An introduction to moduli spaces of curves and its intersection theory

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Abstract. The material of this chapter is based on a series of three lectures for graduate students that the author gave at the *Journées mathématiques de Glanon* in July 2006. We introduce moduli spaces of smooth and stable curves, the tautological cohomology classes on these spaces, and explain how to compute all possible intersection numbers between these classes.

0 Introduction

This chapter is an introduction to the intersection theory on moduli spaces of curves. It is meant to be as elementary as possible, but still reasonably short.

The intersection theory of an algebraic variety M looks for answers to the following questions: What are the interesting cycles (algebraic subvarieties) of M and what cohomology classes do they represent? What are the interesting vector bundles over M and what are their characteristic classes? Can we describe the full cohomology ring of M and identify the above classes in this ring? Can we compute their intersection numbers? In the case of moduli space, the full cohomology ring is still unknown. We are going to study its subring called the "tautological ring" that contains the classes of most interesting cycles and the characteristic classes of most interesting vector bundles.

To give a sense of purpose to the reader, we assume the following goal: after reading this paper, one should be able to write a computer program evaluating

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all intersection numbers between the tautological classes on the moduli space of stable curves. And to understand the foundation of every step of these computations. A program like that was first written by C. Faber [5], but our approach is a little different.

Other good introductions to moduli spaces include [10] and [20].

Section 1 is an informal introduction to moduli spaces of smooth and stable curves. It contains many definitions and theorems and lots of examples, but no proofs.

In Section 2 we define the tautological cohomology classes on the moduli spaces. Simplest computations of intersection numbers are carried out.

In Section 3 we explain how to reduce the computations of all intersection numbers of all tautological classes to those involving only the so-called ψ-classes. This involves a variety of useful techniques from algebraic geometry, in particular the Grothendieck-Riemann-Roch formula.

Finally, in Section 4 we formulate Witten's conjecture (Kontsevich's theorem) that allows one to compute all intersection numbers among the ψ-classes. Explaining the proof of Witten's conjecture is beyond the scope of this paper.

The chapter is based on a series of three lectures for graduate students that the author gave at the *Journées mathématiques de Glanon* in July 2006. I am deeply grateful to the organizers for the invitation. I would also like to thank M. Kazarian whose unpublished notes on moduli spaces largely inspired the third section of the chapter.

Contents

1 From Riemann surfaces to moduli spaces

1.1 Riemann surfaces

Terminology. The main objects of our study are the *smooth compact complex curves* also called *Riemann surfaces* with n marked numbered pairwise distinct points. Unless otherwise specified they are assumed to be connected.

Every compact complex curve has an underlying structure of a 2-dimensional oriented smooth compact surface, that is uniquely characterized by its genus g.

Example 1.1. The sphere possesses a unique structure of Riemann surface up to isomorphism: that of a complex projective line $\mathbb{C}\mathrm{P}^1$ (see [6], IV.4.1). A complex curve of genus 0 is called a *rational curve.* The automorphism group of $\mathbb{C}\mathrm{P}^1$ is $\mathrm{PSL}(2,\mathbb{C})$ acting by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Proposition 1.2. *The automorphism group* $\mathrm{PSL}(2,\mathbb{C})$ *of* $\mathbb{C}\mathrm{P}^1$ *allows one to send any three distinct points* x_1, x_2, x_3 *to* 0, 1, *and* ∞ *respectively in a unique way.*

We leave the proof as an exercise to the reader.

Example 1.3. Up to isomorphism every structure of Riemann surface on the torus is obtained by factorizing $\mathbb{C}$ by a lattice $L \simeq \mathbb{Z}^2$ (see [6], IV.6.1). A complex curve of genus 1 is called an *elliptic curve*. The automorphism group $\mathrm{Aut}(E)$ of any elliptic curve E contains a subgroup isomorphic to E itself acting by translations.

Proposition 1.4. *Two elliptic curves* $\mathbb{C}/L_1$ *and* $\mathbb{C}/L_2$ *are isomorphic if and only if* $L_2 = aL_1$, $a \in \mathbb{C}^*$.

Sketch of proof. An isomorphism between these two curves is a holomorphic function on $\mathbb{C}$ that sends any two points equivalent modulo L_1 to two points equivalent modulo L_2. Such a holomorphic function is easily seen to have at most linear growth, so it is of the form $z \mapsto az + b$.

1.2 Moduli spaces

Moduli spaces of Riemann surfaces of genus g with n marked points can be defined as *smooth Deligne-Mumford stacks* (in the algebraic-geometric setting) or as *smooth complex orbifolds* (in an analytic setting). The latter notion is simpler and will be discussed in the next section. For the time being we define moduli spaces as sets.

Definition 1.5. For $2 - 2g - n < 0$, the *moduli space* $\mathcal{M}_{g,n}$ is the set of isomorphism classes of Riemann surfaces of genus g with n marked points.

Remark 1.6. The automorphism group of any Riemann surface satisfying $2 - 2g - n < 0$ is finite (see [6], V.1.2, V.1.3). On the other hand, every Riemann surface with $2 - 2g - n \geq 0$ has an infinite group of marked point preserving automorphisms. For reasons that will become clear in Section 1.3, this makes it impossible to define the moduli spaces $\mathcal{M}_{0,0}$, $\mathcal{M}_{0,1}$, $\mathcal{M}_{0,2}$, and $\mathcal{M}_{1,0}$ as orbifolds. (They still make sense as sets, but this is of little use.)

Example 1.7. Let $g = 0$, $n = 3$. Every rational curve (C, x_1, x_2, x_3) with three marked points can be identified with $(\mathbb{C}\mathrm{P}^1, 0, 1, \infty)$ in a unique way. Thus $\mathcal{M}_{0,3} =$ point.

Example 1.8. Let $g = 0$, $n = 4$. Every curve (C, x_1, x_2, x_3, x_4) can be uniquely identified with $(\mathbb{C}\mathrm{P}^1, 0, 1, \infty, t)$. The number $t \neq 0, 1, \infty$ is determined by the positions of the marked points on C. It is called the *modulus* and gave rise to the term "moduli space". If $C = \mathbb{C}\mathrm{P}^1$, then t is the cross-ratio of x_1, x_2, x_3, x_4. The moduli space $\mathcal{M}_{0,4}$ is the set of values of t, that is $\mathcal{M}_{0,4} = \mathbb{C}\mathrm{P}^1 \setminus \{0, 1, \infty\}$.

Example 1.9. Generalizing the previous example, take $g = 0$ and an arbitrary n. The curve $(C, x_1, \ldots, x_n)$ can be uniquely identified with $(\mathbb{C}\mathrm{P}^1, 0, 1, \infty, t_1, \ldots, t_{n-3})$. The moduli space $\mathcal{M}_{0,n}$ is given by

$$\mathcal{M}_{0,n} = \{(t_1, \ldots, t_{n-3}) \in \left(\mathbb{C}\mathrm{P}^1\right)^{n-3} \mid \ t_i \neq 0, 1, \infty, \ \ t_i \neq t_j\}.$$

Example 1.10. According to Example 1.3, every elliptic curve is isomorphic to the quotient of $\mathbb{C}$ by a rank 2 lattice L. The image of $0 \in \mathbb{C}$ is a natural marked point on E. Thus $\mathcal{M}_{1,1} = \{\text{lattices}\}/\mathbb{C}^*$. Consider a direct basis (z_1, z_2) of a lattice L. Multiplying L by $1/z_1$ we obtain a lattice with basis $(1, \tau)$, where τ lies in the upper half-plane $\mathbb{H}$. Choosing another basis of the same lattice we obtain another point $\tau' \in \mathbb{H}$. Thus the group $\mathrm{SL}(2, \mathbb{Z})$ of direct base changes in a lattice acts on $\mathbb{H}$. This action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}.$$

We have $\mathcal{M}_{1,1} = \mathbb{H}/\mathrm{SL}(2, \mathbb{Z})$. The matrix $-\mathrm{Id} \in \mathrm{SL}(2, \mathbb{Z})$ acts trivially on $\mathbb{H}$. The group $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\pm\mathrm{Id}$ has a fundamental domain shown in the figure. The moduli space $\mathcal{M}_{1,1}$ is obtained from the fundamental domain by identifying the arcs AB and AB' and the half-lines BC and $B'C'$.

Example 1.11. Let $g = 2$, $n = 0$. By Riemann-Roch's theorem, every Riemann surface of genus g carries a g-dimensional vector space Λ of abelian differentials (that is, holomorphic differential 1-forms). Each abelian differential has $2g - 2$ zeroes. (See [6], III.4.)

For $g = 2$, we have $\dim \Lambda = 2$. Let (α, β) be a basis of Λ, and consider the map $f : C \to \mathbb{C}\mathrm{P}^1$ given by the quotient $f = \alpha/\beta$. (In intrinsic terms the image of f is the projectivization of the dual vector space of Λ. Choosing a basis in Λ identifies it with $\mathbb{C}\mathrm{P}^1$.) The map f is of degree at most 2, because both α and β have two zeroes and no poles. But f cannot be a constant (because then α and β would be proportional to each other and would not form a basis of Λ) and cannot be of degree 1 (because then it would establish an isomorphism between its genus 2 domain and its genus 0 target, with is not possible). Thus $\deg f = 2$. The involution of C that interchanges the two sheets of f or, in other words, interchanges the two zeroes of every holomorphic differential, is called the *hyperelliptic involution*. By an Euler characteristic count (or applying the Riemann-Hurwitz formula, which is the same) we obtain that f must have 6 ramification points, that is, 6 distinct points in $\mathbb{C}\mathrm{P}^1$ that have one double preimage rather than two simple preimages. The 6 preimages of these points on C are the fixed points of the hyperelliptic involution and are called *Weierstrass points*.

Summarizing, we see that giving a genus 2 Riemann surface is equivalent to giving 6 distinct nonnumbered points on a rational curve. Thus $\mathcal{M}_{2,0} = \mathcal{M}_{0,6}/S_6$, where S_6 is the symmetric group. However, this equality only holds for sets. The moduli spaces $\mathcal{M}_{2,0}$ and $\mathcal{M}_{0,6}/S_6$ actually have different orbifold structures, because every genus 2 curve has an automorphism that the genus 0 curve with 6 marked points does not have: namely, the hyperelliptic involution.

1.3 Orbifolds

Here we give a minimal set of definitions necessary for our purposes. Readers interested in learning more about orbifolds and stacks are referred to [2, 15, 21].

A smooth complex n-dimensional orbifold is locally isomorphic to an open ball in $\mathbb{C}^n$ factorized by a finite group action. Let us give a precise definition.

Let X be a topological space.

