gamma = \operatorname{Spec} \mathbb{C}[x,y]^\Gamma$ is an irreducible normal 2-dimensional variety with an isolated singularity at the origin. Such a variety is known to have a minimal resolution, unique up to isomorphism.

The following result is a reformulation of a result of P. Kronheimer in the language of quiver moduli, cf. [CS].

Theorem 4.6.3. (i) *There is a natural isomorphism $\mathcal{M}_0(\delta) \cong \mathbb{C}^2/\Gamma$, of algebraic varieties.*

(ii) *Assume that $\theta \in \mathbb{Z}^I$ does not belong to root hyperplanes of the affine root system. Then, the variety $\mathcal{M}_\theta(\delta)$ is smooth and the canonical map $\pi : \mathcal{M}_\theta(\delta) \to \mathcal{M}_0(\delta) = \mathbb{C}^2/\Gamma$ is the minimal resolution of $\mathbb{C}^2/\Gamma$.*

5. Nakajima varieties

5.1. We now combine together all the previous constructions. Thus, we fix a quiver Q and consider the quiver $\overline{Q^\heartsuit}$, the double of $\overline{Q}$. Given any dimension vector $(\mathbf{v}, \mathbf{w}) \in \mathbb{Z}^I \times \mathbb{Z}^I$, choose a pair of I-graded vector spaces $V = \oplus_{i \in I} V_i$ and $W = \oplus_{i \in I} W_i$ such that $\dim_I V = \mathbf{v}$ and $\dim_I W = \mathbf{w}$. By definition, we have

$$\operatorname{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w}) = T^* \operatorname{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$$
$$= \operatorname{Rep}(Q, \mathbf{v}) \times \operatorname{Rep}(Q^{\mathrm{op}}, \mathbf{v}) \times \operatorname{Hom}_{\mathbb{C}I}(V, W) \times \operatorname{Hom}_{\mathbb{C}I}(W, V).$$

Thus, one may view an element of $\operatorname{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$ as a quadruple $(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j})$, where $\mathbf{x} \in \operatorname{Rep}(Q, \mathbf{v})$, $\mathbf{y} \in \operatorname{Rep}(Q^{\mathrm{op}}, \mathbf{v})$, $\mathbf{i} \in \operatorname{Hom}_{\mathbb{C}I}(W, V)$, and $\mathbf{j} \in \operatorname{Hom}_{\mathbb{C}I}(V, W)$.

In particular, we find

$$\dim \operatorname{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w}) = A_{\overline{Q}} \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w}. \tag{5.1.1}$$

The vector space $\operatorname{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$ has the symplectic structure of a cotangent bundle and the group $G_{\mathbf{v}}$ acts on $\operatorname{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$ by symplectic automorphisms, via $G_{\mathbf{v}} \ni g : (\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \mapsto (g\mathbf{x}g^{-1}, g\mathbf{y}g^{-1}, g \circ \mathbf{i}, \mathbf{j} \circ g^{-1})$. The associated moment map is given by

$$\mu : \operatorname{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w}) \to \mathfrak{g}_{\mathbf{v}}^* = \mathfrak{g}_{\mathbf{v}}, \quad (\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \mapsto \sum [x, y] + \mathbf{i} \circ \mathbf{j} \in \mathfrak{g}_{\mathbf{v}}. \tag{5.1.2}$$

Here, we write $\mathbf{i} \circ \mathbf{j} := \sum_{i \in I} \mathbf{i}_i \circ \mathbf{j}_i$ where, for each i, $\mathbf{i}_i \circ \mathbf{j}_i : V_i \to V_i$ is a rank one operator.

For any $\lambda \in \mathbb{C}^I$ we have

$$\mu^{-1}(\lambda) = \{(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \in \operatorname{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w}) \mid [\mathbf{x}, \mathbf{y}] + \mathbf{i} \otimes \mathbf{j} = \lambda\}. \tag{5.1.3}$$

From now on, whenever we discuss varieties involving (5.1.3), **we will always assume that $\mathbf{w} \neq 0$.**

Remark 5.1.4. Recall the quiver $Q^{\mathbf{w}}$ introduced by Crawley-Boevey, see §3.1. One can use the isomorphism in (4.3.5) for the quiver $Q^{\mathbf{w}}$ to identify the scheme (5.1.3) with $\operatorname{Rep}(\Pi_{\widehat{\lambda}}, \widehat{\mathbf{v}})$, the representation scheme of the preprojective algebra for the the quiver $Q^{\mathbf{w}}$ where $\widehat{\lambda}$ is an appropriate parameter. $\qquad \diamond$

Given $\theta \in \mathbb{Z}^I$, we may apply general Definition 2.3.1 to the variety $\mu^{-1}(\lambda)$ and the character χ_θ of the group $G_{\mathbf{v}}$. This way, one proves

Proposition 5.1.5. A quadruple $(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \in \mu^{-1}(\lambda)$ is θ-semistable if and only if the following holds:

For any collection of vector subspaces $S = (S_i)_{i \in I} \subset V = (V_i)_{i \in I}$ *which is stable under the maps* $\mathbf{x}$ *and* $\mathbf{y}$, *we have*

$$S_i \subset \operatorname{Ker} \mathbf{j}_i, \quad \forall i \in I \quad \implies \quad \theta(\dim_I S) \leq 0; \tag{5.1.6}$$

$$S_i \supset \operatorname{Image} \mathbf{i}_i, \quad \forall i \in I \quad \implies \quad \theta(\dim_I S) \leq \theta(\dim_I V). \tag{5.1.7}$$

Example 5.1.8. In the case $\theta = 0$, any point in $\mu^{-1}(\lambda)$ is θ-semistable. Such a point is θ-stable if and only if the only collection of subspaces $S = (S_i)_{i \in I} \subset V = (V_i)_{i \in I}$ which is stable under the maps $\mathbf{x}$ and $\mathbf{y}$, is $S = 0$ or $S = V$.

The above proposition implies, in particular, the following result

Corollary 5.1.9. *In the special case where* $\theta = \pm\theta^+$, *the point* $(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \in \mu^{-1}(\lambda)$ *is* θ-*semistable if and only if, in the notation of Proposition 5.1.5(i), we have*

$$S_i \subset \operatorname{Ker} \mathbf{j}_i, \quad \forall i \in I \quad \implies \quad S = 0 \quad \text{if } \theta = \theta^+, \text{ resp.}$$

$$S_i \supset \operatorname{Image} \mathbf{i}_i, \quad \forall i \in I \quad \implies \quad S = V \quad \text{if } \theta = -\theta^+.$$

Definition 5.1.10. *The variety* $\mathcal{M}_{\lambda,\theta}(\mathbf{v}, \mathbf{w}) := \mu^{-1}(\lambda)_\theta^{ss} /_{\chi_\theta} G_{\mathbf{v}}$ *is called the* Nakajima variety *with parameters* (λ, θ). *Let* $\mathcal{M}_{\lambda,\theta}^s(\mathbf{v}, \mathbf{w}) \subset \mathcal{M}_{\lambda,\theta}(\mathbf{v}, \mathbf{w})$ *denote the Zariski open subset corresponding to stable points.*

Thanks to the general formalism of Hamiltonian reduction, the symplectic structure on the manifold $\operatorname{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$ gives $\mathcal{M}_{\lambda,\theta}(\mathbf{v}, \mathbf{w})$ the canonical structure of a (not necessarily smooth) algebraic Poisson variety.

Remark 5.1.11. The equation $[\mathbf{x}, \mathbf{y}] + \mathbf{i} \otimes \mathbf{j} = \lambda$, in (5.1.3), is often called the *moment map equation*, or the *ADHM-equation*, since an equation of this form was first considered by Atiyah, Hitchin, Drinfeld, and Manin in their work on instantons on $\mathbb{P}^2$, cf. [ADHM] and also [Na3].

From that point of view, it is natural to view (5.1.3) as part of a larger system of hyper-Kähler moment map equations, cf. Remark 4.5.2. Accordingly, we will refer to the pair (λ, θ), viewed as an element of the real vector space $\mathbb{R}^3 \otimes \mathbb{R}^{\mathbf{v}} = (\mathbb{C} \oplus \mathbb{R}) \otimes \mathbb{R}^{\mathbf{v}} = \mathbb{C}^{\mathbf{v}} \oplus \mathbb{R}^{\mathbf{v}}$, as a 'hyper-Kähler parameter'. $\diamondsuit$

5.2. To formulate the main properties of Nakajima varieties, fix a quiver Q and write C_Q for the Cartan matrix of Q. We introduce the following set,

$$R' := \{\mathbf{v} \in \mathbb{Z}^I \smallsetminus \{0\} \mid C_Q \mathbf{v} \cdot \mathbf{v} \leq 2, \ \forall i \in I\}.$$

If Q is a quiver of either finite Dynkin or extended Dynkin types, then $R'_Q = R_Q$ is the set of roots associated with the Cartan matrix C_Q. This is not necessarily true for more general quivers.

For $\alpha \in \mathbb{R}^I$, write $\mathbf{v}^\perp := \{\lambda \in \mathbb{R}^I \mid \lambda \cdot \mathbf{v} = 0\}$.

Given a dimension vector $\mathbf{v} \in \mathbb{Z}_{\geq 0}^I$, the parameter $(\lambda, \theta) \in \mathbb{C}^I \times \mathbb{Z}^I$ will be called $\mathbf{v}$-*regular* if, viewed as a hyper-Kähler parameter $(\lambda, \theta) \in \mathbb{R}^3 \otimes \mathbb{R}^I$, it satisfies, cf. [Na6, §1(iii)],

$$(\lambda, \theta) \in (\mathbb{R}^3 \otimes \mathbb{R}^I) \smallsetminus \bigcup_{\{\alpha \in R'_Q \mid 0 \leq \alpha \leq \mathbf{v}\}} \mathbb{R}^3 \otimes \alpha^\perp. \tag{5.2.1}$$

We note that $(\lambda, \theta) := (0, \theta^+)$ is a $\mathbf{v}$-regular parameter for any dimension vector.

Theorem 5.2.2. *Fix $\lambda \in \mathbb{C}^I$, $\theta \in \mathbb{Z}^I$, where $\lambda \cdot \mathbf{v} = 0$. Then, we have*

(i) We have $\mathcal{M}_{\lambda,0}(\mathbf{v}, \mathbf{w}) = \mu^{-1}(\lambda)//G_{\mathbf{v}}$; this is an affine variety and there is a canonical projective morphism $\pi : \mathcal{M}_{\lambda,\theta}(\mathbf{v}, \mathbf{w}) \to \mathcal{M}_{\lambda,0}(\mathbf{v}, \mathbf{w})$, which respects the Poisson brackets.

(ii) Let the parameter (λ, θ) be $\mathbf{v}$-regular. Then any θ-semistable point in $\mu^{-1}(\lambda)$ is θ-stable, so $\mathcal{M}_{\lambda,\theta}(\mathbf{v}, \mathbf{w}) = \mathcal{M}_{\lambda,\theta}^s(\mathbf{v}, \mathbf{w})$, cf. [Na1, §2.8],[Na1, §3(ii)]. Furthermore, this variety is smooth and connected variety of dimension

$$\dim \mathcal{M}_{\lambda,\theta}(\mathbf{v}, \mathbf{w}) \;=\; 2\mathbf{w} \cdot \mathbf{v} - C_Q \mathbf{v} \cdot \mathbf{v}.$$

The Poisson structure on $\mathcal{M}_{\lambda,\theta}(\mathbf{v}, \mathbf{w})$ is nondegenerate making it an algebraic symplectic manifold.

(iii) The variety $\mathcal{M}_{0,\theta+}(\mathbf{v}, \mathbf{w})$ contains $T^*\mathcal{R}_{\theta+}(\mathbf{v}, \mathbf{w})$ as a Zariski open subset.

Sketch of Proof. Part (i) is clear. To prove (ii), one shows that the isotropy group of any point $(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \in \mu^{-1}(\lambda)$ that satisfies conditions (5.1.6)-(5.1.7) is trivial, provided the parameter (λ, θ) is $\mathbf{v}$-regular. It follows, in particular, that the $G_{\mathbf{v}}$-orbit of a semistable point $(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \in \mu^{-1}(\lambda)$ must be an orbit of maximal dimension equal to $\dim G_{\mathbf{v}}$. We conclude that one semistable orbit can not be contained in the closure of another semistable orbit. Thus, all semistable orbits are closed in $\mu^{-1}(\lambda)^{ss}$, hence any semistable point is actually stable.