Definition 1.12. An *orbifold chart* on X is the following data:

$$U/G \xrightarrow{\varphi} V \subset X,$$

where $U \subset \mathbb{C}^n$ is a contractible open set endowed with a bi-holomorphic action of a finite group G, $V \subset X$ is an open set, and φ is a homeomorphism from U/G to V.

Sometimes the chart will be denoted simply by V if this does not lead to ambiguity. Note that a nontrivial subgroup of G can act trivially on U.

Definition 1.13. A chart

$$U'/G' \xrightarrow{\varphi'} V' \subset X$$

is called a *subchart* of

$$U/G \xrightarrow{\varphi} V \subset X,$$

if V' is a subset of V and there is a group homomorphism $G' \to G$ and a holomorphic embedding $U' \hookrightarrow U$ such that (i) the embedding and the group morphism commute with the group actions; (ii) the G'-stabilizer of every point in U' is isomorphic to the G-stabilizer of its image in U; (iii) the embedding commutes with the isomorphisms φ and φ'.

The following figure shows a typical example of a sub-chart.

Definition 1.14. Two orbifold charts V_1 and V_2 are called *compatible* if every point of $V_1 \cap V_2$ is contained in some chart V_3 that is a subchart of both V_1 and V_2.

Note that any attempt to define a chart $V_1 \cap V_2$ would lead to problems, because $V_1 \cap V_2$ is, in general, not connected and the preimages of $V_1 \cap V_2$ in U_1 and in U_2 are not necessarily contractible.

Definition 1.15. A *smooth complex orbifold* is a topological space X entirely covered by a family of compatible charts.

Definition 1.16. Let X be an orbifold and $x \in X$ a point. The *stabilizer* of x is the stabilizer in G of a preimage of x in U under φ in some chart. (By definition it does not depend on the chart or of the preimage.)

Example 1.17. If M is a smooth complex manifold endowed with an action of a finite group G, then $X = M/G$ has a natural orbifold structure.

All notions related to manifolds and possessing a local definition can be automatically extended to orbifolds.

For instance, a differential form α on a chart V is defined as a G-invariant differential form α_U on U. The integral of α over a chain $C \subset V$ is defined as

$$\frac{1}{|G|} \int\limits_{\varphi^{-1}(C)} \alpha_U .$$

Further, a vector bundle over a chart V is defined as a vector bundle over the open set U together with a fiberwise linear lifting of the G-action to the total space of the bundle. We can define a connection on a vector bundle and the curvature of the connection in the natural way.

Defining global characteristics of orbifolds, for instance, their cohomology rings or their homotopy groups, is more delicate. It is possible to define the ring $H^*(X, \mathbb{Z})$ for an orbifold X, but we won't do it here. Instead, we content ourselves with the straightforward definition of the cohomology ring over $\mathbb{Q}$.

Definition 1.18. The *homology and cohomology groups of an orbifold over* $\mathbb{Q}$ are defined as the homology cohomology groups of its underlying topological space (also over $\mathbb{Q}$).

Theorem 1.19 ([3]). *Poincaré duality holds for homology and cohomology groups over $\mathbb{Q}$ of smooth compact orbifolds.*

Remark 1.20. Let X be an orbifold and Y an irreducible sub-orbifold. Denote by $\widehat{X}$ and $\widehat{Y}$ the underlying topological spaces. By convention, the homology class $[Y] \in H_*(X, \mathbb{Q}) = H_*(\widehat{X}, \mathbb{Q})$ is equal to $\frac{1}{|G_Y|}[\widehat{Y}] \in H_*(\widehat{X}, \mathbb{Q})$, where G_Y is the stabilizer of a generic point of Y.

Example 1.21. Consider the action of $\mathbb{Z}/k\mathbb{Z}$ on $\mathbb{C}\mathrm{P}^1$ by rotations and let X be the quotient orbifold. Then the class $[0] \in H_0(X, \mathbb{Q})$ is $1/k$ times the class of a generic point.

It turns out that the moduli space $\mathcal{M}_{g,n}$ (for $2 - 2g - n < 0$) possesses a natural structure of a smooth complex $(3g - 3 + n)$-dimensional orbifold. Moreover, the stabilizer of a point $t \in \mathcal{M}_{g,n}$ is equal to the automorphism group of the corresponding Riemann surface with n marked points C_t. Let us explain how to endow the moduli space with an orbifold structure.

We say that $p : \mathcal{C} \to B$ is a *family of genus g Riemann surfaces with n marked points* if p is endowed with n disjoint sections $s_i : B \to \mathcal{C}$ (so that $p \circ s_i = \mathrm{Id}$) and every fiber of p is a smooth Riemann surface. The intersections of the sections with every fiber of p are the marked points of the fiber. If we have two families $p_1 : \mathcal{C}_1 \to B_1$ and $p_2 : \mathcal{C}_2 \to B_2$ and a subset $B_2' \subset B_2$, we say that *the restriction of p_2 to B_2' is a pull-back of p_1* if there exists a morphism $\varphi : B_2' \to B_1$ such that $\mathcal{C}_2$ restricted to B_2' is isomorphic to the pull-back of $\mathcal{C}_1$ under φ.

Theorem 1.22 ([10], 2.C). *Let C be a genus g Riemann surface with n marked points. Let G be its (finite) isomorphism group.*

There exists (a) an open bounded simply connected domain $U \subset \mathbb{C}^{3g-3+n}$; (b) a family $p : \mathcal{C} \to U$ of genus g Riemann surfaces with n marked points; (c) a group G with an action on $\mathcal{C}$ commuting with p and thus descending to an action of G on U satisfying the following conditions: (i) The fiber C_0 over $0 \in C^{3g-3+m}$ is isomorphic to C. (ii) The action of G preserves C_0 and acts as the symmetry group of C_0. (iii) For any family of smooth curves with n marked points $\mathcal{C}_B \to B$ such that C_b is isomorphic to C for some $b \in B$, the restriction of the family to some open subset $b \in B' \subset B$ is a pull-back of the family $\mathcal{C} \to U$.

Theorem 1.22 leads to a construction of two smooth orbifolds.

The first one, $\mathcal{M}_{g,n}$, covered by the charts U/G, is the moduli space. It follows from the theorem that the stabilizer of $t \in \mathcal{M}_{g,n}$ is isomorphic to the symmetry group of the surface C_t.

The second one, $\mathcal{C}_{g,n}$ is covered by the open sets $\mathcal{C}$ (these are not charts, because they are not simply connected, but it is easy to subdivide them into charts). There is an orbifold morphism $p : \mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$ between the two.

Definition 1.23. The map $p : \mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$ is called the *universal curve* over $\mathcal{M}_{g,n}$.

The fibers of the universal curve are Riemann surfaces with n marked points, and each such surface appears exactly once among the fibers.

If we consider the induced map of underlying topological spaces $\hat{p} : C_{g,n} \to M_{g,n}$, then its fibers are of the form C/G, where C is a Riemann surface and G its automorphism group.

Example 1.24. As we explained in Example 1.10, the moduli space $\mathcal{M}_{1,1}$ is isomorphic to $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$. The stabilizer of a lattice L in $\mathrm{SL}(2,\mathbb{Z})$ is the group of basis changes of L that amount to homotheties of $\mathbb{C}$. These can be viewed as isomorphisms of the elliptic curve $\mathbb{C}/L$. Thus the stabilizer of a point in the moduli space is indeed isomorphic to the automorphism group of the corresponding curve.

The stabilizer of a generic lattice L (case A) is the group $\mathbb{Z}/2\mathbb{Z}$ composed of the identity and the central symmetry.

The stabilizer of the lattice $\mathbb{Z} + i\mathbb{Z}$ (case B) is the group $\mathbb{Z}/4\mathbb{Z}$ of rotations by multiples of 90°.

The stabilizer of the lattice $\mathbb{Z} + \frac{1+i\sqrt{3}}{2}\mathbb{Z}$ (case C) is the group $\mathbb{Z}/6\mathbb{Z}$ of rotations by multiples of 60°.

By abuse of language we will often "forget" that the moduli spaces are orbifolds and treat them as manifolds, bearing in mind the above definitions.

1.4 Stable curves and the Deligne-Mumford compactification

As the examples of Section 1.2 show, the moduli space $\mathcal{M}_{g,n}$ is, in general, not compact. We are now going to compactify it by adding new points that correspond to the so-called "stable curves". Let us start with an example.

1.4.1 The case $g = 0$, $n = 4$ As explained in Example 1.8, the moduli space $\mathcal{M}_{0,4}$ is isomorphic to $\mathbb{C}\mathrm{P}^1 \setminus \{0, 1, \infty\}$. A point $t \in \mathbb{C}\mathrm{P}^1 \setminus \{0, 1, \infty\}$ encodes the following curve C_t:

$$(C, x_1, x_2, x_3, x_4) \simeq (\mathbb{C}\mathrm{P}^1, 0, 1, \infty, t).$$

What will happen as $t \to 0$? At first sight, we will simply obtain a curve with four marked points, two of which coincide: $x_1 = x_4$. However, such an approach is unjust with respect to the points x_1 and x_4. Indeed, without changing the curve C_t, we can change its local coordinate via the map $x \mapsto x/t$ and obtain the curve

$$(C, x_1, x_2, x_3, x_4) \simeq (\mathbb{C}\mathrm{P}^1, 0, 1/t, \infty, 1).$$

What we see now in the limit is that x_1 and x_4 do not glue together any longer, but this time x_2 and x_3 do tend to the same point.

Since there is no reason to prefer one local coordinate to the other, neither of the pictures is better than the other one. Thus the right thing to do is to include *both* limit curves in the description of the limit:

The right-hand component corresponds to the initial local coordinate x, while the left-hand component corresponds to the local coordinate x/t.

In can be, at first, difficult to imagine, how a sphere can possibly tend to a curve consisting of two spheres. To make this more visual, consider the following example. Let $xy = t$ be a family of curves in $\mathbb{C}\mathrm{P}^2$ parameterized by t. On each of these curves we mark the following points:

$$(x_1, y_1) = (0, \infty), \quad (x_2, y_2) = (1, t), \quad (x_3, y_3) = (\infty, 0), \quad (x_4, y_4) = (t, 1).$$

Then, for $t \neq 0$, the curve is isomorphic to $\mathbb{C}\mathrm{P}^1$ with four marked points, while for $t = 0$ it degenerates into a curve composed of two spheres (the coordinate axes) with two marked points on each sphere.

Now we go back to the general case.

1.4.2 Stable curves Stable curves are complex algebraic curves that are allowed to have exactly one type of singularities, namely, simple nodes. The simplest example of a curve with a node is the plane curve given by the equation $xy = 0$, that has a node at the origin. The neighborhood of a node is diffeomorphic to two discs with identified centers. A node can be desingularized in two different ways. We say that a node is *normalized* if the two discs with identified centers that form its neighborhood are unglued, *i.e.*, replaced by disjoint discs. A node is *resolved* if the two discs with identified centers that form its neighborhood are replaced by a cylinder.

Definition 1.25. A *stable curve* C with n marked points $x_1, \dots, x_n$ is a complex algebraic curve satisfying the following conditions. (i) The only singularities of C are simple nodes. (ii) The marked points are distinct and do not coincide with the nodes. (iii) The curve $(C, x_1, \dots, x_n)$ has a finite number of automorphisms.

Unless stated otherwise, stable curves are assumed to be connected.

The *genus* of a stable curve C is the genus of the surface obtained from C by resolving all its nodes.

The *normalization* of a stable curve C is the smooth not necessarily connected curve obtained from C by normalizing all its nodes.

Condition (iii) in the above definition can be reformulated as follows. Let $C_1, \dots, C_k$ be the connected components of the normalization of C. Let g_i be the genus of C_i and n_i the number of *special points*, *i.e.*, marked points and preimages of the nodes on C_i. Then Condition (iii) is satisfied if and only if $2 - 2g_i - n_i < 0$ for all i. In this form, the condition is, of course, much easier to check.

The stable curve in the picture below is of genus 4.

Proposition 1.26. *Let C be a stable curve of genus g with n marked points. Then the Euler characteristic of $C \setminus$(marked points and nodes) equals $2-2g-n$.*

Corollary 1.27. *There is only a finite number of topological types of stable curves of genus g with n marked points.*

We leave the proof as an exercise to the reader.

Theorem 1.28 ([10], Chapter 4). *There exists a smooth compact complex $(3g-3+n)$-dimensional orbifold $\overline{\mathcal{M}}_{g,n}$, a smooth compact complex $(3g-2+n)$-dimensional orbifold $\overline{\mathcal{C}}_{g,n}$, and a map $p : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ such that* (i) *$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is an open dense sub-orbifold and $\mathcal{C}_{g,n} \subset \overline{\mathcal{C}}_{g,n}$ its preimage under p; (ii) the fibers of p are stable curves of genus g with n marked points; (iii) each stable curve is isomorphic to exactly one fiber; (iv) the stabilizer of a point $t \in \overline{\mathcal{M}}_{g,n}$ is isomorphic to the automorphism group of the corresponding stable curve C_t.*

Definition 1.29. The space $\overline{\mathcal{M}}_{g,n}$ is called the *Deligne-Mumford compactification* of the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces. The family $p : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is called the *universal curve.*

Definition 1.30. The set $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ parametrizing singular stable curves is called the *boundary* of $\overline{\mathcal{M}}_{g,n}$.

The boundary is a sub-orbifold of $\overline{\mathcal{M}}_{g,n}$ of codimension 1, in other words, a divisor. The term "boundary" may lead one to think that $\overline{\mathcal{M}}_{g,n}$ has a singularity at the boundary, but this is not true: as we have already stated, $\overline{\mathcal{M}}_{g,n}$ is a smooth orbifold, and the boundary points are as smooth as any other points of $\overline{\mathcal{M}}_{g,n}$. A generic point of the boundary corresponds to a stable curve with exactly one node. If a point t of the boundary corresponds to a stable curve C_t with k nodes, there are k local components of the boundary that intersect transversally at t. Each of these components is obtained by resolving $k-1$ out of k nodes of C_t. Thus the boundary is a divisor with normal crossings in $\overline{\mathcal{M}}_{g,n}$. The figure below shows two components of the boundary divisor in $\overline{\mathcal{M}}_{g,n}$ and the corresponding stable curves.

1.4.3 Examples

Example 1.31. We have $\overline{\mathcal{M}}_{0,3} = \mathcal{M}_{0,3} =$ point. Indeed, the unique stable genus 0 curve with 3 marked points is smooth.

Example 1.32. Consider the projection $\tilde{p} : \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ on the first factor. Consider further four distinguished sections $\tilde{s}_i : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$: $\tilde{s}_1(t) = (t, 0)$, $\tilde{s}_2(t) = (t, 1)$, $\tilde{s}_3(t) = (t, \infty)$, $s_4(t) = (t, t)$. Now take the blow-up X of $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ at the three points $(0, 0)$, $(1, 1)$, and (∞, ∞) where the fourth section intersects the three others. We obtain a map $p : X \to \mathbb{C}\mathrm{P}^1$ endowed with four nonintersecting sections. Its fiber over $t \in \mathbb{C}\mathrm{P}^1 \setminus \{0, 1, \infty\}$ is the Riemann sphere with four marked points 0, 1, ∞, and t. The three special fibers over 0, 1, and ∞ are singular stable curves. Thus the map $p : X \to \mathbb{C}\mathrm{P}^1$ is actually the universal curve $\overline{\mathcal{C}}_{0,4} \to \overline{\mathcal{M}}_{0,4}$.

$\mathbb{C}\mathrm{P}^1$ s_4

∞ s_3

1 s_2 $\overline{\mathcal{C}}_{0,4}$

0 s_1 p

$\mathbb{C}\mathrm{P}^1 = \overline{\mathcal{M}}_{0,4}$

0 1 ∞

Example 1.33. The moduli space $\overline{\mathcal{M}}_{1,1}$ is obtained from $\mathcal{M}_{1,1}$ by adding one point corresponding to the singular stable curve:

Example 1.34. In Example 1.11 we saw that $\mathcal{M}_{2,0}$ is isomorphic to $\mathcal{M}_{0,6}/S_6$ up to a $\mathbb{Z}/2\mathbb{Z}$ action. One can prove that $\overline{\mathcal{M}}_{2,0}$ is isomorphic to $\overline{\mathcal{M}}_{0,6}/S_6$ up to a $\mathbb{Z}/2\mathbb{Z}$. As an exercise the reader can enumerate the topological types of genus 2 stable curves and show that the hyperelliptic involution extends to all of them uniquely.

1.4.4 The universal curve at the neighborhood of a node. As in Section 1.4.1, consider the map $p : \mathbb{C}^2 \to \mathbb{C}$ given by $(x, y) \mapsto t = xy$. Then the fibers of p over $t \neq 0$ are smooth (and isomorphic to $\mathbb{C}^*$) while the fiber over $t = 0$ has a node (and is isomorphic to two copies of $\mathbb{C}$ glued together at the origin).

It turns out that this example gives a local model for every node in every universal curve.

Proposition 1.35 (See [10], 3.B, Deformations of stable curves). *Let $p : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve and $z \in \overline{\mathcal{C}}_{g,n}$ a node in a singular fiber. Then there is a neighborhood of z in $\overline{\mathcal{C}}_{g,n}$ with a system of local coordinates $T_1, \ldots, T_{3g-4+n}, x, y$ and a neighborhood of $p(z)$ in $\overline{\mathcal{M}}_{g,n}$ with a system of local coordinates $t_1, \ldots, t_{3g-3+n}$ such that in these coordinates p is given by*

$$t_i = T_i \quad (1 \leq i \leq 3g - 4 + n), \qquad t_{3g-3+n} = xy.$$

1.4.5 The compactness of $\overline{\mathcal{M}}_{g,n}$ illustrated We do not prove the compactness of $\overline{\mathcal{M}}_{g,n}$ here, but to get a feeling of it we give several examples of families of smooth or stable maps and find their limits in $\overline{\mathcal{M}}_{g,n}$.

Example 1.36. Let C a smooth curve of genus 2 and $x_1(t), x_2(t) \in C$ two marked points depending on a parameter t. Suppose that, as $t \to 0$, x_1 and x_2 tend to the same point x. Then the limit stable curve of this family looks as follows:

The curve C "sprouts" a sphere, on which lie the points x_1 and x_2. This sphere is attached to C at the point x.

This limit can be explained as follows. If we choose a fixed local coordinate at the neighborhood of x on C, then, in this local coordinate, the two points tend to x, so we obtain the picture that we see on the genus 2 component of the limit curve. If, however, we choose a local coordinate that depends on t in such a way that x_1 and x_2 remain fixed, then in this local coordinate all the "non-trivial" part of C moves further and further away to ∞. So we end up with a rational curve containing the points x_1 and x_2, while the genus 2 curve seems to be "concentrated" at the point ∞ of this rational curve. This is the picture we see on the genus 0 component of the limit curve.

Example 1.37. Consider the curve $C = \mathbb{C}\mathrm{P}^1$ with 5 marked points depending on a parameter t: $x_1 = 0$, $x_2 = 1$, $x_3 = \infty$, $x_4 = t$, $x_5 = t^2$. As $t \to 0$, this curve tends to

The coordinate on the rightmost sphere is the initial coordinate x. The coordinate on the central sphere is x/t. That on the leftmost sphere is x/t^2.

Example 1.38. Let C be the genus 2 curve obtained as a 2-sheeted covering of $\mathbb{CP}^1$ ramified over the points $x_1, \dots, x_6 \in \mathbb{CP}^1$. If x_5 and x_6 tend to the same point x, then C tends to the following stable curve:

2 Cohomology classes on $\overline{\mathcal{M}}_{g,n}$

In this section we introduce several natural cohomology classes on the moduli space. The ring generated by these classes is called the *tautological cohomology ring* of $\overline{\mathcal{M}}_{g,n}$. Although it is known that for large g and n the rank of the tautological ring is much smaller than that of the full cohomology ring of $\overline{\mathcal{M}}_{g,n}$, most natural geometrically defined cohomology classes happen to be tautological and it is actually not so simple to construct examples of nontautological cohomology classes [8].

2.1 Forgetful and attaching maps

2.1.1 Forgetful maps. The idea of a forgetful map is to assign to a genus g stable curve $(C, x_1, \dots, x_{n+m})$ the curve $(C, x_1, \dots, x_n)$, where we have "forgotten" m marked points out of $n+m$. The main problem is that the resulting curve $(C, x_1, \dots, x_n)$ is not necessarily stable. Assume that $2 - 2g - n < 0$. Then, either the curve $(C, x_1, \dots, x_n)$ is stable, or it has at least one genus 0 component with 1 or 2 special points. In the latter case this component can be contracted into a point. For the curve thus obtained we can once again ask ourselves if it is stable or not, and if not find another component to contract. Since the number of irreducible components decreases with each operation, in the end we will obtain a stable curve. This curve is called the *stabilization* of $(C, x_1, \dots, x_n)$.