Further, by Corollary 4.1.7, the triviality of stabilizers implies that the set $\mu^{-1}(\lambda)^{ss}$ of θ-stable points is smooth and $\mu^{-1}(\lambda)^{ss}/G_{\mathbf{v}}$ is a symplectic manifold. Therefore, using the dimension formula (5.1.1), we compute

$$\dim \left(\mu^{-1}(\lambda)^{ss}/G_{\mathbf{v}} \right) = 2\mathbf{w} \cdot \mathbf{v} + (2\mathrm{Id} - C_Q)\mathbf{v} \cdot \mathbf{v} - 2\dim G_{\mathbf{v}} = 2\mathbf{w} \cdot \mathbf{v} - C_Q \mathbf{v} \cdot \mathbf{v}.$$

(note that unlike the situation considered in Theorem 4.5.6 the $G_{\mathbf{v}}$-action on $\mathrm{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$ does not factor through the quotient $G_{\mathbf{v}}/\mathbb{C}^\times$. Therefore, it is the dimension of the group $G_{\mathbf{v}}$, rather than that of $G_{\mathbf{v}}/\mathbb{C}^\times$, that enters the dimension count above).

Finally, the connectedness of the varieties $\mathcal{M}_{\lambda,\theta}(\mathbf{v}, \mathbf{w})$ is a much more difficult result proved by Crawley-Boevey. The proof is based on the irreducibility statement in Theorem 4.4.2(iii) and on a 'hyper-Kähler rotation' trick (Remark 5.2.3 below). For more details see comments at [CB1], p.261. $\qquad \square$

Remark 5.2.3. For a $\mathbf{v}$-regular parameter (λ, θ), the Nakajima variety $\mathcal{M}_{\lambda,\theta}(\mathbf{v}, \mathbf{w})$ comes equipped with a structure of hyper-Kähler manifold, cf. Remark 4.5.2. In particular, one can show that there is a choice of complex structure on the C^∞-manifold $\mathcal{M}_{\lambda,\theta}(\mathbf{v}, \mathbf{w})$ that makes it a smooth and *affine* algebraic variety, see [Na1, §§3.1, 4.2]. $\qquad \diamondsuit$

Observe next that the group $G_{\mathbf{w}} = \prod_{i \in I} GL(w_i)$ acts naturally on $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ and on $\mathrm{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$. Furthermore, the $G_{\mathbf{w}}$-action on the latter space is Hamiltonian and each fiber $\mu^{-1}(\lambda)$ of the moment map (4.1.2) is a $G_{\mathbf{w}}$-stable subvariety. Also, the $G_{\mathbf{w}}$-action clearly preserves any stability condition hence descends, for any (λ, θ), to a well defined $G_{\mathbf{w}}$-action on the Nakajima variety $\mathcal{M}_{\lambda,\theta}(\mathbf{v}, \mathbf{w})$ by Poisson automorphisms.

Assume now that $\lambda = 0$. In this special case, there are two natural ways to define an additional $\mathbb{C}^\times$-action on $\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w})$ that makes it a $G_{\mathbf{w}} \times \mathbb{C}^\times$-variety. Each of these actions comes from a $\mathbb{C}^\times$-action on $\mathrm{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$ that keeps the fiber $\mu^{-1}(0)$ stable and commutes with the $G_{\mathbf{v}}$-action on the fiber. The first $\mathbb{C}^\times$-action on $\mathrm{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$ is the dilation action given by the formula $\mathbb{C}^\times \ni t : (\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \mapsto (t \cdot \mathbf{x}, t \cdot \mathbf{y}, t \cdot \mathbf{i}, t \cdot \mathbf{j})$. This action rescales the symplectic 2-form ω as $t : \omega \mapsto t^2 \cdot \omega$.

The second $\mathbb{C}^\times$-action on $\mathrm{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$ corresponds, via the identification $\mathrm{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w}) = T^*\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$, to the natural $\mathbb{C}^\times$-action by dilations along the fibers of the contangent bundle. This $\mathbb{C}^\times$-action is defined by the formula $\mathbb{C}^\times \ni t : (\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \mapsto (\mathbf{x}, t \cdot \mathbf{y}, \mathbf{i}, t \cdot \mathbf{j})$. The latter action keeps the subvariety $\mu^{-1}(0)$ stable and commutes with the $G_{\mathbf{v}}$-action on $\mathrm{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$. Therefore,

for any θ, there is an induced $G_{\mathbf{v}}$-action $\mathbb{C}^\times \ni t : z \mapsto t(z)$, on $\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w})$. Furthermore, the map π becomes a $G_{\mathbf{v}}$-equivariant morphism of $G_{\mathbf{v}}$-varieties, and the fiber $\pi^{-1}(0) \subset \mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w})$ becomes a $G_{\mathbf{v}}$-stable subvariety.

The symplectic form ω on $T^* \mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$ gets rescaled under the above $\mathbb{C}^\times$-action as follows $\mathbb{C}^\times \ni t : \omega \mapsto t \cdot \omega$. Hence, the induced symplectic form on $\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w})$, to be denoted by ω again, transforms in a similar way.

For any parameters (λ, θ), the canonical projective morphism $\pi : \mathcal{M}_{\lambda,\theta} \to \mathcal{M}_{\lambda,0}$ is $G_{\mathbf{w}}$-equivariant. In the case $\lambda = 0$ this morphism is also $\mathbb{C}^\times$-equivariant with respect to either of the two $\mathbb{C}^\times$-actions defined above.

5.3.   Let $\mu^{-1}(\lambda)^\circ \subset \mu^{-1}(\lambda)$ be the subset of points with trivial isotropy group. We let $\mathcal{M}^\circ_{\lambda,0}(\mathbf{v}, \mathbf{w})$ $\subset \mathcal{M}_{\lambda,0}(\mathbf{v}, \mathbf{w})$ be the image of this set in $\mu^{-1}(\lambda)//G_{\mathbf{v}}$. Nakajima uses the notation $\mathcal{M}^{\mathrm{reg}}_{\lambda,0}(\mathbf{v}, \mathbf{w})$ for $\mathcal{M}^\circ_{\lambda,0}(\mathbf{v}, \mathbf{w})$, cf. [Na2, §3(v)]. He verifies that $\mathcal{M}^\circ_{\lambda,0}(\mathbf{v}, \mathbf{w})$ is a Zariski open (possibly empty) subset of $\mathcal{M}_{\lambda,0}(\mathbf{v}, \mathbf{w})$.

One has the following result, cf. [Na2], Proposition 3.24.

Proposition 5.3.1. *Assume the quiver Q has no edge loops and that (λ, θ) is a $\mathbf{v}$-regular parameter. Then, one has*

(i) *Any point in $\mu^{-1}(\lambda)^\circ$ is θ-stable.*

(ii) *If the set $\mathcal{M}^\circ_{\lambda,0}(\mathbf{v}, \mathbf{w})$ is nonempty then it is dense in $\mathcal{M}_{\lambda,0}(\mathbf{v}, \mathbf{w})$ and, we have:*

- *The canonical projective morphism $\pi : \mathcal{M}_{\lambda,\theta} \to \mathcal{M}_{\lambda,0}$ is a symplectic resolution;*
- *The set $\pi^{-1}\big(\mathcal{M}^\circ_{\lambda,0}(\mathbf{v}, \mathbf{w})\big)$ is dense in $\mathcal{M}_{\lambda,\theta}(\mathbf{v}, \mathbf{w})$, and the map π restrics to an isomorphism*

$$\pi :\ \pi^{-1}\big(\mathcal{M}^\circ_{\lambda,0}(\mathbf{v}, \mathbf{w})\big) \ \xrightarrow{\sim}\ \mathcal{M}^\circ_{\lambda,0}(\mathbf{v}, \mathbf{w}). \hspace{2cm} \square$$

There is a combinatorial criterion for the set $\mathcal{M}^\circ_{\lambda,0}$ to be nonempty, see [Na2, Proposition 10.5 and Corollary 10.8]. Also, using Theorem 7.2.4 Nakajima proves, see [Na1, Corollary 6.11],

Proposition 5.3.2. *If the quiver Q has no loop edges then the map $\pi :\ \mathcal{M}_{\lambda,\theta}(\mathbf{v}, \mathbf{w}) \to \pi\big(\mathcal{M}_{\lambda,0}(\mathbf{v}, \mathbf{w})\big)$ is semismall for any $\mathbf{v}$-regular parameter (λ, θ).*

Example 5.3.3 (Type $\mathbf{A}$ Dynkin quiver). Let Q be an $\mathbf{A}_n$-quiver, and let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (r, 0, 0, \ldots, 0)$, where $r > v_1 > v_2 > \ldots > v_n > 0$, as in Example 3.2.7. Thus, a representation of the quiver $\overline{Q^\heartsuit}$ looks like

$$\mathrm{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w}) :\quad \overset{W_1}{\bullet} \underset{\mathbf{j}}{\overset{\mathbf{i}}{\rightleftarrows}} \overset{V_1}{\bullet} \underset{\mathbf{x}}{\overset{\mathbf{y}}{\rightleftarrows}} \overset{V_2}{\bullet} \underset{\mathbf{x}}{\overset{\mathbf{y}}{\rightleftarrows}} \cdots \underset{\mathbf{x}}{\overset{\mathbf{y}}{\rightleftarrows}} \overset{V_{n-2}}{\bullet} \underset{\mathbf{x}}{\overset{\mathbf{y}}{\rightleftarrows}} \overset{V_{n-1}}{\bullet} \underset{\mathbf{x}}{\overset{\mathbf{y}}{\rightleftarrows}} \overset{V_n}{\bullet}$$

Write $W := W_1$. The assignment $(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \mapsto \mathbf{j} \circ \mathbf{i}$ gives a map $\varpi :\ \mathrm{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w}) \to \mathfrak{gl}(W)$. Let ϖ denote the restriction of this map to $\mu^{-1}(0) \subset \mathrm{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$, the zero fiber of the moment map, and let $X := \varpi(\mu^{-1}(0))$ be the image of ϖ.

Recall that, according to the discussion in Example 3.2.7, we have $\mathcal{R}_{\theta+}(\mathbf{v}, \mathbf{w}) \cong \mathcal{F}(n, W)$.

Proposition 5.3.4. (i) *The map ϖ induces an isomorphism $\mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w}) = \mu^{-1}(0)//G_{\mathbf{v}} \xrightarrow{\sim} X$, cf. [Na1] and [Sh, Theorem 2.1].*

(ii) *One has an isomorphism $\mathcal{M}_{0,\theta+}(\mathbf{v}, \mathbf{w}) \cong T^*\mathcal{R}_{\theta+}(\mathbf{v}, \mathbf{w}) = T^*\mathcal{F}(n, W)$ such that the canonical map $\pi :\ \mathcal{M}_{0,\theta+}(\mathbf{v}, \mathbf{w}) \to \mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})$ gets identified with natural moment map $T^*\mathcal{F}(n, W) \to X \subset \mathfrak{gl}(W)$, see [Na1, Theorem 7.2].*

Here, the isomorphism $\mathcal{M}_{0,\theta+}(\mathbf{v}, \mathbf{w}) \cong T^*\mathcal{R}_{\theta+}(\mathbf{v}, \mathbf{w})$, of part (ii), is a particularly nice case of the situation considered in Theorem 5.2.2(iii).

Following an observation made by Shmelkin, we alert the reader that the map $T^*\mathcal{R}_{\theta+}(\mathbf{v}, \mathbf{w}) \to X$ need not be surjective, in general. More precisely, one has the following result, which is essentially due Kraft and Procesi [KP], cf. also [Sh, Proposition 2.2(ii)] and [Na1, §7]:

Proposition 5.3.5. *In the above setting, assume in addition that the following (stronger) inequalities hold*

$$r - v_1 \geq v_1 - v_2 \geq v_2 - v_3 \geq \ldots \geq v_{n-1} - v_n \geq v_n. \tag{5.3.6}$$

Then, the map $T^*\mathcal{R}_{\theta+}(\mathbf{v}, \mathbf{w}) \to X$ is surjective. Furthermore, the set $\mathcal{M}_{0,0}^\circ(\mathbf{v}, \mathbf{w})$ gets identified, under the isomorphism $\mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w}) \cong X$, with the unique Zariski open and dense $GL(W)$-conjugacy class in X.

Thus, in this case the affine variety $\mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})$ gets identified with the closure of a nilpotent $GL(W)$-conjugacy class in $\mathfrak{gl}(W)$.