Definition 2.1. The *forgetful map* $p : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ is the map that assigns to a curve $(C, x_1, \dots, x_{n+m})$ the stabilization of the curve $(C, x_1, \dots, x_n)$.

The following picture illustrates the action of $p : \overline{\mathcal{M}}_{3,8} \to \overline{\mathcal{M}}_{3,2}$.

Proposition 2.2. *The universal curve $\overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ and the forgetful map $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ are isomorphic as families over $\overline{\mathcal{M}}_{g,n}$.*

Proof. A point $t \in \overline{\mathcal{M}}_{g,n+1}$ encodes a stable curve $(C, x_1, \dots, x_{n+1})$. Denote by $(\widehat{C}, \hat{x}_1, \dots, \hat{x}_n)$ the stabilization of $(C, x_1, \dots, x_n)$ and by $y \in \widehat{C}$ the image of x_{n+1} under the stabilization. Then $(\widehat{C}, \hat{x}_1, \dots, \hat{x}_n, y)$ is an element of $\overline{\mathcal{C}}_{g,n}$.

To understand this isomorphism more precisely, let us distinguish three cases.

$\overline{\mathcal{C}}_{g,n+1}$ $\overline{\mathcal{C}}_{g,n}$

x_i x_{n+1}

x_{n+1} $\hat{x}_i$

x_{n+1}

$\overline{\mathcal{M}}_{g,n+1}$ $\overline{\mathcal{M}}_{g,n}$

(i) Suppose the curve $(C, x_1, \ldots, x_n)$ is stable. Then $(\widehat{C}, \hat{x}_1, \ldots, \hat{x}_n) = (C, x_1, \ldots, x_n)$. In this case $y = x_{n+1}$ on the curve $C = \widehat{C}$.

(ii) Suppose x_{n+1} lies on a genus 0 component C_0 of C that contains another marked point x_i, a node, and no other special points. Then $\widehat{C}$ is obtained from C by contracting the component C_0, and $\hat{x}_i$ is the image of C_0. In this case, $y = \hat{x}_i$.

(iii) Finally, suppose x_{n+1} lies on a genus 0 component C_0 of C that, in addition, contains two nodes, and no other special points. Then $\widehat{C}$ is obtained from C by contracting the component C_0. In this case, y is the image of C_0 and it is a node of $\widehat{C}$.

It is easy to construct the inverse map and thus to prove that we have constructed an isomorphism.

The figure shows three points in $\overline{\mathcal{M}}_{g,n+1}$ and their images in $\overline{\mathcal{C}}_{g,n}$. $\diamondsuit$

The following proposition is an example of application of forgetful maps.

Proposition 2.3. *The following cohomological relation holds in* $H^2(\overline{\mathcal{M}}_{0,6}, \mathbb{Q})$*:*

(Each picture represents the divisor whose points encode the curves as shown or more degenerate stable curves.)

Proof. The right-hand side and the left-hand side are pull-backs under the forgetful map $p : \overline{\mathcal{M}}_{0,6} \to \overline{\mathcal{M}}_{0,4}$ of the divisors

respectively. Both represent the class of a point in $\overline{\mathcal{M}}_{0,4}$. $\diamondsuit$

2.1.2 Attaching maps. Let $I \sqcup J$ be a partition of the set $\{1, \ldots, n+2\}$ in two disjoint subsets such that $n+1 \in I$, $n+2 \in J$. Choose two integers g_1 and g_2 in such a way that $g_1 + g_2 = g$. Denote by $\overline{\mathcal{M}}_{g_1,I}$ the moduli space of stable curves whose marked points are labeled by the elements of I, and likewise for $\overline{\mathcal{M}}_{g_2,J}$.

Definition 2.4. The *attaching map of separating kind* $q : \overline{\mathcal{M}}_{g_1,I} \times \overline{\mathcal{M}}_{g_2,J} \to \overline{\mathcal{M}}_{g,n}$ assigns to two stable curves the stable curve obtained by identifying the marked points with numbers $n+1$ and $n+2$.

The *attaching map of nonseparating kind* $q : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ assigns to a stable curve the stable curve obtained by identifying the marked points with numbers $n+1$ and $n+2$.

2.1.3 Tautological rings: preliminaries We will now start introducing tautological classes, see also the chapter by G. Mondello in Volume II of this Handbook [17].

Before going into details let us give a definition that motivates the appearance of these classes.

Definition 2.5. The minimal family of subrings $R^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$ stable under the pull-backs and push-forwards under the forgetful and attaching maps is called the family of *tautological rings* of the moduli spaces of stable curves.

Thus $1 \in H^0(\overline{\mathcal{M}}_{g,n})$ lies in the tautological ring (since a subring contains the unit element by definition), the classes represented by boundary strata lie in the tautological ring (since they are images of 1 under attaching maps), the self-intersection of a boundary stratum lies in the tautological ring, and so on. Now we will give an explicit construction of other tautological classes.

The relative cotangent line bundle. Let $p : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve and $\Delta \subset \overline{\mathcal{C}}_{g,n}$ the set of nodes in the singular fibers. Over $\overline{\mathcal{C}}_{g,n} \setminus \Delta$ there is a line bundle $\mathcal{L}$ cotangent to the fibers of the universal curve. We are going to extend this line bundle to the whole universal curve. To do that, it is enough to consider the local picture $p : (x, y) \mapsto xy$ (see Section 1.4.4). In coordinates (x, y), the line bundle $\mathcal{L}$ is generated by the sections $\frac{dx}{x}$ and $\frac{dy}{y}$ modulo the relation $\frac{d(xy)}{xy} = \frac{dx}{x} + \frac{dy}{y} = 0$. Since the restriction of the 1-form $d(xy)$ on every fiber of p vanishes, the line bundle thus obtained is indeed identified with the cotangent line bundle to the fibers of $\overline{\mathcal{C}}_{g,n}$.

Definition 2.6. The line bundle $\mathcal{L}$ extended to the whole universal curve is called the *relative cotangent line bundle*.

The restriction of $\mathcal{L}$ to a fiber C of the universal curve is a line bundle over C. If C is smooth, then $\mathcal{L}|_C$ is the cotangent line bundle and its holomorphic sections are the abelian differentials, that is, the holomorphic differential 1-forms. By extension, the holomorphic sections of $\mathcal{L}|_C$ are called abelian differentials for any stable curve. They can be described as follows.

Definition 2.7. An *abelian differential on a stable curve* C is a meromorphic 1-form α on each component of C satisfying the following properties: (i) the only poles of α are at the nodes of C, (ii) the poles are at most simple, (iii) the residues of the poles on two branches meeting at a node are opposite to each other.

Remark 2.8. More generally, when we speak about meromorphic forms on a stable curve with poles of orders $k_1, \dots, k_n$ at the marked points $x_1, \dots, x_n$, we will actually mean meromorphic sections of $\mathcal{L}$ with poles as above, or, in algebro-geometric notation, the sections of $\mathcal{L}(\sum k_i x_i)$. In other words, in addition to the poles at the marked points, we allow the 1-forms to have simple poles at the nodes with opposite residues on the two branches.

Example 2.9. The figure in Example 1.33 represents a stable curve obtained by identifying two points of the Riemann sphere. On the Riemann sphere we introduce the coordinate z such that the marked point is situated at $z = 1$, while the identified points are $z = 0$ and $z = \infty$. In this coordinate, the abelian differentials on the curve have the form $\lambda \frac{dz}{z}$. The residues of this differential at 0 and ∞ equal λ and $-\lambda$ respectively.

Proposition 2.10. *The abelian differentials on any genus g stable curve form a vector space of dimension g.*

Sketch of the proof. Use the Riemann-Roch formula on every component of the normalization of the curve. ♢

It follows from standard algebraic-geometric arguments (see [11], Exercise 5.8) that, since the dimension of these vector spaces is the same for every curve, they actually form a rank g holomorphic vector bundle over $\overline{\mathcal{M}}_{g,n}$.

Definition 2.11. The *Hodge bundle* Λ is the rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose fiber over $t \in \overline{\mathcal{M}}_{g,n}$ is constituted by the abelian differentials on the curve C_t.

2.2 The ψ-classes

Definition of ψ-classes. First we construct n holomorphic line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$ over $\overline{\mathcal{M}}_{g,n}$. The fiber of $\mathcal{L}_i$ over a point $x \in \overline{\mathcal{M}}_{g,n}$ is the *cotangent* *line to the curve C_x at the ith marked point.* More precisely, let $s_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{C}}_{g,n}$ the section corresponding to the ith marked point (so that $p \circ s_i = \mathrm{Id}$). Then $\mathcal{L}_i = s_i^*(\mathcal{L})$.

Definition 2.12. The *ψ-classes* are the first Chern classes of the line bundles $\mathcal{L}_i$,

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

2.2.1 Expression ψ_i as a sum of divisors for $g = 0$ Over $\overline{\mathcal{M}}_{0,n}$ it is possible to construct an explicit section of the line bundle $\mathcal{L}_i$ and to express its first Chern class ψ_i as a linear combination of divisors.

For pairwise distinct $i, j, k \in \{1, \ldots, n\}$, denote by $\delta_{i|jk}$ the set of stable genus 0 curves with a node separating the ith marked point from the jth and kth marked points.

The set $\delta_{i|jk}$ is a divisor on $\overline{\mathcal{M}}_{0,n}$ and we denote by $[\delta_{i|jk}] \in H^2(\overline{\mathcal{M}}_{0,n})$ its Poincaré dual cohomology class.

Proposition 2.13. *On $\overline{\mathcal{M}}_{0,n}$ we have $\psi_i = [\delta_{i|jk}]$ for any j, k.*

Proof. We construct an explicit meromorphic section α of the dual cotangent line bundle $\mathcal{L}$ over the universal curve. Its restriction to the ith section s_i of the universal curve will give us a holomorphic section of $\mathcal{L}_i$. The class ψ_i is then represented by the divisor of its zeroes.

The meromorphic section α of $\mathcal{L}$ is constructed as follows. On each fiber of the universal curve (*i.e.*, on each stable curve) there is a unique meromorphic 1-form (in the sense of Remark 2.8) with simple poles at the jth and the kth marked points with residues 1 and -1 respectively. This form gives us a section of $\mathcal{L}$ on each stable curve. Their union is the section α of $\mathcal{L}$ over the whole universal curve.

In order to determine the zeroes of the restriction $\alpha|_{s_i}$ let us study α in more detail. A stable curve C of genus 0 is a tree of spheres. One of the spheres contains the jth marked point, another one (possibly the same) contains the kth marked point. There is a chain of spheres connecting these two spheres (shown in grey in the figure).

On every sphere of the chain, the 1-form $\alpha|_C$ has two simple poles: one with residue 1 (at the jth marked point or at the node leading to the jth marked point) and one with residue -1 (at the kth marked point or at the node leading to the kth marked point) The 1-form vanishes on the spheres that do not belong to the chain. Thus α determines a nonvanishing cotangent vector at the ith marked point if and only if the ith marked point lies on the chain. In other words, $\alpha|_{s_i}$ vanishes if and only if the curve C contains a node that separates the ith marked point from the jth and the kth marked points. But this is precisely the description of $\delta_{i|jk}$.

We conclude that the divisor $\delta_{i|jk}$ represents the class ψ_i. $\diamondsuit$

Example 2.14. We have

$$\int_{\overline{\mathcal{M}}_{0,4}} \psi_1 = 1,$$

because the divisor $\delta_{1|23}$ is composed of exactly one point corresponding to the curve:

Example 2.15. Let us compute the integral

$$\int_{\overline{\mathcal{M}}_{0,5}} \psi_1\psi_2.$$

It is possible to express both classes in divisors and then study the intersection of these divisors, but this method is rather complicated, because it involves a struggle with self-intersections. A better idea is to express the ψ-classes in terms of divisors one at a time. We have

$$\psi_1 = [\delta_{1|23}] = \qquad .$$

Now we must compute the integral of ψ_2 over $\delta_{1|23}$. Each of the three components of $\delta_{1|23}$ is isomorphic to $\overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,4}$, and we see that ψ_2 is the pull-back of a ψ-class either from $\overline{\mathcal{M}}_{0,3}$ (for the first component) or from $\overline{\mathcal{M}}_{0,4}$ (for the second and the third components). In the first case, the integral of ψ_2 vanishes, while in the second and the third cases it is equal to 1 according to Example 2.14. We conclude that

$$\int_{\overline{\mathcal{M}}_{0,5}} \psi_1\psi_2 = 2.$$

Proposition 2.16. *We have*

$$\int_{\overline{\mathcal{M}}_{0,3}} 1 = 1; \qquad \int_{\overline{\mathcal{M}}_{0,4}} \psi_1 = 1; \qquad \int_{\overline{\mathcal{M}}_{0,5}} \psi_1^2 = 1; \qquad \int_{\overline{\mathcal{M}}_{0,5}} \psi_1\psi_2 = 2;$$

$$\int_{\overline{\mathcal{M}}_{0,6}} \psi_1^3 = 1; \qquad \int_{\overline{\mathcal{M}}_{0,6}} \psi_1^2\psi_2 = 3; \qquad \int_{\overline{\mathcal{M}}_{0,6}} \psi_1\psi_2\psi_3 = 6.$$

Proposition 2.17. *Let p be the forgetful map $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. Consider the set of stable curves that contain a spherical component with exactly three special points: a node and the marked points number i and $n+1$.*

The points encoding such curves form a divisor $\delta_{(i,n+1)}$ of $\overline{\mathcal{M}}_{g,n+1}$. Now we can consider the class ψ_i ($1 \le i \le n$) both on $\overline{\mathcal{M}}_{g,n}$ and on $\overline{\mathcal{M}}_{g,n+1}$. We have

$$\psi_i - p^*(\psi_i) = [\delta_{(i,n+1)}].$$

Proposition 2.18. *Let D_i be the divisor of the ith special section in the universal curve $p : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. Then we have $p_*(D_i^{k+1}) = (-\psi_i)^k$.*

We leave the proofs as an exercise to the reader.

2.2.2 Modular forms and the class ψ_1 on $\overline{\mathcal{M}}_{1,1}$ Recall that a lattice $L \subset \mathbb{C}$ is an additive subgroup of $\mathbb{C}$ isomorphic to $\mathbb{Z}^2$.

Definition 2.19. A *modular form of weight* $k \in \mathbb{N}$ is a function F on the set of lattices such that (i) $F(cL) = F(L)/c^k$ for $c \in \mathbb{C}^*$ and (ii) the function $f(\tau) = F(\mathbb{Z} + \tau\mathbb{Z})$ is holomorphic on the upper half-plane $\operatorname{Im}\tau > 0$, and (iii) $f(\tau)$ is bounded on the half-plane $\operatorname{Im}\tau \ge C$ for any positive constant C.

Since the lattice $\mathbb{Z} + \tau\mathbb{Z}$ is the same as $\mathbb{Z} + (\tau+1)\mathbb{Z}$, the function f is periodic with period 1. Therefore there exists a function $\varphi(q)$, holomorphic on the open punctured unit disc, such that $f(\tau) = \varphi(e^{2\pi i\tau})$. This fucntion is bounded at the neighborhood of the origin, therefore it can be extended to a holomorphic function on the whole unit disc and expanded into a power series in q at 0, which is the usual way to represent a modular form.

Since $-L = L$ for every lattice L, we see that there are no nonzero modular forms of odd weight. On the other hand, there exists a nonzero modular form of any even weight $k \ge 4$, given by

$$E_k(L) = \sum_{z \in L \setminus \{0\}} \frac{1}{z^k}.$$

(For odd k this sum vanishes, while for $k = 2$ it is not absolutely convergent.) The value of the corresponding function $\varphi_k(q)$ at $q = 0$ is equal to

$$\varphi_k(0) = \lim_{\operatorname{Im}\tau \to \infty} E_k(\mathbb{Z} + \tau\mathbb{Z}) = \sum_{z \in \mathbb{Z} \setminus \{0\}} \frac{1}{z^k} = 2\zeta(k).$$

The relation between modular forms and the ψ-class on $\overline{\mathcal{M}}_{1,1}$ comes from the following proposition.

Proposition 2.20. *The space of modular forms of weight k is naturally identified with the space of holomorphic sections of $\mathcal{L}_1^{\otimes k}$ over $\overline{\mathcal{M}}_{1,1}$.*

Proof. Let F be a modular form of weight k. We claim that $F(L)dz^k$ is a well-defined holomorphic section of $\mathcal{L}_1^{\otimes k}$ over $\overline{\mathcal{M}}_{1,1}$.

First of all, if $\mathbb{C}/L$ is any elliptic curve, then the value of $F(L)dz^k$ at the marked point (the image of $0 \in \mathbb{C}$) is indeed a differential k-form, that is, an element of the fiber of $\mathcal{L}_1^{\otimes k}$. If we apply a homothety $z \mapsto cz$, replacing L by cL, we obtain an isomorphic elliptic curve. However, the k-form $F(L)dz^k$ does not change, because $F(L)$ is divided by c^k, while dz^k is multiplied by c^k. Thus $F(L)dz^k$ is a well-defined section of $\mathcal{L}_1^{\otimes k}$.

The fact that this section is holomorphic over $\mathcal{M}_{1,1}$ follows from the fact that $f(\tau)$ is holomorphic. The fact that it is also holomorphic at the boundary point follows from the fact the function $\varphi(q)$ is holomorphic at $q = 0$.

Conversely, if s is a holomorphic section of $\mathcal{L}_1^{\otimes k}$, then taking the value of s over the curve $\mathbb{C}/L$ and dividing by dz^k, we obtain a function on lattices L. The same argument as above shows that it is a modular form of weight k. ♢

Proposition 2.21. *We have*

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}.$$

Proof. We are going to give three similar computations leading to the same result. Denote by $f_k(\tau)$ and $\varphi_k(q)$ the functions associated with the modular form E_k. One can check (see, for instance [19], chapter VII) that in the modular figure (*i.e.*, on $\mathcal{M}_{1,1}$) the function f_4 has a unique simple zero at $\tau = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$, while f_6 has a unique simple zero at $\tau = i$. (The fact that these are indeed zeroes is an easy exercise for the reader.) Further, the function

$$\left(\frac{\varphi_4}{2\zeta(4)}\right)^3 - \left(\frac{\varphi_6}{2\zeta(6)}\right)^2$$

has a unique zero at $q = 0$. The stabilizers of the corresponding points in $\overline{\mathcal{M}}_{1,1}$ are $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z}$ respectively (see Example 1.24). Thus the first Chern class of $\mathcal{L}_1^{\otimes 4}$ equals $1/6$, that of $\mathcal{L}_1^{\otimes 6}$ equals $1/4$, that of $\mathcal{L}_1^{\otimes 12}$ equals $1/2$. In every case we find that the first Chern class of $\mathcal{L}_1$ equals $\psi_1 = 1/24$. ♢

Proposition 2.22. *Denote by $\delta_{(\mathrm{irr})}, \delta_{(1)} \subset \overline{\mathcal{M}}_{1,n}$ the divisors*

$\delta_{(\mathrm{irr})}$ $\delta_{(1)}$

In other words, the points of $\delta_{(\mathrm{irr})}$ encode curves with at least one nonseparating node; the points of $\delta_{(1)}$ encode curves with a separating node dividing the curve *into a stable curve of genus* 1 *and a stable curve of genus* 0 *containing the marked point number 1. Then the class* ψ_1 *on* $\overline{\mathcal{M}}_{1,n}$ *equals*

$$\psi_1 = \frac{1}{12}[\delta_{(\mathrm{irr})}] + [\delta_{(1)}].$$

The proof follows from the computation of ψ_1 on $\overline{\mathcal{M}}_{1,1}$ and from Proposition 2.17.

2.3 Other tautological classes

All cohomology classes we consider are with rational coefficients.