5.4. **A Lagrangian subvariety.** We recall the following standard

Definition 5.4.1. *A locally closed subvariety Λ of a symplectic manifold (M, ω) is called Lagrangian if the tangent space to Λ at any smooth point $x \in \Lambda$ is a maximal isotropic subspace of $T_x M$ (the tangent space to M at x) with respect to the symplectic 2-form ω.*

From now on, we fix a quiver Q, and we let $\lambda = 0$. Below, we will use the second of the two $\mathbb{C}^\times$-actions on Nakajima varieties, introduced in §5.2. Recall that this action is given by the formula $\mathbb{C}^\times \ni t : \phi = (\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \mapsto t(\phi) = (\mathbf{x}, t \cdot \mathbf{y}, \mathbf{i}, t \cdot \mathbf{j})$. Thus, for any $\theta \in \mathbb{Z}^I$, we have the canonical $\mathbb{C}^\times$-equivariant projective morphism $\pi : \mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w}) \to \mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})$.

We define $\Lambda_\theta(\mathbf{v}, \mathbf{w}) := [\pi^{-1}(\mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})^{\mathbb{C}^\times})]_{\mathrm{red}}$, the preimage of the $\mathbb{C}^\times$-fixed point set equipped with reduced scheme structure. Thus, $\Lambda_\theta(\mathbf{v}, \mathbf{w}) \subset \mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w})$ is a reduced closed subscheme.

Theorem 5.4.2. *For a $\mathbf{v}$-regular parameter $(0, \theta)$, we have:*

(i) *Each irreducible component of the variety $\Lambda_\theta(\mathbf{v}, \mathbf{w})$ is a Lagrangian subvariety of $\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w})$, a symplectic manifold.*

(ii) *Assume, in addition, that $\theta = \theta^+$ and the quiver Q has no oriented cycles. Then $\Lambda_\theta(\mathbf{v}, \mathbf{w}) = \pi^{-1}(0)$; furthermore, the $G_\mathbf{v}$-orbit of a quadruple $(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \in \mu^{-1}(0)^{ss}$ represents a point of $\Lambda_\theta(\mathbf{v}, \mathbf{w})$ if and only if we have $\mathbf{i} = 0$ and the $G_\mathbf{v}$-orbit of the pair $(\mathbf{x}, \mathbf{y}) \in \mathrm{Rep}(\overline{Q}, \mathbf{v})$ contains the pair $(0, 0) \in \mathrm{Rep}(\overline{Q}, \mathbf{v})$ in its closure.*

Remark 5.4.3. The statement in (ii) motivates the name 'nilpotent variety' for the variety $\pi^{-1}(0)$.$\Diamond$

We will now proceed with the proof of Theorem 5.4.2(i).

First of all, observe that for any representation $\phi = (\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \in \mathrm{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$, we have $\lim_{t \to 0} t(\phi) = \lim_{t \to 0} (\mathbf{x}, t \cdot \mathbf{y}, \mathbf{i}, t \cdot \mathbf{j}) = (\mathbf{x}, 0, \mathbf{i}, 0)$. The image of the point $(\mathbf{x}, 0, \mathbf{i}, 0)$ in $\mathrm{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})//G_\mathbf{v}$, the categorical quotient, is clearly a $\mathbb{C}^\times$-fixed point. Thus, we conclude that the $\mathbb{C}^\times$-action provides a contraction of $\mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})$ to $\mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})^{\mathbb{C}^\times}$, the fixed point set.

Further, the fixed point set of the $\mathbb{C}^\times$-action in the smooth variety $\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w})$ is a (necessarily smooth) subvariety $F := \mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w})^{\mathbb{C}^\times} \subset \mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w})$. We write $F_1, \ldots, F_r$ for the connected components of F, and introduce the following sets

$$\Lambda_s := \{z \in \mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w}) \mid \lim_{t \to \infty} t(z) \text{ exists, and we have } \lim_{t \to \infty} t(z) \in F_s\}, \quad s = 1, \ldots, r. \tag{5.4.4}$$

Lemma 5.4.5. *For any quiver Q, the set F is contained in $\pi^{-1}(\mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})^{\mathbb{C}^\times})$, and there is a decomposition $\pi^{-1}(\mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})^{\mathbb{C}^\times}) = \bigsqcup_{1 \leq s \leq r} \Lambda_s.$*

Proof. Since π is a $\mathbb{C}^\times$-equivariant morphism, we have $\pi\big(\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})^{\mathbb{C}^\times}\big) \subset \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})^{\mathbb{C}^\times}$. In particular, one has $F \subset \pi^{-1}\big(\mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})^{\mathbb{C}^\times}\big)$.

Now, fix $\widetilde{z} \in \mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$ and let $z = \pi(\widetilde{z}) \in \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$. We consider the maps $\mathbb{C}^\times \to \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$, $t \mapsto t(z)$, resp. $\mathbb{C}^\times \to \mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$, $t \mapsto t(\widetilde{z})$. It is clear that if the limit of $t(z)$, $t \to \infty$, exists then this limit is a $\mathbb{C}^\times$-fixed point.

Assume first that $\widetilde{z} \in L_\theta(\mathbf{v},\mathbf{w})$. Then, z is a $\mathbb{C}^\times$-fixed point and $t(\widetilde{z}) \in \pi^{-1}(z)$ for any t. It follows, since $\pi^{-1}(z)$ is a complete variety, that the map $t \mapsto t(\widetilde{z})$ extends to a regular map $\mathbb{P}^1 \to \mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$. Thus, for any $\widetilde{z} \in \pi^{-1}\big(\mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})^{\mathbb{C}^\times}\big)$, the limit of $t(\widetilde{z})$, $t \to \infty$, exists and we have $\lim_{t\to\infty} t(\widetilde{z}) \in F$. We conclude that $L_\theta(\mathbf{v},\mathbf{w}) \subset \cup_{1 \le s \le r} \Lambda_s$.

Assume next that $\widetilde{z} \in \cup_{1 \le s \le r} \Lambda_s$, so we have $\lim_{t\to\infty} t(\widetilde{z}) \in F$. It follows that the map $t \mapsto \pi(t(\widetilde{z})) = t(z)$ also has a limit as $t \to \infty$. Therefore, the map $\mathbb{C}^\times \to \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$, $t \mapsto t(z)$ extends to the point $t = \infty$. On the other hand, by the observation made before Lemma 5.4.5, the latter map automatically extends to the point $t = 0$. Therefore, we get a regular map $\mathbb{P}^1 \to \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$. Such a map must be a constant map, since $\mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})$ is affine. Thus, we must have $\pi(\widetilde{z}) = z \in \mathcal{M}_{0,0}(\mathbf{v},\mathbf{w})^{\mathbb{C}^\times}$. We conclude that $\widetilde{z} \in L_\theta(\mathbf{v},\mathbf{w})$. The result follows. $\qquad\square$

Remark 5.4.6. We have shown that the $\mathbb{C}^\times$-action provides a contraction of the variety $\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$ to the fixed point set F.

Theorem 5.4.2(i) is clearly a consequence of the following more precise result

Proposition 5.4.7. *Each piece Λ_s is a smooth, connected, locally closed Lagrangian subvariety of $\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$.*

Furthermore, the closures $\overline{\Lambda}_s$, $s = 1, \dots, r$, are precisely the irreducible components of Λ.

Proof. The pieces defined by equation (5.4.4) are known as the Bialinicki-Birula pieces. The Bialinicki-Birula pieces are shown by Bialinicki-Birula to be smooth, connected, and locally closed subsets, for any $\mathbb{C}^\times$-action on a smooth quasi-projective variety X, provided the action contracts X to $X^{\mathbb{C}^\times}$. The first statement of the proposition follows.

Next, we fix a connected component F_s and a point $\phi \in F_s$. The tangent space to $\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$ at ϕ has a weight decomposition with respect to the $\mathbb{C}^\times$-action

$$T_\phi\big(\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})\big) = \bigoplus_{m\in\mathbb{Z}} H_m, \tag{5.4.8}$$

such that $t \in \mathbb{C}^\times$ acts on the direct summand H_m via multiplication by t^m. In particular, we see that $H_0 = T_\phi F$, is the tangent space to the fixed point set F.

Recall that the symplectic 2-form ω on $\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})$ has weight $+1$ with respect to the $\mathbb{C}^\times$-action. Hence, a pair of direct summands H_k and H_l are ω-orthogonal unless $k + l = 1$; furthermore, the 2-form gives a perfect pairing $\omega : H_m \times H_{1-m} \to \mathbb{C}$, for any $m \in \mathbb{Z}$. We see, in particular, that $\bigoplus_{m\le 0} H_m$ is a Lagrangian subspace in $\bigoplus_{m\in\mathbb{Z}} H_m$.

To complete the proof, pick $z \in \Lambda_s$ such that $\lim_{t\to\infty} t(z) = \phi$. It is clear that, for the curve $t \mapsto t(z)$ to have a limit as $t \to \infty$, the tangent vector to the curve at $t = \infty$ must belong to the span of nonpositive weight subspaces. In other words, we must have

$$\frac{d(t(z))}{dt}\Big|_{t=\infty} \in \bigoplus_{m<0} H_m.$$

Since Λ_s is smooth at ϕ, we deduce the equation $T_\phi(\Lambda_s) = \bigoplus_{m\le 0} H_m$. It follows, by the above, that $T_\phi(\Lambda_s)$ is a Lagrangian subspace in $T_\phi\big(\mathcal{M}_{0,\theta}(\mathbf{v},\mathbf{w})\big)$, and the first statement of the proposition is proved.

Now, the decomposition of Lemma 5.4.5 presents Λ as a union of irreducible varieties of equal dimensions, and the second statement of the proposition follows. $\qquad\square$

We also prove the following result which is part of statement (ii) of Theorem 5.4.2.

Lemma 5.4.9. *If the quiver Q has no oriented cycles, then one has $\mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})^{\mathbb{C}^\times} = \{0\}$, the only fixed point.*

Thus, in this case, we have $L_\theta(\mathbf{v}, \mathbf{w}) = \pi^{-1}(0)$.

Proof. A $\mathbb{C}^\times$-fixed point in $\mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w}) = \mu^{-1}(0)/G_\mathbf{v}$ is represented, by definition, by a $\mathbb{C}^\times$-stable and closed $G_\mathbf{v}$-orbit $\mathcal{O} \subset \mu^{-1}(0)$. Hence, for any point $(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \in \mathcal{O}$, in this closed orbit, we must have $\lim_{t\to\infty} (\mathbf{x}, t\cdot\mathbf{y}, \mathbf{i}, t\cdot\mathbf{j}) \in \mathcal{O}$. Thus, we get $(\mathbf{x}, 0, \mathbf{i}, 0) \in \mathcal{O}$. We conclude that the orbit $\mathcal{O}$ is contained in $\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})$, the zero section of the cotangent bundle $T^* \mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w}) = \mathrm{Rep}(\overline{Q^\heartsuit}, \mathbf{v}, \mathbf{w})$.

Observe next that, for any homogeneous polynomial $f \in \mathbb{C}[\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})]^{G_\mathbf{v}}$, of positive degree, we have $f|_{\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})} = 0$, since Q has no oriented cycles, see Proposition 2.1.1. Also, the restriction map $\mathbb{C}[\mathrm{Rep}(Q^\heartsuit, \mathbf{v}, \mathbf{w})]^{G_\mathbf{v}} \to \mathbb{C}[\mu^{-1}(0)]^{G_\mathbf{v}}$ is a surjection, since $\mu^{-1}(0)$ is a closed subvariety and the group $G_\mathbf{v}$ is reductive. It follows from this that any homogeneous invariant polynomial $f \in \mathbb{C}[\mu^{-1}(0)]^{G_\mathbf{v}}$, of positive degree, vanishes on the orbit $\mathcal{O}$. But $G_\mathbf{v}$-invariant polynomials are known to separate closed $G_\mathbf{v}$-orbits. Thus, $\mathcal{O} = \{0\}$. $\qquad\square$

5.5. **Proof of Proposition 1.2.2.** The cohomology vanishing is a standard application of the Grauert-Riemenschneider theorem. The latter theorem says that higher derived direct images of the canonical sheaf under a proper morphism vanish. We apply this to the proper morphism $\pi : \widetilde{X} \to X$. The canonical sheaf of $\widetilde{X}$ is isomorphic to $\mathcal{O}_{\widetilde{X}}$ since $\widetilde{X}$ is a symplectic manifold. Hence, the Grauert-Riemenschneider theorem yields $R^i\pi_*\mathcal{O}_{\widetilde{X}} = 0$ for all $i > 0$. It follows that the complex $R\pi_*\mathcal{O}_{\widetilde{X}}$ representing the derived direct image is quasi-isomorphic to $\pi_*\mathcal{O}_{\widetilde{X}}$, an ordinary direct image.

Now, using the above, for any $i > 0$, we compute

$$H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = \mathbb{H}^i(X, R\pi_*\mathcal{O}_{\widetilde{X}}) = H^i(X, \pi_*\mathcal{O}_{\widetilde{X}}) = 0,$$

where $\mathbb{H}^i(-)$ stands for the hyper-cohomology of a complex of sheaves and where the rightmost equality above holds since $\pi_*\mathcal{O}_{\widetilde{X}}$ is a coherent sheaf on an *affine* variety.