2.3.1 The classes on the universal curve. On the universal curve we define the following classes.

- D_i is the divisor given by the ith section of the universal curve. In other words, the intersection of D_i with a fiber C of $\overline{\mathcal{C}}_{g,n}$ is the ith marked point on C. By abuse of notation we denote by $D_i \in H^2(\overline{\mathcal{C}}_{g,n})$ the cohomology class Poincaré dual to the divisor.
- $D = \sum_{i=1}^n D_i$.
- $\omega = c_1(\mathcal{L})$.
- $K = c_1(\mathcal{L}^{\log}) = \omega + D \in H^2(\overline{\mathcal{C}}_{g,n})$, where $\mathcal{L}^{\log}$ is the line bundle $\mathcal{L}$ twisted by the divisor D.
- Δ is the codimension 2 subvariety of $\overline{\mathcal{C}}_{g,n}$ consisting of the nodes of the singular fibers. By abuse of notation, $\Delta \in H^4(\overline{\mathcal{C}}_{g,n})$ will also denote the Poincaré dual cohomology class.
- Let N be the normal vector bundle to Δ in $\overline{\mathcal{C}}_{g,n}$. Then we denote by

$$\Delta_{k,l} = (-c_1(N))^k \Delta^{l+1}.$$

To simplify the notation, we introduce two symbols ν_1 and ν_2 with the convention $\nu_1 + \nu_2 = -c_1(N)$, $\nu_1\nu_2 = c_2(N)$. Since $c_2(N)\Delta = \Delta^2$, we also identify $\nu_1\nu_2$ with Δ. Thus, even though the symbols ν_1 and ν_2 separately are meaningless, every symmetric polynomial in ν_1 and ν_2 divisible by $\nu_1\nu_2$ determines a well-defined cohomology class. For instance, we have

$$\Delta_{k,l} = \Delta \cdot (\nu_1 + \nu_2)^k (\nu_1\nu_2)^l = (\nu_1 + \nu_2)^k (\nu_1\nu_2)^{l+1}.$$

Since Δ is the set of nodes in the singular fibers of $\overline{\mathcal{C}}_{g,n}$, it has a natural 2-sheeted (unramified) covering $p : \widetilde{\Delta} \to \Delta$ whose points are couples (node + choice of a branch). Over $\widetilde{\Delta}$ we can define two natural line bundles $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ cotangent, respectively, to the first and to the second branch at the node. The pull-back p^*N of N to $\widetilde{\Delta}$ is naturally identified with $\mathcal{L}_\alpha^\vee \oplus \mathcal{L}_\beta^\vee$. Thus, if $P(\nu_1, \nu_2)$ is a symmetric polynomial, we have $p^*(\Delta \cdot P(\nu_1, \nu_2)) = \widetilde{\Delta} \cdot P(c_1(\mathcal{L}_\alpha), c_1(\mathcal{L}_\beta))$.

2.3.2 Intersecting classes on the universal curve

Proposition 2.23. *For all $1 \le i, j \le n$, $i \neq j$ we have*

$$KD_i = D_iD_j = K\Delta = D_i\Delta = 0 \in H^*(\overline{\mathcal{C}}_{g,n}).$$

Proof. The divisors D_i and D_j do not intersect, so the intersection of the corresponding classes vanishes. Similarly, the divisor D_i does not meet Δ, so their intersection vanishes. The restriction of the line bundle $\mathcal{L}^{\log}$ to D_i is trivial. Indeed, the sections of $\mathcal{L}^{\log}$ are 1-forms with simple poles at the marked points, and the fiber at the marked point is the line of residues, so it is canonically identified with $\mathbb{C}$. The intersection KD_i is the first Chern class of the restriction of $\mathcal{L}^{\log}$ to D_i. Therefore it vanishes. The restriction of $\mathcal{L}^{\log}$ to Δ is not necessarily trivial. However its pull-back to the double-sheeted covering $\widetilde{\Delta}$ is trivial (because the fiber is the line of residues identified with $\mathbb{C}$). Alternatively, one can say that $(\mathcal{L}^{\log})^{\otimes 2}$ is trivial. Therefore $K\Delta = 0$. ♢

Remark 2.24. A line bundle whose tensor power is trivial is called *rationally trivial.* Although it is not necessarily trivial itself, all it characteristic classes over $\mathbb{Q}$ vanish. This is the case of $\mathcal{L}|_\Delta = \mathcal{L}^{\log}|_\Delta$.

Corollary 2.25. *Every polynomial in the classes $D_i, K, \Delta_{k,l}$ on $\overline{\mathcal{C}}_{g,n}$ can be written in the form*

$$P_K(K) + \sum_{i=1}^{n} P_i(D_i) + \Delta \cdot P_\Delta(\nu_1, \nu_2),$$

while P_K and P_i, $1 \le i \le n$ are arbitrary polynomials, while P_Δ is a symmetric polynomial with the convention $\nu_1\nu_2 = \Delta$, $\nu_1 + \nu_2 = -c_1(N)$.

Proof. Given a polynomial in $D_i, K, \Delta_{k,l}$, we can, according to the proposition, cross out the "mixed terms", that is, the monomials containing products D_iD_j, D_iK, $K\Delta_{k,l}$ or $D_i\Delta_{k,l}$. We end up with a sum of powers of D_i, powers of K, and products of $\Delta_{k,l}$. Now, by definition, $\Delta_{k_1,l_1}\Delta_{k_2,l_2} = \Delta_{k_1+k_2,l_1+l_2+1}$. Therefore a polynomial in variables $\Delta_{k,l}$ can be rewritten in the form $\Delta P_\Delta(\nu_1, \nu_2)$, where P_Δ is a symmetric polynomial. ♢

2.3.3 The classes on the moduli space. Let $p : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve. On the moduli space $\overline{\mathcal{M}}_{g,n}$ we define the following classes.

- $\kappa_m = p_*(K^{m+1}) \in H^{2m}(\overline{\mathcal{M}}_{g,n})$.
- $\psi_i = -p_*(D_i^2) \in H^2(\overline{\mathcal{M}}_{g,n})$.
- $\delta_{k,l} = p_*(\Delta_{k,l}) \in H^{k+2l+1}(\overline{\mathcal{M}}_{g,n})$.

- $\lambda_i = c_i(\Lambda) \in H^{2i}(\overline{\mathcal{M}}_{g,n})$, where Λ is the Hodge bundle and c_i the ith Chern class.

Note that for the definition of the ψ-class we use Proposition 2.18. Thus, with the exception of the λ-classes, the tautological classes on $\overline{\mathcal{M}}_{g,n}$ are push-forwards of tautological classes on $\overline{\mathcal{C}}_{g,n}$ and their products. For example, $\delta_{0,0}$ is the boundary divisor on $\overline{\mathcal{M}}_{g,n}$, *i.e.*, $\delta_{0,0} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$.

Example 2.26. As an exercise, the reader can show that over $\overline{\mathcal{M}}_{1,1}$ the line bundles Λ and $\mathcal{L}_1$ are isomorphic. Hence

$$\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = \frac{1}{24}.$$

Theorem 2.27. *The classes ψ_i, κ_m, $\delta_{k,l}$, and λ_i lie in the tautological ring in the sense of Definition 2.5.*

Proof. Let $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map. Then $\psi_i = -p_*(\delta^2_{(i,n+1)})$, where $\delta_{(i,n+1)}$ is defined in Propositions 2.17, while $\kappa_m = p_*(\psi_{n+1}^{m+1})$. The class $\delta_{k,l}$, is the sum of push-forwards under the attaching maps of the class $(\psi_{n+1} + \psi_{n+2})^k (\psi_{n+1}\psi_{n+2})^l$. Thus all these classes lie in the tautological ring. The class λ_i is expressed via the ψ-, κ-, and δ-classes in Theorem 3.16. It follows that it too lies in the tautological ring. ♢

3 Algebraic geometry on moduli spaces

In the previous section we introduced a wide range of tautological classes on the moduli space $\overline{\mathcal{M}}_{g,n}$, namely, the ψ-, κ-, δ-, and λ-classes. Now we would like to learn to compute all possible intersection numbers between these classes.

This is done in three steps.

First, by applying the Grothendieck-Riemann-Roch (GRR) formula we express λ-classes in terms of ψ-, κ-, and δ-classes. This gives us an opportunity to introduce the GRR formula and to give an example of its application in a concrete situation.

Second, by studying the pull-backs of the ψ-, κ-, and δ-classes under attaching and forgetful maps, we will be able to eliminate one by one the κ- and δ-classes from intersection numbers.

The remaining problem of computing intersection numbers of the ψ-classes is much more difficult. The answer was first conjectured by E. Witten [22]. It is formulated below in Theorems 4.4 and 4.5. Witten's conjecture now has at least 5 different proofs (the most accessible to a non-specialist is probably [13]), and all of them use nontrivial techniques. In this note we will not prove Witten's conjecture, but give its formulation and say a few words about how it appeared. Witten's conjecture is also discussed in Mondello's chapter of this Hanbook [17].

3.1 Characteristic classes and the GRR formula

In this section we present the Grothendieck-Riemann-Roch (GRR) formula. But first we recall the necessary information on characteristic classes of vector bundles, mostly without proofs.

3.1.1 The first Chern class.

Definition 3.1. Let $L \to B$ be a holomorphic line bundle over a complex manifold B. Let s be a nonzero meromorphic section of L and $Z - P$ the associated divisor: the set of zeroes minus the set of poles of s. Then $[Z]-[P] \in H^2(B, \mathbb{Z})$ is called the *first Chern class* of L and denoted by $c_1(L)$.

The first Chern class is well-defined, *i.e.*, it does not depend on the choice of the section. Moreover, $c_1(L)$ is a *topological* invariant of L. In other words, it only depends on the topological type of L and B, but not on the complex structure of B nor on the holomorphic structure of L. Actually, there exists a different definition of first Chern classes (which we won't use) that does not involve the holomorphic structure at all.

3.1.2 Total Chern class, Todd class, Chern character. Let $V \to B$ be a vector bundle of rank k.

Definition 3.2. We say that V can be *exhausted* by line bundles if we can find a line subbundle L_1 of V, then a line subbundle L_2 of the quotient $V_1 = V/L_1$, then a line subbundle L_3 of the quotient $V_2 = V_1/L_2$, and so on, until the last quotient is itself a line bundle L_k. The simplest case is when $V = \oplus L_i$.

If V is exhausted by line bundles, the first Chern classes $r_i = c_1(L_i)$ are called the *Chern roots* of V.

Definition 3.3. Let V be a vector bundle with Chern roots $r_1, \ldots, r_k$. Its *total Chern class* is defined by

$$c(V) = \prod_{i=1}^{k} (1 + r_i);$$

Its *Todd class* is defined by

$$\mathrm{Td}(V) = \prod_{i=1}^{k} \frac{r_i}{1 - e^{-r_i}};$$

Its *Chern character* is defined by

$$\mathrm{ch}(V) = \sum_{i=1}^{k} e^{r_i}.$$

The homogeneous parts of degree i of these classes are denoted by c_i, Td_i, and ch_i respectively.

If we know the total Chern class of a vector bundle, we can compute its Todd class and Chern character (except ch_0 that is equal to the rank of the bundle). For instance, let us compute ch_3. We have

$$\begin{aligned} \mathrm{ch}_3 &= \frac{1}{6} \sum r_i^3 \\ &= \frac{1}{6} \Big(\sum r_i\Big)^3 - \frac{1}{2} \Big(\sum r_i\Big) \Big(\sum_{i<j} r_i r_j\Big) + \frac{1}{2} \sum_{i<j<k} r_i r_j r_k \\ &= \frac{1}{6} c_1^3 - \frac{1}{2} c_1 c_2 + \frac{1}{2} c_3. \end{aligned}$$

As an exercise, the reader can compute the expressions for ch_1, ch_2, Td_1, and Td_2 in terms of Chern classes c_1 and c_2.