To prove the second statement of Proposition 1.2.2 one uses the following argument due to Wierzba. Write ω for the algebraic symplectic 2-form on $\widetilde{X}$ and let $\overline{\omega}$ be its complex conjugate, an anti-holomorphic 2-form. This 2-form gives a Dolbeau cohomology class $[\overline{\omega}] \in H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. The latter class is in fact equal to zero since we have shown that $H^2(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$.

Now, let $x \in X$. We must prove that the restriction of the 2-form ω, equivalently, the restriction of the 2-form $\overline{\omega}$, to $\pi^{-1}(x)$ vanishes. To this end, let $Y \twoheadrightarrow \pi^{-1}(x)$ be a resolution of singularities of the fiber, and write $f : Y \to \widetilde{X}$ for the composite $Y \twoheadrightarrow \pi^{-1}(x) \hookrightarrow \widetilde{X}$. Thus, $f^*\overline{\omega}$ is an anti-holomorphic 2-form on Y and, in Dolbeau cohomology of Y, we have $[f^*\overline{\omega}] = f^*[\overline{\omega}] = 0$.

On the other hand, Y is a smooth and projective variety. Hence, by Hodge theory, we have $H^2(Y, \mathcal{O}_Y) \xrightarrow{\sim} H^{0,2}(Y, \mathbb{C}) \subset H^2(Y, \mathbb{C})$. It is clear that the Dolbeau cohomology class of the 2-form $f^*\overline{\omega}$ goes, under this isomorphism, to the de Rham cohomology class of $f^*\overline{\omega}$. Thus, in de Rham cohomology of Y, we have $[f^*\overline{\omega}] = 0$. But any nonzero anti-holomorphic differential form on a Kähler manifold gives a nonzero de Rham cohomology class, thanks to Hodge theory. It follows that the 2-form $f^*\overline{\omega}$ vanishes, hence $\overline{\omega}|_{\pi^{-1}(x)} = 0$, and we are done. $\qquad\square$

We remark that, for a quiver without edge-loops, the inequality $\dim \pi^{-1}(0) \leq \frac{1}{2} \dim \mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w})$ is an immediate consequence of Proposition 1.2.2 and Proposition 5.3.1.

Here is another approach to the proof of this inequality (in the special case of quiver varieties). The argument below, based on the 'hyper-Kähler rotation' trick, was suggested to me by Nakajima.

In more detail, using a hyper-Kähler rotation, cf. Remark 5.2.3, we may view $\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w})$ as a smooth and affine algebraic variety of complex dimension $2d$, say. It follows that singular homology groups, $H_i(\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w}), \mathbb{R})$, vanish for all $i > 2d$, by a standard argument from Morse theory. On the other hand, $\Lambda_\theta(\mathbf{v}, \mathbf{w})$ is a compact subset of $\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w})$. Hence, each irreducible component of $\Lambda_\theta(\mathbf{v}, \mathbf{w})$ of complex dimension n (in the original complex structure) gives a nonzero homology class in $H_{2n}(\mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w}), \mathbb{R})$. Thus, we must have $2n \leq 2d$, and the required dimension inequality follows.

5.6. **Hilbert scheme of points.** Let Q be the Jordan quiver, and let $\mathbf{v} \in \mathbb{Z}$ be a positive integer.

In the setting of Example 4.3.4, the fiber of the moment map over a central element $\lambda \cdot \mathrm{Id} \in \mathfrak{gl}_\mathbf{v}$ equals

$$\mu^{-1}(\lambda \cdot \mathrm{Id}) = \{(\mathbf{x}, \mathbf{y}) \in \mathfrak{gl}_\mathbf{v} \times \mathfrak{gl}_\mathbf{v} \mid [\mathbf{x}, \mathbf{y}] = \lambda \cdot \mathrm{Id}\}.$$

This variety is empty for $\lambda \neq 0$ since we have $\mathrm{Tr}([\mathbf{x}, \mathbf{y}]) = 0$. For $\lambda = 0$, we get $\mu^{-1}(0) = \mathcal{Z}$, the *commuting variety* of the Lie algebra $\mathfrak{gl}_\mathbf{v}$.

Let $\imath : \mathbb{C}^\mathbf{v} \hookrightarrow \mathfrak{gl}_\mathbf{v}$ be the imbedding of diagonal matrices. Since any two diagonal matrices commute, we get a closed imbedding $\imath \times \imath : \mathbb{C}^\mathbf{v} \times \mathbb{C}^\mathbf{v} \hookrightarrow \mathcal{Z}$. The group $\mathbb{S}_\mathbf{v} \subset GL_\mathbf{v}$, of permutation matrices, acts diagonally on $\mathcal{Z} \subset \mathfrak{gl}_\mathbf{v} \times \mathfrak{gl}_\mathbf{v}$, and clearly preserves the image of the map $\imath \times \imath$. Therefore, restriction of $\mathrm{Ad}\, GL_\mathbf{v}$-invariant functions induces an algebra map $(\imath \times \imath)^* : \mathbb{C}[\mathcal{Z}]^{\mathrm{Ad}\, GL_\mathbf{v}} \to \mathbb{C}[\mathbb{C}^\mathbf{v} \times \mathbb{C}^\mathbf{v}]^{\mathbb{S}_\mathbf{v}}$. The latter map can be shown to be an algebra isomorphism.

Thus, we deduce

$$\mathcal{M}_{0,0}(\mathbf{v}) = \mu^{-1}(0)//GL_\mathbf{v} = \mathrm{Spec}\, \mathbb{C}[\mathcal{Z}]^{\mathrm{Ad}\, GL_\mathbf{v}} = \mathrm{Spec}\, \mathbb{C}[\mathbb{C}^\mathbf{v} \times \mathbb{C}^\mathbf{v}]^{\mathbb{S}_\mathbf{v}} = (\mathbb{C}^\mathbf{v} \times \mathbb{C}^\mathbf{v})/\mathbb{S}_n. \qquad (5.6.1)$$

Next, we study Nakajima varieties $\mathcal{M}_{\lambda,\theta}(\mathbf{v}, \mathbf{w})$ for the Jordan quiver Q. We have

Therefore, writing $M_\lambda(\mathbf{v}, \mathbf{w}) := \mu^{-1}(\lambda \cdot \mathrm{Id})$ for the corresponding fiber of the moment map, we get

$$M_\lambda(\mathbf{v}, \mathbf{w}) = \{(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \in \mathfrak{gl}_\mathbf{v} \times \mathfrak{gl}_\mathbf{v} \times \mathrm{Hom}(\mathbb{C}^\mathbf{w}, \mathbb{C}^\mathbf{v}) \times \mathrm{Hom}(\mathbb{C}^\mathbf{v}, \mathbb{C}^\mathbf{w}) \mid [\mathbf{x}, \mathbf{y}] + \mathbf{i} \otimes \mathbf{j} = \lambda \cdot \mathrm{Id}\}.$$

Here, $\mathbf{i} \otimes \mathbf{j}$ denotes a rank one linear operator $\mathbb{C}^\mathbf{v} \to \mathbb{C}^\mathbf{v}$, $u \mapsto \langle \mathbf{j}, u \rangle \cdot \mathbf{i}$.

The above variety $M_\lambda(\mathbf{v}, \mathbf{w})$ is nonempty for any $\lambda \in \mathbb{C}$. Below, we restrict ourselves to the special case $\mathbf{w} = 1$. In this case, we may view $\mathbf{i}$ as a vector in $\mathbb{C}^\mathbf{v} = \mathrm{Hom}(\mathbb{C}, \mathbb{C}^\mathbf{v})$, resp. $\mathbf{j}$ as a covector in $(\mathbb{C}^\mathbf{v})^* = \mathrm{Hom}(\mathbb{C}^\mathbf{v}, \mathbb{C})$.

Assume first that $\lambda \neq 0$. Then, one proves that there is no proper subspace $0 \neq S \subsetneq \mathbb{C}^\mathbf{v}$ such that $\mathbf{i} \in S$ and such that S is stable under the maps $\mathbf{x}, \mathbf{y}$. It follows, by Corollary 4.5.3(i) that $M_\lambda(\mathbf{v}, 1)$ is a smooth affine variety and that the $GL_\mathbf{v}$-action on this variety is free. Therefore, each $GL_\mathbf{v}$-orbit in $M_\lambda(\mathbf{v}, 1)$ is closed. We conclude

$$\mathcal{M}_{\lambda,0}(\mathbf{v}, 1) = M_\lambda(\mathbf{v}, 1)//GL_\mathbf{v} = M_\lambda(\mathbf{v}, 1)/GL_\mathbf{v} =: \text{Calogero-Moser space.}$$

Also, we compute

$$\dim \mathcal{M}_{\lambda,0}(\mathbf{v}, 1) = \dim M_\lambda(\mathbf{v}, 1) - \dim GL_\mathbf{v} = (2\mathbf{v}^2 + 2\mathbf{v} - \mathbf{v}^2) - \mathbf{v}^2 = 2\mathbf{v}.$$

Next, let $\lambda = 0$. Then, from the equation $[\mathbf{x}, \mathbf{y}] + \mathbf{i} \otimes \mathbf{j} = 0$ we deduce

$$\langle \mathbf{j}, \mathbf{i} \rangle = \mathrm{Tr}(\mathbf{i} \otimes \mathbf{j}) = -\mathrm{Tr}([\mathbf{x}, \mathbf{y}]) = 0.$$

It follows that $\mathbf{i} \otimes \mathbf{j}$ is a nilpotent rank one linear operator in $\mathbb{C}^\mathbf{v}$.

The following result of linear algebra will play an important role in our analysis, see [EG, Lemma 12.7].

Lemma 5.6.2. *Let* $\mathbf{x}, \mathbf{y} \in \mathfrak{gl}_{\mathbf{v}}$ *be a pair of linear operators such that* $[\mathbf{x}, \mathbf{y}]$ *is a nilpotent rank one operator. Then there exists a basis of* $\mathbb{C}^{\mathbf{v}}$ *such the matrices of* $\mathbf{x}$ *and* $\mathbf{y}$ *in that basis are both upper-triangular.* □

In the case $\lambda = 0$, the variety $M_0(\mathbf{v}, 1)$ is neither smooth nor irreducible. Thus, to get a good quotient one has to impose a stability condition. First, let $\theta = 0$, so $\mathcal{M}_{0,0}(\mathbf{v}, 1)$ is an affine algebraic variety, by Theorem 5.2.2.

To describe this variety explicitly, one shows using the above lemma, that the assignment sending a quadruple $(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \in M_0(\mathbf{v}, 1)$ to the joint spectrum $(\operatorname{Spec} \mathbf{x}, \operatorname{Spec} \mathbf{y}) \in \mathbb{C}^{\mathbf{v}} \times \mathbb{C}^{\mathbf{v}}$, of the operators $\mathbf{x}$ and $\mathbf{y}$, written in an upper-triangular form provided by lemma 5.6.2, gives a well-defined morphism $M_0(\mathbf{v}, 1) \to (\mathbb{C}^{\mathbf{v}} \times \mathbb{C}^{\mathbf{v}})/\mathbb{S}_{\mathbf{v}}$, of algebraic varieties. Moreover, this morphism turns out to induce an algebra isomorphism $\mathbb{C}[\mathbb{C}^{\mathbf{v}} \times \mathbb{C}^{\mathbf{v}}]^{\mathbb{S}_{\mathbf{v}}} \xrightarrow{\sim} \mathbb{C}[M_0(\mathbf{v}, 1)]^{G_{\mathbf{v}}}$.

We conclude that the Nakajima variety with parameters $(\lambda, \theta) = (0, 0)$ is an affine variety

$$\mathcal{M}_{0,0}(\mathbf{v}, 1) \cong (\mathbb{C}^{\mathbf{v}} \times \mathbb{C}^{\mathbf{v}})/\mathbb{S}_{\mathbf{v}}. \tag{5.6.3}$$

Remark 5.6.4. The $\mathbb{C}^{\times}$-action on $\mathcal{M}_{0,0}(\mathbf{v}, 1)$ that has been defined in the previous subsection goes under the above isomorphsm to a $\mathbb{C}^{\times}$-action on $(\mathbb{C}^{\mathbf{v}} \times \mathbb{C}^{\mathbf{v}})/\mathbb{S}_{\mathbf{v}}$. The latter action is given by $\mathbb{C}^{\times} \ni t : (u, v) \mapsto (u, t \cdot v)$. The fixed points of that action form a subset $(\mathbb{C}^{\mathbf{v}}/\mathbb{S}_{\mathbf{v}}) \times \{0\} \subset (\mathbb{C}^{\mathbf{v}} \times \mathbb{C}^{\mathbf{v}})/\mathbb{S}_{\mathbf{v}}$. Note the subset in question does not reduce to a single point. Indeed, the Jordan quiver has an edge loop and, therefore, Lemma 5.4.9 does not apply in our present situation. ◊

Next, we take $\theta := -\theta^{+} = -1 \in \mathbb{Z}$. With this choice of θ, a point $(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \in M_0(\mathbf{v}, 1)$ is stable if and only if condition (5.1.7) holds, and we have

Proposition 5.6.5. *The set of* θ-*semistable points equals*

$$M_0(\mathbf{v}, 1)^s_{-\theta^+} = \{(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \mid [\mathbf{x}, \mathbf{y}] = 0, \ \mathbf{j} = 0, \ \mathbf{i} \ \textit{is a cyclic vector for } (\mathbf{x}, \mathbf{y})\}. \tag{5.6.6}$$

Proof. According to Theorem 5.1.5, a point $(\mathbf{x}, \mathbf{y}, \mathbf{i}, \mathbf{j}) \in M_0(\mathbf{v}, 1)$ is stable if and only if condition (5.1.7) holds. The condition means that $\mathbf{i}$ is a cyclic vector for $(\mathbf{x}, \mathbf{y})$, i.e., we have $\mathbb{C}\langle \mathbf{x}, \mathbf{y} \rangle \mathbf{i} = \mathbb{C}^{\mathbf{v}}$.

We claim that the last equation implies $\mathbf{j} = 0$. To see this, we observe that for any $a \in \mathfrak{gl}_{\mathbf{v}}$, we have

$$\langle \mathbf{j}, \ a\mathbf{i} \rangle = \operatorname{Tr}\left(a \circ (\mathbf{i} \otimes \mathbf{j})\right) = -\operatorname{Tr}\left(a \circ [\mathbf{x}, \mathbf{y}]\right), \qquad \forall a \in \mathfrak{gl}_{\mathbf{v}}. \tag{5.6.7}$$

Assume now that $a \in \mathbb{C}\langle \mathbf{x}, \mathbf{y} \rangle$, is a noncommutative polynomial in $\mathbf{x}$ and $\mathbf{y}$. Then, we may write the matrices $\mathbf{x}, \mathbf{y}$, and a in an upper-triangular form, by Lemma 5.6.2. In this form, the principal diagonal of the matrix $[\mathbf{x}, \mathbf{y}]$ vanishes, and we get $\operatorname{Tr}\left(a \circ [\mathbf{x}, \mathbf{y}]\right) = 0$. Thus, (5.6.7) implies that the linear function $\mathbf{j}$ vanishes on the vector space $\mathbb{C}\langle \mathbf{x}, \mathbf{y} \rangle \mathbf{i} = \mathbb{C}^{\mathbf{v}}$, and the proposition follows. □

For any commuting pair $(\mathbf{x}, \mathbf{y}) \in \mathfrak{gl}_{\mathbf{v}}$ and any vector $\mathbf{i} \in \mathbb{C}^{\mathbf{v}}$, we introduce a set of polynomials in two indeterminates, x and y, as follows

$$J_{\mathbf{x}, \mathbf{y}, \mathbf{i}} := \{f \in \mathbb{C}[x, y] \mid f(\mathbf{x}, \mathbf{y})\mathbf{i} = 0\}.$$

It is clear that $J_{\mathbf{x}, \mathbf{y}, \mathbf{i}}$ is an ideal of the algebra $\mathbb{C}[x, y]$. Furthermore, this ideal has codimension $\mathbf{v}$ in $\mathbb{C}[x, y]$ if and only if the map $\mathbb{C}[x, y]/J_{\mathbf{x}, \mathbf{y}, \mathbf{i}} \to \mathbb{C}^{\mathbf{v}}$, $f \mapsto f(\mathbf{x}, \mathbf{y})\mathbf{i}$, is surjective. The latter holds if and only if $\mathbf{i}$ is a cyclic vector for the pair $(\mathbf{x}, \mathbf{y})$. In fact, one proves

Corollary 5.6.8. *The assigment* $(\mathbf{x}, \mathbf{y}, \mathbf{i}) \mapsto J_{\mathbf{x}, \mathbf{y}, \mathbf{i}}$ *establishes a bijection between the orbit set* $M_0(\mathbf{v}, 1)^s_{-\theta^+}/GL_{\mathbf{v}}$ *and the set of ideals* $J \subset \mathbb{C}[x, y]$ *such that* $\dim \mathbb{C}[x, y]/J = \mathbf{v}$. □

The set of codimension $\mathbf{v}$ ideals in the algebra $\mathbb{C}[x, y]$ has a natural scheme structure. The resulting scheme $\mathrm{Hilb}^n(\mathbb{C}^2)$ turns out to be a smooth connected variety of dimension $2n$, called the *Hilbert scheme* of n points in the plane. Thus, we see that, for $\mathbf{w} = 1$ and $\lambda = 0$, $\theta = -\theta^+$, one has a natural isomorphism

$$\mathcal{M}_{0,-\theta^+}(\mathbf{v}, 1) \cong \mathrm{Hilb}^n(\mathbb{C}^2).$$

In this case, the canonical projective morphism π, cf. (5.6.3),

$$\pi: \ \mathcal{M}_{0,-\theta^+}(\mathbf{v}, 1) = \mathrm{Hilb}^n(\mathbb{C}^2) \ \longrightarrow \ \mathcal{M}_{0,0}(\mathbf{v}, 1) = (\mathbb{C}^{\mathbf{v}} \times \mathbb{C}^{\mathbf{v}})/\mathbb{S}_{\mathbf{v}},$$

turns out to be a resolution of singularities, called the *Hilbert-Chow morphism.*

Remark 5.6.9. One can show that changing our choice of stability condition from $\theta = -\theta^+$ to $\theta = \theta^+$ leads to isomorphic quiver varieties, because of the isomorphisms of Remark 2.3.10.

6. Convolution in homology

In this section, we review a machinery that produces associative, not necessarily commutative, algebras from certain geometric data. The algebras in question are realized as either homology or K-groups of an appropriate variety, and the corresponding algebra structure is given by an operation on homology, resp. on K-theory, known as 'convolution'.

We refer the reader to [CG], ch. 2 and 5, see also [Gi1], for more information about the convolution operation and for other applications of this construction in representation theory.

In the next section, the formalism developed below will be applied to quiver varieties.

6.1. Convolution.
Let $\mathbb{C}[X]$ denote the vector space of $\mathbb{C}$-valued functions on a finite set X. Characteristic functions of one element subsets form a $\mathbb{C}$-base of $\mathbb{C}[X]$.

Let X_r, $i = 1, 2$, be a pair of finite sets. A linear operator $K: \mathbb{C}[X_1] \to \mathbb{C}[X_2]$ is given, in the bases of characteristic functions, by a rectangular $|X_1| \times |X_2|$-matrix $|K(x_2, x_1)|_{x_i \in X_i}$. We may view this matrix as a $\mathbb{C}$-valued function $(x_1, x_2) \mapsto K(x_1, x_2)$, on $X_1 \times X_2$, called the *kernel of the operator K.*

The action of K is then given, in terms of that kernel, by the formula

$$K: \ f \mapsto K * f, \quad \text{where} \quad (K * f)(x_2) := \sum_{x_1 \in X_1} K(x_2, x_1) \cdot f(x_1). \tag{6.1.1}$$

Now, let X_i, $i = 1, 2, 3$, be a triple of finite sets, and let $K: \mathbb{C}[X_1] \to \mathbb{C}[X_2]$ and $K': \mathbb{C}[X_2] \to \mathbb{C}[X_3]$ be a pair of operators, with kernels $K_{32} \in \mathbb{C}[X_3 \times X_2]$ and $K_{21} \in \mathbb{C}[X_2 \times X_1]$, respectively. One may form the composite operator $K \circ K': \mathbb{C}[X_1] \to \mathbb{C}[X_3]$, $f \mapsto K(K'(f))$.

Explicitly, in terms of the kernels, for any $f \in \mathbb{C}[X_1]$, the function $K(K'(f))$ is given by

$$x_3 \ \mapsto \ K(K'(f))(x_3) = \sum_{x_2 \in X_2} K_{32}(x_3, x_2) \cdot \left(\sum_{x_1 \in X_1} K_{21}(x_2, x_1) \cdot f(x_1) \right)$$

$$= \sum_{x_1 \in X_1} \left(\sum_{x_2 \in X_2} K_{32}(x_3, x_2) \cdot K_{21}(x_2, x_1) \right) \cdot f(x_1).$$

Thus, the kernel of the composite operator $K \circ K'$ is a function $K_{32} * K_{21}$, on $X_3 \times X_1$, given by the formula

$$(x_3, x_1) \ \mapsto \ (K_{32} * K_{21})(x_3, x_1) \ := \ \sum_{x_2 \in X_2} K_{32}(x_3, x_2) \cdot K_{21}(x_2, x_1). \tag{6.1.2}$$

The operation

$$* : \quad \mathbb{C}[X_3 \times X_2] \times \mathbb{C}[X_2 \times X_1] \longrightarrow \mathbb{C}[X_3 \times X_1], \quad K_{32} \times K_{21} \mapsto K_{32} * K_{21} \qquad (6.1.3)$$

is called *convolution* of kernels. Thinking of kernels as of rectangular matrices, the convolution becomes nothing but matrix multiplication. Thus, formula (6.1.2) corresponds to the standard matrix multiplication for $|X_3| \times |X_2|$-matrix by a $|X_2| \times |X_1|$-matrices. So, all we have done so far was a reinterpretation of the fact that composition of linear operators corresponds to a product of corresponding matrices.

Remark 6.1.4. A There is an equivalent, but slightly more elegant, way to write formula (6.1.2) as follows.

For any map $p : X \to Y$, of finite sets, one has a pull-back map $p^* : \mathbb{C}[Y] \to \mathbb{C}[X]$, of functions given by $(p^* f)(x) := p(f(x))$, $\forall x \in X$. We also define a *push-forward* linear map on functions by

$$p_* : \quad \mathbb{C}[X] \to \mathbb{C}[Y], \quad f \mapsto p_* f, \quad \text{where} \quad (p_* f)(y) := \sum_{\{x \in p^{-1}(y)\}} f(x). \qquad (6.1.5)$$

For any pair $i, j \in \{1, 2, 3\}$, let $p_{ij} : X_3 \times X_2 \times X_1 \to X_i \times X_j$ be the projection along the factor not named. It is clear that, with the above notation, formula (6.1.2) may be rewritten as follows

$$K_{32} * K_{21} \quad := \quad (p_{31})_* \big((p_{32}^* K_{32}) \cdot (p_{21}^* K_{21}) \big). \qquad (6.1.6)$$

We will be especially interested in a special case of convolution (6.1.6) where $X_1 = X_2 = X_3 = X$ is a set with n elements. Then, the convolution product (6.1.6) makes $\mathbb{C}[X \times X]$ an associative algebra. According to the preceeding discussion, this algebra is isomorphic to the algebra of $n \times n$-matrices.

One may get more interesting examples of convolution algebras by considering an equivariant version of the above construction, where there is a group G acting on a finite set X. We let G act diagonally on $X \times X$ and let $\mathbb{C}[X \times X]^G \subset \mathbb{C}[X \times X]$ be the subspace of G-invariant functions. This space is clearly isomorphic to $\mathbb{C}[(X \times X)/G]$, the space of functions on the set of G-diagonal orbits in $X \times X$.

It is immediate to check that the convolution product (6.1.2)-(6.1.3) is G-equivariant, hence it makes $\mathbb{C}[X \times X]^G$ a subalgebra of $\mathbb{C}[X \times X]$. The resulting algebra $\big(\mathbb{C}[X \times X]^G, * \big)$ may be shown to be always semisimple. Such an algebra need not be simple, so it is not necessarily isomorphic to a matrix algebra, in general.

Example 6.1.7 (Group algebra). Given a finite group G, we take $X = G$. We let G act on X by left translations, and act diagonally on $G \times G$, as before. Observe that the map $G \times G \to G$, $(g_1, g_2) \mapsto g_1^{-1} \cdot g_2$ descends to a well defined map $(G \times G)/G \to G$. Moreover, the latter map is easily seen to be a bijection.