Not every vector bundle can be exhausted by line bundles. The characteristic classes are defined in full generality by using the following proposition.

Proposition 3.4. *For every vector bundle $V \to B$ there exists a map of complex manifolds $p : B' \to B$ such that $p^*(V)$ can be exhausted by line bundles and the induced morphism $p^* : H^*(B, \mathbb{Z}) \to H^*(B', \mathbb{Z})$ in cohomology is an injection.*

Sketch of proof. Let $V \to B$ be a vector bundle and $p : \mathbb{P}(V) \to B$ its projectivization. Then the tautological line bundle over $\mathbb{P}(V)$ is a subbundle of $p^*(V)$ and p induces an injection from $H^*(B, \mathbb{Z})$ to $H^*(\mathbb{P}(V), \mathbb{Z})$. It suffices to apply this construction several times. ♢

Thus, for instance, $c(V)$ is uniquely defined by the condition $p^*(c(V)) = c(p^*V)$, and similarly for the Todd class and the Chern character.

Remark 3.5. In general, given a power series f, one can assign to every vector bundle the corresponding *character* and *class*. The character is given by $\sum f(r_i)$, while the class is given by $\prod f(r_i)$. Thus for an exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

of vector bundles, we have

$$\mathrm{class}(V_2) = \mathrm{class}(V_1) \cdot \mathrm{class}(V_3),$$

$$\mathrm{character}(V_2) = \mathrm{character}(V_1) + \mathrm{character}(V_3).$$

For instance, the Todd class is defined using the series

$$f(x) = \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{n \geq 1} \frac{B_{2n}}{(2n)!} x^{2n},$$

where B_{2n} are the Bernoulli numbers:

$$B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad \ldots$$

One of the standard references for characteristic classes is [16]; for an algebraic-geometric presentation see [7], Chapter 4.

3.1.3 Cohomology spaces of vector bundles. Our use of cohomology spaces of vector bundles is very limited, so we introduce them in a very brief way. See [9], Chapter 0 for more details.

To a vector bundle V over a compact complex algebraic variety B one assigns its cohomology spaces, $H^k(B, V)$, $k = 0, 1, \ldots \dim B$. These are the cohomology groups $\mathrm{Ker}\, \bar\partial / \mathrm{Im}\, \bar\partial$ of the complex

$$0 \stackrel{\bar\partial}{\to} \mathcal{A}^{0,0}(V) \stackrel{\bar\partial}{\to} \mathcal{A}^{0,1}(V) \stackrel{\bar\partial}{\to} \mathcal{A}^{0,2}(V) \stackrel{\bar\partial}{\to} \ldots$$

The terms of this complex are the spaces of smooth V-valued differential forms of type $(0, k)$. In local coordinates, such a differential form has the form

$$\sum s_{i_1, \ldots, i_k}(z)\, d\bar z_{i_1} \wedge \cdots \wedge d\bar z_{i_k},$$

where $s_{i_1, \ldots, i_k}$ is a smooth section of V. It is a nontrivial fact that $H^k(B, V)$ is a finite-dimensional vector space and that $H^k(B, V) = 0$ for $k > \dim B$.

We will use only two properties of the cohomology groups.

First, $H^0(V)$ is the space of holomorphic sections of V. This follows directly from the definition.

Second, let K be the *canonical* line bundle over B, in other words the highest exterior power of the cotangent vector bundle. Let $V^\vee$ be the dual vector bundle of V. Then $H^k(B, V)$ and $H^{\dim B - k}(B, K \otimes V^\vee)$ are dual vector spaces. This property is called the *Serre duality.*

3.1.4 K^0, p_*, and $p_!$ Let X be a complex manifold. Consider the free additive group generated by the vector bundles over X. To every short exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

we assign the relation $V_1 - V_2 + V_3 = 0$ in this group. The *Grothendieck group* $K^0(X)$ is the factor of the free group by all such relations.

We would obtain the same group if the vector bundles were replaced by coherent sheaves, because every coherent sheaf has a finite resolution by vector bundles.

The Chern character determines a group morphism $\mathrm{ch} : K^0(X) \to H^*(X, \mathbb{Q})$ from the Grothendieck group to the cohomology group of X.

Let $p : X \to Y$ be a morphism of complex manifolds with compact fibers. Then p induces a morphism $p_* : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ of cohomology groups (the fiberwise integration). It is defined by $\langle p_* \alpha, C \rangle = \langle \alpha, p^{-1}(C) \rangle$ for any cycle $C \subset Y$ in general position.

On the other hand, p also determines a morphism $p_! : K^0(X) \to K^0(Y)$, that we now describe.

Denote by X_y the fiber over a point $y \in Y$. Then we can consider the cohomology spaces $H^k(X_y, V)$ (where V actually stands for the restriction of V to X_y). For each k, the vector spaces $H^k(X_y, V)$ form a sheaf over Y. This sheaf is denoted by $R^k(p, V)$. The morphism $p_!$ of Grothendieck groups is now defined by

$$p_!(V) = R^0(p, V) - R^1(p, V) + \dots .$$

Now we have the following diagram of morphisms.

$$\begin{array}{ccc} & & \qquad \mathrm{Td}(p) \\ K^0(X) & \xrightarrow{\mathrm{ch}} & H^*(X, \mathbb{Q}) \\ p_! \big\downarrow & & p_* \big\downarrow \\ K^0(Y) & \xrightarrow{\mathrm{ch}} & H^*(Y, \mathbb{Q}) \end{array}$$

The question is: is it commutative? The answer, given by the GRR formula, is that it is not, but can be made to commute if we add a multiplicative factor $\mathrm{Td}(p)$.

3.1.5 The Grothendieck-Riemann-Roch formula Let $p : X \to Y$ be a morphism of complex manifolds with compact fibers. Let V be a vector bundle over X. Denote by

$$\mathrm{Td}(p) = \frac{\mathrm{Td}(TX)}{\mathrm{Td}(p^*(TY))}.$$

Theorem 3.6 (GRR, see [7], Chapter 9). *We have*

$$\mathrm{ch}(p_! V) = p_* \left[\mathrm{ch}(V)\mathrm{Td}(p)\right].$$

Example 3.7. Apply the GRR formula to the situation $\mathcal{F} \to X \to$ point, where X is a (compact) Riemann surface and $\mathcal{F}$ is a sheaf. We obtain the Riemann-Roch formula

$$h^0(\mathcal{F}) - h^1(\mathcal{F}) = c_1(\mathcal{F}) + 1 - g,$$

where h^0 and h^1 are the dimensions of H^0 and H^1.

Example 3.8. Let $p : X \hookrightarrow Y$ be an embedding of smooth manifolds and $N \to X$ the normal vector bundle to X in Y. Let n be the rank of N. Then we have $p_*(\alpha) = c_n(N) \cdot \alpha$. Indeed, $c_n(N) \cdot [X]$ is the self-intersection of X in the total space of N. Applying GRR to the situation $\mathcal{O}_X \to X \hookrightarrow Y$ we get

$$c_n(N) = \mathrm{Td}(N) \cdot \sum_{k=0}^{n} (-1)^k \mathrm{ch}\left(\bigwedge^k N^\vee\right).$$

As an exercise, the reader can express both sides of this equality in terms of Chern roots of N and check that they coincide.

3.1.6 The Koszul resolution The Koszul resolution (see [4], Chapter 17) is an ingredient that we will need to apply the GRR formula to moduli spaces, so let us introduce it here.

Let $p : V \to X$ be a vector bundle over a smooth complex manifold X. Then X is embedded in the total space of V by the zero section, $X \subset V$. On the total space V, consider the sheaf $\mathcal{O}_X$ supported on X. Its sections over an open set $U \subset V$ are the holomorphic functions on $U \cap X$. Thus the fiber of $\mathcal{O}_X$ over a point of X is $\mathbb{C}$, while its fiber over a point outside X is 0.

It is always unpleasant to work with sheaves that are not vector bundles, therefore we would like to construct a *resolution* of $\mathcal{O}_X$, in other words, an exact sequence of sheaves that contains $\mathcal{O}_X$, but whose all other terms are vector bundles.

Example 3.9. Suppose V is a line bundle. Then over the total space of V there are two sheaves: the sheaf $\mathcal{O}_V$ of holomorphic functions and the sheaf $\mathcal{O}_V(X)$ of holomorphic functions vanishing on X. The short exact sequence

$$0 \to \mathcal{O}_V(X) \to \mathcal{O}_V \to \mathcal{O}_X \to 0$$

is a resolution of $\mathcal{O}_X$.

Our aim is to generalize this example.

Denote by $\mathbf{V}$ the pull-back of the vector bundle V to the total space of V:

$$\begin{array}{ccc} \mathbf{V} & \longrightarrow & V \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

Thus $\mathbf{V}$ is a vector bundle over the total space of V.

Denote by $\mathcal{A}^k(\mathbf{V})$ the sheaf of skew-symmetric k-forms on $\mathbf{V}$. In particular, $\mathcal{A}^1(\mathbf{V})$ is the sheaf of sections of $\mathbf{V}^\vee$, while $\mathcal{A}^0(\mathbf{V}) = \mathcal{O}_V$.

Let x be a point in X, $v \in V_x$ a point in the fiber V_x over x, and α be a skew-symmetric k-form on V_x. Then the form $i_v\alpha$ obtained by substituting v as the first entry of α is a skew-symmetric $(k-1)$-form on V_x. Thus we obtain a natural sheaf morphism $d : \mathcal{A}^k(\mathbf{V}) \to \mathcal{A}^{k-1}(\mathbf{V})$ obtained by substituting v into α.

Denote by p the rank of V.

Theorem 3.10. *The following sequence of sheaves in exact.*

$$0 \to \mathcal{A}^p(\mathbf{V}) \to \mathcal{A}^{p-1}(\mathbf{V}) \to \cdots \to \mathcal{A}^1(\mathbf{V}) \to \mathcal{O}_V \to \mathcal{O}_X \to 0.$$

This exact sequence is called the *Koszul resolution* of the sheaf $\mathcal{O}_X$.

Proof of Theorem 3.10 in the case of a rank 2 vector bundle.

For simplicity we restrict ourselves to the case $p = 2$, since it is the only case that we will need.