We deduce the following chain of vector space isomorphisms

$$\mathbb{C}[G \times G]^G \xrightarrow{\sim} \mathbb{C}[(G \times G)/G] \xrightarrow{\sim} \mathbb{C}[G]. \qquad (6.1.8)$$

It is straightforward to check that the restriction of convolution (6.1.2)-(6.1.3) to $\mathbb{C}[G \times G]^G$ goes, under the composite isomorphism in (6.1.8), to the standard convolution on a group. The latter is given by

$$(f * f')(g) = \sum_{h \in G} f(gh^{-1}) \cdot f'(h), \qquad \forall f, f' \in \mathbb{C}[G].$$

We conclude that the algebra $\big(\mathbb{C}[G \times G]^G, * \big)$, with convolution product (6.1.6), is isomorphic to the *group algebra* of G.

Example 6.1.9 (Hecke algebra). Let $G = G(\mathbb{F})$ be a split reductive group over a finite field $\mathbb{F} = \mathbb{F}_q$. Let $B \subset G$ be a Borel subgroup of G. We put $X := G/B$, and let G act on X by left translations. It is known, thanks to the Bruhat decomposition, that G-diagonal orbits in $G/B \times G/B$ are labelled by the elements of W, the Weyl group of G.

The resulting convolution algebra $H_q(G) := \big(\mathbb{C}[G/B \times G/B]^G, *\big)$ is called the *Hecke algebra* of G.

6.2. Borel-Moore homology.

We are going to extend the constructions of the previous subsection to the case where finite sets are replaced by smooth C^∞-manifolds.

Thus, we let X_i, $i = 1, 2, 3$, be a triple of smooth manifolds. One might try to replace the summation in formula (6.1.2) by integration to get a convolution product of the form $* \colon C^\infty(X_3 \times X_2) \times C^\infty(X_2 \times X_1) \to C^\infty(X_3 \times X_1)$, cf (6.1.3).

To make this work, one still needs additional ingredients. One such ingredient is a *measure* on X_2 that is necessary in order to define the integral that replaces summation in formula (6.1.2).

An alternate approach, that does not require introducing a measure, is to replace functions by differential forms. In this way, one defines a convolution product

$$\Omega^p(X_3 \times X_2) \times \Omega^q(X_2 \times X_1) \to \Omega^{p+q-\dim X_2}(X_3 \times X_1), \;\; K_{32} \times K_{21} \mapsto \int_{X_2} (p_{32}^* K_{32}) \wedge (p_{21}^* K_{21}). \quad (6.2.1)$$

To insure the convergence of the integral in (6.2.1) one may assume, for instance, that the manifold X_2 is compact. A slightly weaker assumption, that is sufficient for (6.2.1) to make sence, is to restrict considerations to differential forms with certain support condition that would insure, in particular, that the set

$$p_{32}^{-1}(\operatorname{supp} K_{32}) \;\cap\; p_{21}^{-1}(\operatorname{supp} K_{21}) \;\; \text{be compact.} \quad (6.2.2)$$

Unfortunately, none of the above works in the examples arising from quiver varieties that we would like to consider below. In those examples, the manifolds X_i, $i = 1, 2, 3$, are the quiver varieties, which are *noncompact* complex algebraic varieties. It turns out that the only natural support condition one could make in those cases in order for (6.2.2) to hold, is to require supports of K_{32} and K_{21}, in (6.2.1), be contained in appropriate *closed* algebraic subvarieties.

Obviously, any C^∞-differential form on a manifold whose support is contained in a closed (proper) submanifold must vanish identically. There are, however, plenty of 'distribution-like' differential forms, called *currents*, which may be supported on closed submanifolds. Indeed, replacing differential forms by currents resolves the convergence problem for integration. Unfortunately, introducing currents creates another problem: the wedge-product operation, which is used in (6.2.1), is not well defined for currents.

All the above difficulties may be resolved by introducing homology. Recall that there is the de Rham differential acting on the (graded) vector space $\Omega^\bullet(X)$, of differential forms on a manifold X. The homology of the resulting de Rham complex $\big(\Omega^\bullet(X),\, d\big)$ is isomorphic to $H^\bullet(X, \mathbb{C})$, the singular cohomology of X with complex coefficients. Similarly, there is a natural de Rham differential on the (graded) vector space of currents on X, and the homology of the resulting complex is known to be isomorphic to $H_\bullet^{BM}(X, \mathbb{C})$, the *Borel-Moore homology* of X with complex coefficients. The latter is the homology theory that we are going to use.

For practical purposes, it is more convenient to use a different (*a posteriori* equivalent) definition of Borel-Moore homology based on Poincaré duality rather than on the de Rham complex of currents. We now recall this definition.

Let M be a smooth *oriented* C^∞-manifold of real dimension m. One defines Borel-Moore homology of a closed subset $X \subset M$ to be the following relative cohomology

$$H_\bullet^{BM}(X) := H^\bullet(M, M \smallsetminus X; \mathbb{C}). \quad (6.2.3)$$

It can be shown that the group on the right is, in fact, independent of the choice of a closed imbedding of X into a smooth manifold.

Notation 6.2.4. **From now on, we drop the superscript 'BM' and let $H_\bullet(X)$ stand for Borel-Moore homology (rather than ordinary homology) of X.**

A property that makes Borel-Moore homology so useful for our purposes is that, for any X, which is either a smooth connected, and *oriented* C^∞-manifold or an irreducible *complex* algebraic variety, the space $H_m(X)$, where $m := \dim_\mathbb{R} X$, is 1-dimensional; furthermore, there is a canonical base element $[X] \in H_m(X)$, called the *fundamental class* of X.

Remark 6.2.5. Note that, in the ordinary homology theory, fundamental classes only exist for *compact* manifolds, while such a compactness condition is not necessary for the fundamental class to exist in Borel-Moore homology. ◇

We record a few basic properties of the Borel-Moore homology theory. First, for any *proper* map $p : X \to Y$, there is a push-forward functor $p_* : H_\bullet(X) \to H_\bullet(Y)$.

Second, there is a cap-product on Borel-Moore homology. In more detail, given two closed subsets $X, Y \subset M$, where M is a smooth oriented manifold of real dimension m, one has a cup product

$$\cup : H^{m-i}(M, M \smallsetminus X;\ \mathbb{C}) \times H^{m-j}(M, M \smallsetminus Y;\ \mathbb{C}) \to H^{2m-i-j}(M, M \smallsetminus (X \cup Y);\ \mathbb{C}).$$

We define a cap-product on Borel-Moore homology by transporting the above cup product via formula (6.2.3); this way we obtain a cap-product pairing

$$\cap :\ H_i(X) \times H_j(Y) \to H_{i+j-m}(X), \qquad m = \dim_\mathbb{R} M. \tag{6.2.6}$$

It should be emphasized that the cap-product so defined *does* depend on the ambient smooth manifold M.

6.3. Convolution in Borel-Moore homology.

There is a convolution product in Borel-Moore homology that provides an adequate generalization, from the case of finite sets to the case of manifolds, of the convolution product (6.1.6).

To define the convolution product , fix M_i, $i = 1, 2, 3$, a triple of smooth oriented manifolds, and let $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ denote the projection along the factor not named, cf. (6.1.6).

Definition 6.3.1. *A pair of closed subsets $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ is said to be* composable *if the following map* (6.3.2) *is* proper

$$p_{13} :\ (p_{12}^{-1} Z_{12}) \cap (p_{23}^{-1} Z_{23}) \to M_1 \times M_3. \tag{6.3.2}$$

Given composable subsets as above, we define their composite *to be*

$$Z_{12} \circ Z_{23} := p_{13}\big[(p_{12}^{-1} Z_{12}) \cap (p_{23}^{-1} Z_{23})\big]\ \subset\ M_1 \times M_3.$$

Now, let $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ be as above, and put $m_i := \dim M_i$.

We use $M := M_1 \times M_2 \times M_3$ as an ambient manifold and apply formula (6.2.6). In this way, we get a cap product map

$$\cap :\ H_{i+m_3}(p_{12}^{-1} Z_{12}) \times H_{j+m_1}(p_{23}^{-1} Z_{23})\ \longrightarrow\ H_{i+j-m_2}((p_{12}^{-1} Z_{12}) \cap (p_{23}^{-1} Z_{23})).$$

Assume further that Z_{12} and Z_{23} are composable. Then, we have a push-forward morphism $(p_{13})_*$, on Borel-Moore homology, induced by the *proper* map (6.3.2).

One defines the convolution in Borel-Moore homology as the following map, cf. (6.1.6), (6.2.1),

$$* :\ H_i(Z_{12}) \times H_j(Z_{23})\ \longrightarrow\ H_{i+j-\dim M_2}(Z_{12} \circ Z_{23}),$$

$$c_{12} \times c_{23}\ \mapsto\ c_{12} * c_{23} := (p_{13})_*\Big((c_{12} \boxtimes [M_3]) \cap ([M_1] \boxtimes c_{23})\Big). \tag{6.3.3}$$

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6.4. Convolution algebra. Fix M, a smooth complex algebraic variety, not necessarily connected, in general. Further, let Y be a (not necessarily smooth) algebraic variety and $\pi : M \to Y$, a *proper* morphism. Thus, we may form a fiber product $Z := M \times_Y M$, a closed subvariety of $M \times M$.

One may apply the convolution in Borel-Moore homology operation in a special case where $M_1 = M_2 = M_3 = M$, and $Z_{12} = Z_{23} = Z$. The assumption the morphism π be proper insures that the set Z is composable with itself in the sense of Definition 6.3.1. Furthermore, it is immediate to check that one has $Z \circ Z = Z$. Thus, the convolution product (6.3.3) gives $H_\bullet(Z)$, the total Borel-Moore homology group of Z, a structure of associative algebra. The fundamental class $[\Delta]$, of the diagonal $\Delta \subset M \times M$, is the unit of the algebra $\left(H_\bullet(Z), \ast\right)$.

Next, pick a point $y \in Y$ and put $M_y := \pi^{-1}(y)$. Consider the setting of section 6.3 in the special case where $M_1 = M_2 = M$, and where $M_3 = pt$ is a point. Thus, we have $M_2 \times M_3 = M_2 \times pt = M$, and put $Z_{12} := M \times_Y M = Z$, as before, and $Z_{23} := M_y = \pi^{-1}(y)$, viewed as a closed subset in $M_2 \times M_3 = M$.

It is immediate to check that the sets Z and M_y are composable and, moreover, one has $Z \circ M_y = M_y$. Therefore, convolution in BM homology gives the space $H_\bullet(M_y)$ an $H_\bullet(Z)$-module structure.

Let $\mathscr{V}$ denote a set that provides a labelling for connectected components of the manifold M. We write $M^{(r)}$ for the connectected component with label $r \in \mathscr{V}$. For any pair $M^{(r)}, M^{(s)}$, of connectected components, we put $Z^{(r,s)} := Z \cap (M^{(r)} \times M^{(s)})$. Similarly, we put $M_y^{(r)} := M_y \cap M^{(r)}$, for any $r \in \mathscr{V}$. Clearly, we have $H_\bullet(Z) = \bigoplus_{r,s \in \mathscr{V}} H_\bullet(Z^{(r,s)})$, resp. $H_\bullet(M_y) = \bigoplus_{r \in \mathscr{V}} H_\bullet(M_y^{(r)})$.

We write $H_{\mathrm{top}}(M_y^{(r)})$ for the top Borel-Moore homology group of $M_y^{(r)}$. This group has a natural basis formed by the fundamental classes of irreducible components of the variety $M_y^{(r)}$ of maximal dimension.

Next, for each pair (r,s), we introduce a new $\mathbb{Z}$-grading on the vector space $H_\bullet(Z^{(r,s)})$ as follows

$$H_{[i]}(Z^{(r,s)}) := H_{d-i}(Z^{(r,s)}) \quad \text{where} \quad d := \tfrac{1}{2}(\dim_\mathbb{R} M^{(r)} + \dim_\mathbb{R} M^{(s)}). \qquad (6.4.1)$$

We extend this grading to $H_\bullet(Z)$ by setting $H_{[i]}(Z) = \bigoplus_{r,s \in \mathscr{V}} H_{[i]}(Z^{(r,s)})$.

The following result is an immediate consequence of formula (6.3.3).

Lemma 6.4.2. (i) *The new grading makes $H_{[\bullet]}(Z)$ a graded algebra with respect to the convolution product, i.e., we have $H_{[i]}(Z) \ast H_{[j]}(Z) \subset H_{[i+j]}(Z)$, for any $i, j \in \mathbb{Z}$. In particular, $H_{[0]}(Z)$ is a subalgebra of the convolution algebra $\left(H_{[\bullet]}(Z), \ast\right)$.*

(ii) *For any $y \in Y$, the vector space $H_{\mathrm{top}}(M_y) := \bigoplus_{r \in \mathscr{V}} H_{\mathrm{top}}(M_y^{(r)})$ is stable under the convolution-action of the subalgebra $H_{[0]}(Z) \subset H_{[\bullet]}(Z)$ on $H_\bullet(M_y)$.* $\qquad\square$

Remark 6.4.3. In the especially important case where M is connected and the map $\pi : M \to Y$ is semismall, eg. the case where π is a symplectic resolution, for the integer d appearing in (6.4.1), we have $d = \dim_\mathbb{R} M = \dim_\mathbb{R} Z$. Thus, in such a case, one has $H_{[0]}(Z) = H_{\mathrm{top}}(Z)$. This group has a natural basis formed by the fundamental classes of irreducible components of the variety Z of maximal dimension. $\qquad\Diamond$

Remark 6.4.4. The material of §§6.3-6.4 is taken from [Gi3]. The general notion of convolution algebra in Borel-Moore homology, as well as the geometric construction of its irreducible representations, was discovered in that paper.

7. KAC-MOODY ALGEBRAS AND QUIVER VARIETIES

7.1. Throughout this subsection, we fix a quiver Q, without edge loops, and a dimension vector $\mathbf{w} \in \mathbb{Z}^I$. We also fix a stability parameter $\theta \in \mathbb{R}^I$ and use simplified notation $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w}) := \mathcal{M}_{0,\theta}(\mathbf{v}, \mathbf{w})$, resp. $\mathcal{M}_0(\mathbf{v}, \mathbf{w}) = \mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})$. Recall that we write $\mathbf{v} \geq \mathbf{v}'$ whenever $\mathbf{v} - \mathbf{v}' \in \mathbb{Z}_{\geq 0}^I$.

Given a pair $0 \leq \mathbf{v}' \leq \mathbf{v}$, of dimension vectors, we choose I-graded vector spaces V, V', and W such that $\dim_I V = \mathbf{v}$, $\dim_I V' = \mathbf{v}'$, and $\dim_I W = \mathbf{w}$. Thus, we identify Nakajima's varieties of relevant dimesions with corresponding Hamiltonian reductions of the representation spaces $\mathrm{Rep}(\overline{Q^\heartsuit}, V, W)$, resp. $\mathrm{Rep}(\overline{Q^\heartsuit}, V', W)$. Therefore, a choice of I-graded vector space isomorphism $\phi : V' \oplus V'' \xrightarrow{\sim} V$ clearly induces a vector space imbedding $\jmath_\phi : \mathrm{Rep}(\overline{Q^\heartsuit}, V', W) \to \mathrm{Rep}(\overline{Q^\heartsuit}, V, W)$, $(\mathbf{x}', \mathbf{y}', \mathbf{i}', \mathbf{j}') \mapsto (\mathbf{x}', \mathbf{y}', \mathbf{i}', \mathbf{j}') \oplus 0''$, where $0'' \in \mathrm{Rep}(\overline{Q^\heartsuit}, V'', W)$ denotes the zero quadruple. The latter imbedding induces a morphism $\mathcal{M}_0(\mathbf{v}', \mathbf{w}) \to \mathcal{M}_0(\mathbf{v}, \mathbf{w})$, of the corresponding categorical quotients.

Remark 7.1.1. Note that the map $\jmath_\phi$ does *not* give rise to any natural morphism $\mathcal{M}_\theta(\mathbf{v}', \mathbf{w}) \to \mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$ because the stability conditions involved in the definitions of these spaces are not compatible, in general. $\diamondsuit$

We observe that, for any other I-graded vector space isomorphism $\psi : V' \oplus V'' \xrightarrow{\sim} V$, there exists an element $g \in G_\mathbf{v}$ such that one has $\jmath_\psi = g \circ \jmath_\phi$. It follows, that the maps $\jmath_\phi$ and $\jmath_\psi$ induce the *same* morphism $\imath_{\mathbf{v}', \mathbf{v}} : \mathcal{M}_0(\mathbf{v}', \mathbf{w}) \to \mathcal{M}_0(\mathbf{v}, \mathbf{w})$. Thus, the latter morphism is defined canonically. Furthermore, according to [Na4, Lemma 2.5.3], one has

Lemma 7.1.2. *For any dimension vectors* $\mathbf{v}' \leq \mathbf{v}$, *the canonical morphism* $\imath_{\mathbf{v}', \mathbf{v}} : \mathcal{M}_0(\mathbf{v}', \mathbf{w}) \to \mathcal{M}_0(\mathbf{v}, \mathbf{w})$ *is a closed imbedding.* $\square$

Thus, for any $\mathbf{v}' \leq \mathbf{v}$, one has natural inclusions $\mathcal{M}_0^\circ(\mathbf{v}', \mathbf{w}) \hookrightarrow \mathcal{M}_0(\mathbf{v}', \mathbf{w}) \hookrightarrow \mathcal{M}_0(\mathbf{v}, \mathbf{w})$. Here, the first inclusion is an open, resp. the second inclusion $\imath_{\mathbf{v}'\mathbf{v}}$ is a closed, imbedding. Below, we will identify $\mathcal{M}_0^\circ(\mathbf{v}', \mathbf{w})$ with a subset of $\mathcal{M}_0(\mathbf{v}, \mathbf{w})$ via the composite imbedding and put

$$\mathcal{M}_0^{\mathrm{good}}(\mathbf{v}, \mathbf{w}) := \cup_{0 \leq \mathbf{v}' \leq \mathbf{v}} \mathcal{M}_0^\circ(\mathbf{v}', \mathbf{w}). \tag{7.1.3}$$

Remark 7.1.4. (i) If Q is a finite Dynkin quiver of type $\mathbf{A}, \mathbf{D}, \mathbf{E}$, cf. Example 4.5.9, then, according to [Na2, Remark 3.28], one has $\mathcal{M}_0(\mathbf{v}, \mathbf{w}) = \mathcal{M}_0^{\mathrm{good}}(\mathbf{v}, \mathbf{w})$.

(ii) Let Q be an extended Dynkin quiver and let $\Gamma \subset SL_2(\mathbb{C})$ be the finite subgroup associated with Q via the McKay correspondence, cf. §4.6. Then, one can show that there is a natural decomposition

$$\mathcal{M}_0(\mathbf{v}, \mathbf{w}) = \bigcup_{\{\mathbf{v}' \in \mathbb{Z}_{\geq 0}^I, \ k \geq 0 \ | \ \mathbf{v}' + k \cdot \delta \leq \mathbf{v}\}} \mathcal{M}_0^\circ(\mathbf{v}', \mathbf{w}) \times \mathrm{Sym}^k(\mathbb{C}^2/\Gamma),$$

where δ denotes the minimal imaginary root.

Furthermore, the set $\mathcal{M}_0^{\mathrm{good}}(\mathbf{v}, \mathbf{w})$ equals the union of pieces in the above decomposition corresponding to $k = 0$. Thus, in this case, we have $\mathcal{M}_0(\mathbf{v}, \mathbf{w}) \neq \mathcal{M}_0^{\mathrm{good}}(\mathbf{v}, \mathbf{w})$, in general.

7.2. **A Steinberg type variety.** Let $\mathbf{v}', \mathbf{v} \in \mathbb{Z}_{\geq 0}^I$ be a pair of dimension vectors, and identify $\mathcal{M}_0(\mathbf{v}', \mathbf{w})$ with a closed subset of $\mathcal{M}_0(\mathbf{v} + \mathbf{v}', \mathbf{w})$ via the canonical imbedding. Thus, we get a well defined composite $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w}) \to \mathcal{M}_0(\mathbf{v}, \mathbf{w}) \hookrightarrow \mathcal{M}_0(\mathbf{v}, \mathbf{w})$, where the first map is the canonical projective morphism.

Definition 7.2.1. *Given* $\theta \in \mathbb{Z}^I$ *and any pair* $\mathbf{v}, \mathbf{v}' \in \mathbb{Z}^I$, *of dimension vectors, we define an associated* Steinberg variety

$$Z_\theta(\mathbf{v}, \mathbf{v}', \mathbf{w}) := \mathcal{M}_\theta(\mathbf{v}, \mathbf{w}) \times_{\mathcal{M}_0(\mathbf{v}+\mathbf{v}', \mathbf{w})} \mathcal{M}_\theta(\mathbf{v}', \mathbf{w}) \subset \mathcal{M}_\theta(\mathbf{v}, \mathbf{w}) \times \mathcal{M}_\theta(\mathbf{v}', \mathbf{w}), \tag{7.2.2}$$

as a fiber product in the following diagram

$$Z(\mathbf{v}, \mathbf{v}', \mathbf{w}) \tag{7.2.3}$$

$$\mathcal{M}_\theta(\mathbf{v}, \mathbf{w}) \qquad\qquad\qquad \mathcal{M}_\theta(\mathbf{v}', \mathbf{w})$$

$$\downarrow^\pi \qquad\qquad\qquad\qquad\qquad \downarrow^\pi$$

$$\mathcal{M}_0(\mathbf{v}, \mathbf{w}) \overset{\iota_{\mathbf{v},\mathbf{v}+\mathbf{v}'}}{\hookrightarrow} \mathcal{M}_0(\mathbf{v}+\mathbf{v}', \mathbf{w}) \overset{\iota_{\mathbf{v}',\mathbf{v}+\mathbf{v}'}}{\longleftarrow} \mathcal{M}_0(\mathbf{v}', \mathbf{w})$$

By definition, the morphisms π induce a natural projective morphism $\pi_Z : Z_\theta(\mathbf{v}, \mathbf{v}', \mathbf{w}) \rightarrow \mathcal{M}_0(\mathbf{v} + \mathbf{v}', \mathbf{w})$.

The Steinberg variety is typically quite singular and has many irreducible components. In the special case where $\mathbf{v} = \mathbf{v}'$, the diagonal $\mathcal{M}_\theta(\mathbf{v}, \mathbf{w}) \subset \mathcal{M}_\theta(\mathbf{v}, \mathbf{w}) \times \mathcal{M}_\theta(\mathbf{v}, \mathbf{w})$ is one such component, which is smooth provided θ is $\mathbf{v}$-regular.

Assume now that $\theta = \theta^+$, and write $\mathcal{M}(\mathbf{v}, \mathbf{w}) := \mathcal{M}_{\theta+}(\mathbf{v}, \mathbf{w})$, resp. $Z(\mathbf{v}, \mathbf{v}', \mathbf{w}) := Z_\theta(\mathbf{v}, \mathbf{v}', \mathbf{w})$. Then, $\mathcal{M}(\mathbf{v}, \mathbf{w})$, resp. $\mathcal{M}(\mathbf{v}', \mathbf{w})$, is a smooth symplectic algebraic variety with symplectic 2-form ω, resp. ω'. We equip the cartesian product $\mathcal{M}(\mathbf{v}, \mathbf{w}) \times \mathcal{M}(\mathbf{v}', \mathbf{w})$ with the symplectic 2-form $\omega + (-\omega')$.

Next, recall the set introduced in (7.1.3) and put

$$Z^{\mathrm{good}}(\mathbf{v}, \mathbf{v}', \mathbf{w}) := Z(\mathbf{v}, \mathbf{v}', \mathbf{w}) \smallsetminus \overline{\pi_Z^{-1}\big(\mathcal{M}_0(\mathbf{v}+\mathbf{v}', \mathbf{w}) \smallsetminus \mathcal{M}_0^{\mathrm{good}}(\mathbf{v}, \mathbf{w})\big)},$$

where bar stands for the closure. Thus, $Z^{\mathrm{good}}(\mathbf{v}, \mathbf{v}', \mathbf{w})$ is an open subset of $Z(\mathbf{v}, \mathbf{v}', \mathbf{w})$.

According to [Na2], Theorem 7.2, one has

Theorem 7.2.4. (i) *Any irreducible component of $Z^{\mathrm{good}}(\mathbf{v}, \mathbf{v}', \mathbf{w})$ is a (locally closed) Lagrangian subvariety of $\mathcal{M}(\mathbf{v}, \mathbf{w}) \times \mathcal{M}(\mathbf{v}', \mathbf{w})$.*

(ii) *The dimension of any irreducible component of $Z(\mathbf{v}, \mathbf{v}', \mathbf{w})$ is $\leq \frac{1}{2}\big(\dim \mathcal{M}(\mathbf{v}, \mathbf{w}) + \dim \mathcal{M}(\mathbf{v}', \mathbf{w})\big)$.*

Nakajima also proves, cf. [Na2, Corollary 10.11].

Proposition 7.2.5. *Let Q be either a finite Dynkin or an extended Dynkin quiver. Then, each irreducible component of $Z(\mathbf{v}, \mathbf{v}', \mathbf{w})$ has dimension equal to $\frac{1}{2}\big(\dim \mathcal{M}(\mathbf{v}, \mathbf{w}) + \dim \mathcal{M}(\mathbf{v}', \mathbf{w})\big)$.*

In the case of finite Dynkin quivers, the result follows from Remark 7.1.4. The extended Dynkin case may be proved using the fact that a similar result is known to hold for the Jordan quiver, see [Na6, Remark 1.23].[3]

7.3. **Geometric construction of $\widetilde{U}(\mathfrak{g})$.** We keep the assumption and notation of the previous subsection, in particular, we take $(\lambda, \theta) = (0, \theta^+)$. and write $\mathcal{M}(\mathbf{v}, \mathbf{w}) := \mathcal{M}_{0,\theta+}(\mathbf{v}, \mathbf{w})$, etc. We use simplified notation $\mathcal{M}(\mathbf{v}, \mathbf{w}) := \mathcal{M}_{0,\theta+}(\mathbf{v}, \mathbf{w})$, and $\mathcal{M}_0(\mathbf{v}, \mathbf{w}) = \mathcal{M}_{0,0}(\mathbf{v}, \mathbf{w})$, etc.

We introduce the following disconnected varieties

$$M(\mathbf{w}) := \bigsqcup_{\mathbf{v} \in \mathbb{Z}^I} \mathcal{M}(\mathbf{v}, \mathbf{w}), \quad M_0(\mathbf{w}) := \bigsqcup_{\mathbf{v} \in \mathbb{Z}^I} \mathcal{M}_0(\mathbf{v}, \mathbf{w}), \quad Z(\mathbf{w}) := \bigsqcup_{(\mathbf{v}, \mathbf{v}') \in \mathbb{Z}^I \times \mathbb{Z}^I} Z(\mathbf{w}, \mathbf{v}, \mathbf{v}').$$

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[3]I am grateful to H. Nakajima for clarifying this point to me.

Thus, the morphisms $\pi : \mathcal{M}(\mathbf{v}, \mathbf{w}) \to \mathcal{M}_0(\mathbf{v}, \mathbf{w})$ may be assembled together to give a morphism $M(\mathbf{w}) \to M_0(\mathbf{w})$, and we have $Z(\mathbf{w}) = M(\mathbf{w}) \times_{M_0(\mathbf{w})} M(\mathbf{w})$. Also, we define

$$H_{\mathbf{w}} := \bigoplus_{m \geq 0} \left(\prod_{\{(\mathbf{v},\mathbf{v}') \in \mathbb{Z}_{\geq 0}^I \times \mathbb{Z}_{\geq 0}^I \,|\, |\mathbf{v}-\mathbf{v}'| \leq m\}} H_{[0]}\big(Z(\mathbf{w}, \mathbf{v}, \mathbf{v}')\big) \right),$$

where, for any $\mathbf{v}, \mathbf{v}' \in \mathbb{Z}^I$, we write $|\mathbf{v} - \mathbf{v}'| := \sum_{i \in I} |v_i - v_i'|$.

Thus, $H_{\mathbf{w}}$ is a certain completion of the direct sum $\bigoplus_{\mathbf{v},\mathbf{v}'} H_{[0]}\big(Z(\mathbf{w}, \mathbf{v}, \mathbf{v}')\big)$ whose elements are, in general, infinite sums; at a heuristic level, one has $H_{\mathbf{w}} = H_{[0]}(Z(\mathbf{w}))$. It is easy to see that convolution in Borel-Moore homology for various pairs of spaces $Z(\mathbf{w}, \mathbf{v}, \mathbf{v}')$ extends to a well defined operation on $H_{\mathbf{w}}$ that makes $(H_{\mathbf{w}}, *)$ an assiciative $\mathbb{C}$-algebra.

We also let $\Lambda(\mathbf{v}, \mathbf{w}) = \pi^{-1}(0)$ be the zero fiber of the morphism π, cf. §5.4. We put

$$\Lambda_{\mathbf{w}} := \bigsqcup_{\mathbf{v} \in \mathbb{Z}^I} \Lambda(\mathbf{v}, \mathbf{w}), \ \text{ resp. } \ L_{\mathbf{w}} := \bigoplus_{\mathbf{v} \in \mathbb{Z}^I} H_{\text{top}}\big(\Lambda(\mathbf{v}, \mathbf{w})\big).$$

Thus, heuristacally, one has $L_{\mathbf{w}} = H_{\text{top}}(\Lambda_{\mathbf{w}})$.

Recall next that, associated with the Cartan matrix C_Q, of the quiver Q, there is a canonically defined Kac-Moody Lie algebra $\mathfrak{g}_Q$, with Chevalley generators e_i, h_i, f_i, $i \in I$, see [Ka]. We write $\mathfrak{h}$ for the Cartan subalgebra of $\mathfrak{g}_Q$. For each $i \in I$, let $\alpha_i \in \mathfrak{h}^*$ denote the corresponding simple root, resp. $\varpi_i \in \mathfrak{h}$ denote the corresponding fundamental weight such that $\varpi_i(h_i) = 1$ and $\varpi_i(h_j) = 0$ for any $j \neq i$.

Let $U(\mathfrak{g}_Q)$ be the universal enveloping algebra of $\mathfrak{g}_Q$. There is a convenient modification of this algebra where the Cartan part in the standard triangular decomposition of $U(\mathfrak{g}_Q)$ is replaced by the weight lattice. The resulting algebra $\widetilde{U}(\mathfrak{g}_Q)$, called the *modified enveloping algebra*, was first introduced by Lusztig, cf. [L1].

One of the main results of Nakajima's theory reads, see [Na2, Theorem 9.4 and §11]

Theorem 7.3.1. (i) *There is a natural algebra homomorphism* $\Psi : \widetilde{U}(\mathfrak{g}_Q) \to H_{\mathbf{w}}$.

(ii) *The* $\widetilde{U}(\mathfrak{g}_Q)$-*action on the vector space* $L_{\mathbf{w}}$, *induced by the homomorphism* Ψ *via Lemma 6.4.2(ii), makes the latter a simple integrable* $\mathfrak{g}_Q$-*module with highest weight* $\sum_{i \in I} w_i \cdot \varpi_i$.

Remark 7.3.2. Theorem 5.4.2 implies that $\Lambda_{\mathbf{w}}$ is a (disconnected) *Lagrangian* subvariety of $M(\mathbf{w})$, a disconnected symplectic manifold. It follows that the fundamental classes of *all* irreducible components of the variety $\Lambda_{\mathbf{w}}$ form a natural basis in the vector space $L_{\mathbf{w}} = H_{\text{top}}(\Lambda_{\mathbf{w}})$. This basis goes, via the identification provided by Theorem 7.3.1(ii), to a so-called *semicanonical* basis in the corresponding simple $U(\mathfrak{g}_Q)$-module, cf. [L4].

Remark 7.3.3. In the special case where Q is a Dynkin quiver of type **A**, Theorem 7.3.1 reduces to an earlier result obtained in [Gi3], where the corresponding Steinberg variety was introduced.

Many interesting interconnections arising specifically in the case of quivers of type **A** are discussed in [MV].

Hint on proof of Theorem 7.3.1. The homomorphism Ψ, of Theorem 7.3.1(i), is constructed by sending each of the Chevalley generators e_i, h_i, f_i, $i \in I$ to an appropriate explicit linear combination of the fundamental classes of some carefully chosen *smooth* irreducible components of the Steinberg variety $Z(\mathbf{w})$.

Specifically, fix $i \in I$ and let $\mathbf{e}^i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^I$ denote the i-th coordinate vector. Then, the generator h_i is sent to a linear combination of the form $\sum_{\mathbf{v}} a_{\mathbf{v}} \cdot [\mathcal{M}(\mathbf{v}, \mathbf{w})]$, where $[\mathcal{M}(\mathbf{v}, \mathbf{w})]$ denotes the fundamental class of the diagonal $\mathcal{M}(\mathbf{v}, \mathbf{w}) \subset \mathcal{M}(\mathbf{v}, \mathbf{w}) \times \mathcal{M}(\mathbf{v}, \mathbf{w})$, and $a_{\mathbf{v}} \in \mathbb{Q}$ are certain rational coefficients.

The generator e_i is sent to a linear combination of the form $\sum_{\mathbf{v}} b_{\mathbf{v}} \cdot [Z^i(\mathbf{v}, \mathbf{w})]$. Here $Z^i(\mathbf{v}, \mathbf{w}) \subset \mathcal{M}(\mathbf{v}, \mathbf{w}) \times \mathcal{M}(\mathbf{v}+\mathbf{e}^i, \mathbf{w})$, is a smooth irreducible component of the Steinberg varietiy $Z(\mathbf{v}, \mathbf{v}+\mathbf{e}^i, \mathbf{w})$, and $b_i \in \mathbb{Q}$ are some coefficients. Similarly, the generator f_i is sent to a linear combination of the form $\sum_{\mathbf{v}} c_{\mathbf{v}} \cdot [Z^i(\mathbf{v}-\mathbf{e}^i, \mathbf{w})^{\mathrm{op}}]$, $c_{\mathbf{v}} \in \mathbb{Q}$. In the last formula, $Z^i(\mathbf{v}-\mathbf{e}^i, \mathbf{w})^{\mathrm{op}} \subset \mathcal{M}(\mathbf{v}, \mathbf{w}) \times \mathcal{M}(\mathbf{v}-\mathbf{e}^i, \mathbf{w})$ is a subvariety which is obtained from the variety $Z^i(\mathbf{v} - \mathbf{e}^i, \mathbf{v}) \subset \mathcal{M}(\mathbf{v} - \mathbf{e}^i, \mathbf{w}) \times \mathcal{M}(\mathbf{v}, \mathbf{w})$, involved in the formula for the generator e_i, by the flip-isomorphism $\mathcal{M}(\mathbf{v} - \mathbf{e}^i, \mathbf{w}) \times \mathcal{M}(\mathbf{v}, \mathbf{w}) \cong \mathcal{M}(\mathbf{v}, \mathbf{w}) \times \mathcal{M}(\mathbf{v} - \mathbf{e}^i, \mathbf{w})$. $\square$

The following result was proved in [Na2, Theorem 10.2]

Theorem 7.3.4. *Let $x \in \mathcal{M}_0^\circ(\mathbf{v}', \mathbf{w})$ for some $0 \leq \mathbf{v}' \leq \mathbf{v}$, and view x as a point in $\mathcal{M}_0(\mathbf{v}, \mathbf{w})$. Then, one has*

(i) *The fiber $\mathcal{M}(\mathbf{v}, \mathbf{w})_x = \pi^{-1}(x)$ is equi-dimensional;*

(ii) *The convolution product makes $H_{\mathrm{top}}(\mathcal{M}(\mathbf{v}, \mathbf{w})_x)$ an $\widetilde{U}(\mathfrak{g}_Q)$-module. This is an integrable simple $\widetilde{U}(\mathfrak{g}_Q)$-module with the highest weight equal to $\sum_{i \in I} (w_i \cdot \varpi_i - v_i' \cdot \alpha_i)$.*

Remark 7.3.5. Note that part (ii) of the above theorem reduces, in the special case $\mathbf{v}' = 0$, to Theorem 7.3.1(ii). $\diamond$

In the paper [Na4], Nakajima proves analogues of Theorems 7.3.1 and 7.3.4, where the algebra $\widetilde{U}(\mathfrak{g}_Q)$ is replaced by $\widetilde{U}_q(\widehat{\mathfrak{g}}_Q)$, the (modified) quantized enveloping algebra of the *affinization* of the Kac-Moody algebra $\mathfrak{g}_Q$. Accordingly, Borel-Moore homology is replaced in [Na4] by equivariant K-theory; in particular, the algebra $H_{\mathbf{w}}$ is replaced by (a completion of) $K^{G_{\mathbf{w}} \times \mathbb{C}^\times}(Z(\mathbf{w}))$, the $G_{\mathbf{w}} \times \mathbb{C}^\times$-equivariant K-group of the Steinberg variety.

In the special case where Q is a Dynkin quiver of type $\mathbf{A}_{n-1}$ we have $\mathfrak{g} = \mathfrak{sl}_n$. One can use the description of the corresponding quiver varieties in terms of partial flag manifolds provided by Proposition 5.3.4. The results of Nakajima [Na4] reduce, in this case, to the results obtained earlier in [GV], cf. also [V] (in these papers, the authors consider the algebra $\widetilde{U}_q(\widehat{\mathfrak{gl}}_n)$ rather than $\widetilde{U}_q(\widehat{\mathfrak{sl}}_n)$, but the difference is not very essential).

Remark 7.3.6. The use of equivariant K-theory by Nakajima was strongly motivated by a similar approach to representations of affine Hecke algebras that has been known at the time, see [KL] and [CG].

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Department of Mathematics, University of Chicago, Chicago, IL 60637, USA

E-mail address: ginzburg@math.uchicago.edu