Choose an open chart in X and a trivialization of the vector bundle V over this chart. Let $t = (t_1, t_2, \dots)$ be the local coordinates on the chart and (x, y) the coordinates in the fibers of V. The maps d of the Koszul resolution can be explicitly written out as

$$\begin{aligned} 0 &\overset{1}{\mapsto} 0 \cdot dx \wedge dy, \\ f(t;x,y)\, dx \wedge dy &\overset{2}{\mapsto} -yf(t;x,y)\, dx + xf(t;x,y)\, dy, \\ g_x(t;x,y)\, dx + g_y(t;x,y)\, dy &\overset{3}{\mapsto} xg_x(t;x,y) + yg_y(t;x,y), \\ h(t;x,y) &\overset{4}{\mapsto} h(t;0,0), \\ h(t;0,0) &\overset{5}{\mapsto} 0. \end{aligned}$$

We must check that the image of each map coincides with the kernel of the next map.

The image of map 1 and the kernel map 2 vanish.

The kernel of map 3 is given by the condition $xg_x + yg_y = 0$. In particular, this implies that g_x is divisible by y, while g_y is divisible by x. Hence an element of the kernel lies in the image of the first map, since we can take $f = -g_x/y = g_y/x$. Conversely, if $g_x = -yf$ and $g_y = xf$, then $xg_x + yg_y = 0$, hence the image of map 2 is included in the kernel of map 3.

The kernel of map 4 and the image of map 3 are composed of maps that vanish at $x = y = 0$.

Finally, map 4 is surjective and map 5 takes everything to zero. $\diamondsuit$

We leave the generalization to the arbitrary rank as an exercise to the reader.

3.2 Applying GRR to the universal curve

The aim of this section is to apply the GRR formula to the case where the morphism of complex manifolds is the universal curve $p : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, while the vector bundle over $\overline{\mathcal{C}}_{g,n}$ is the relative cotangent line bundle $\mathcal{L}$ from Definition 2.6. We follow Mumford's paper [18].

A careful reader may wonder whether the GRR formula is applicable to orbifolds. Such worries are well-founded, because in general it is not. As an example, consider the line bundle $\mathcal{L}_1$ over $\overline{\mathcal{M}}_{1,1}$. As we saw in Section 2.2.2, the holomorphic sections of $\mathcal{L}_1^k$ are the modular forms of weight k. It is well-known (see, for instance, [19], VII, 3, Theorem 4) that modular forms are homogeneous polynomials in E_4 and E_6, which allows one to find the dimension of $H^0(\mathcal{L}_1^k)$ and to work out the necessary modifications of the Riemann-Roch formula in this example.

However, the GRR formula applies without changes to a morphism between two *orbifolds* if every *fiber* of the morphism is a compact *manifold*, *i.e.*, the orbifold structure of the fibers is trivial. This is, of course, true for the universal curve, because its fibers are stable curves. If the fibers have a nontrivial orbifold structure, the GRR formula must be modified to take into account the stabilizers of different points of the fibers.

Taking a look at the GRR formula, we see that we must compute three things: $\mathrm{ch}(p_!\mathcal{L})$, $\mathrm{ch}(\mathcal{L})$, and $\mathrm{Td}(p)$. Here "compute" means express either via the ψ-, κ-, δ-, and λ-classes (on the moduli space) or via the classes D_i, K, Δ and ν_1, ν_2 (on the universal curve).

Proposition 3.11. *We have* $p_!\mathcal{L} = \Lambda - \mathbb{C}$*, where* Λ *is the Hodge bundle and* $\mathbb{C}$ *the trivial line bundle over* $\overline{\mathcal{M}}_{g,n}$*.*

Proof. Let C be a stable curve. Then $H^0(C, \mathcal{L})$ is the space of abelian differentials on C. This means that $R^0(p, \mathcal{L})$ is the Hodge bundle Λ (Definition 2.11). By Serre's duality, the space $H^1(C, \mathcal{L})$ is dual to $H^0(C, \mathcal{L} \otimes \mathcal{L}^\vee) = H^0(C, \mathbb{C}) = \mathbb{C}$. (In Section 3.1.3 we only described the Serre duality over smooth manifolds, but it can be extended to stable curves by using the line bundle $\mathcal{L}$ instead of the cotangent line bundle.) Thus the space $H^1(C, \mathcal{L})$ is naturally identified with $\mathbb{C}$, so $R^1(p, \mathcal{L}) = \mathbb{C}$ is the trivial line bundle. $\diamondsuit$

Corollary 3.12. *We have*

$$\mathrm{ch}(p_!\mathcal{L}) = \mathrm{ch}(\Lambda) - 1.$$

Thus the left-hand side of the GRR formula involves λ-classes. Once we have computed the right-hand side we will be able to express the λ-classes via other classes.

Proposition 3.13. *We have*

$$\mathrm{ch}(\mathcal{L}) = e^{\omega}.$$

Proof. Obvious. ♢

3.2.1 Computing Td(p) Computing the first two ingredients of the GRR formula was easy, but the computation of Td(p) requires more work. If all the fibers of the universal curve were smooth, there would have been an exact sequence relating the vector bundles $T\overline{\mathcal{M}}_{g,n}$, $T\overline{\mathcal{C}}_{g,n}$ and $\mathcal{L}$. The Todd class Td(p) would then be equal to the Todd class of $-\mathcal{L}$. In reality, however, some fibers of p are singular and the singularity locus is Δ. Therefore we will get a more complicated expression for Td(p), involving $\mathcal{L}$ and the sheaf $\mathcal{O}_\Delta$. We will then compute the Todd class of $\mathcal{O}_\Delta$ using the Koszul resolution.

For practical reasons, it is easier to write an exact sequence involving the cotangent bundles $T^\vee\overline{\mathcal{M}}_{g,n}$ and $T^\vee\overline{\mathcal{C}}_{g,n}$ to the moduli space and the universal curve. Let $\mathcal{O}_\Delta$ be the sheaf over $\overline{\mathcal{C}}_{g,n}$ whose sections are functions on Δ (so the sheaf is supported on Δ).

Proposition 3.14. *The sequence*

$$0 \to p^*(T^\vee\overline{\mathcal{M}}_{g,n}) \overset{(dp)^\vee}{\longrightarrow} T^\vee\overline{\mathcal{C}}_{g,n} \to \mathcal{L} \to \mathcal{O}_\Delta \otimes \mathcal{L}|_\Delta \to 0$$

is an exact sequence of sheaves over $\overline{\mathcal{C}}_{g,n}$.

Proof. The maps in this exact sequence are (i) the adjoint map of the differential of p, (ii) the restriction of a 1-form on the tangent space to $\overline{\mathcal{C}}_{g,n}$ to the tangent space of a fiber, and (iii) restricting a section of $\mathcal{L}$ to Δ.

First consider the map p at the neighborhood of a point $z \in \overline{\mathcal{C}}_{g,n} \setminus \Delta$ and let $t = p(z) \in \overline{\mathcal{M}}_{g,n}$ be its image in the moduli space. The cotangent space to $\overline{\mathcal{M}}_{g,n}$ at t is injected in the cotangent space to $\overline{\mathcal{C}}_{g,n}$ at z by $dp^\vee$. The cokernel is the cotangent space to the fiber, *i.e.*, $\mathcal{L}$. The fiber of $\mathcal{O}_\Delta$ is equal to 0, so we obtain a short exact sequence.

Now take a point $z \in \Delta$ and let $t = p(z) \in \overline{\mathcal{M}}_{g,n}$ be its image in the moduli space. Consider the model local picture where p has the form $p : (X,Y) \mapsto T = XY$, where z is the point $X = Y = 0$ and t is the point $T = 0$. According to Section 1.4.4, in the general case the local picture is the direct product of our model local picture with a trivial map $p : B \to B$.

In the model local picture, the vector bundle $T^\vee\overline{\mathcal{M}}_{g,n}$ is the line bundle generated by dT. The vector bundle $T^\vee\overline{\mathcal{C}}_{g,n}$ is generated by dX and dY. Finally, the line bundle $\mathcal{L}$ is generated by the sections $\frac{dX}{X}$ and $\frac{dY}{Y}$ modulo the relation $\frac{dX}{X} + \frac{dY}{Y} = 0$. The maps of the sequence can be explicitly written out as follows.

$$\begin{aligned} f(T)dT &\overset{(i)}{\mapsto} Y\,f(XY)\,dX + X\,f(XY)\,dY, \\ g_X(X,Y)\,dX + g_Y(X,Y)\,dY &\overset{(ii)}{\mapsto} \Big(Xg_X(X,Y) - Yg_Y(X,Y)\Big)\frac{dX}{X} \\ &= \Big(-Xg_X(X,Y) + Yg_Y(X,Y)\Big)\frac{dY}{Y}, \\ h(X,Y)\frac{dX}{X} &\overset{(iii)}{\mapsto} h(0,0). \end{aligned}$$

These maps are very close to the maps of the Koszul resolution from Theorem 3.10 and the proof of the exactness is literally the same. ♢

Define

$$\mathrm{Td}(p^\vee) = \frac{\mathrm{Td}(T^\vee\overline{\mathcal{C}}_{g,n})}{\mathrm{Td}(p^*T^\vee\overline{\mathcal{M}}_{g,n})}.$$

Then $\mathrm{Td}(p^\vee)$ is obtained from $\mathrm{Td}(p)$ by changing the signs of the odd degree terms. Thus computing $\mathrm{Td}(p)$ is equivalent to computing $\mathrm{Td}(p^\vee)$.

Corollary 3.15. *We have*

$$\mathrm{Td}(p^\vee) = \frac{\mathrm{Td}(\mathcal{L})}{\mathrm{Td}(\mathcal{O}_\Delta)}.$$

Proof. Using Proposition 3.14, we obtain

$$\mathrm{Td}(p^\vee) = \frac{\mathrm{Td}(\mathcal{L})}{\mathrm{Td}(\mathcal{O}_\Delta \otimes \mathcal{L})},$$

because the Todd class is multiplicative. On the other hand, we have $\mathrm{Td}(\mathcal{O}_\Delta \otimes \mathcal{L}) = \mathrm{Td}(\mathcal{O}_\Delta)$. Indeed, as we know, the restriction of the line bundle $\mathcal{L}$ to Δ is rationally trivial. Hence all characteristic classes of $\mathcal{L}$ on Δ are the same as for a trivial line bundle. ♢

$\mathrm{Td}(\mathcal{L})$ is immediately evaluated to be

$$\mathrm{Td}(\mathcal{L}) = \frac{\omega}{1 - e^{-\omega}}.$$

Evaluating the Todd class of $\mathcal{O}_\Delta$ will be the last step of our computation. To do that we will use the Koszul resolution.

Let N be the normal vector bundle to Δ in $\overline{\mathcal{C}}_{g,n}$. Rather than evaluating $\mathrm{Td}(\mathcal{O}_\Delta)$ on $\overline{\mathcal{C}}_{g,n}$ we can evaluate $\mathrm{Td}(\mathcal{O}_\Delta)$ for the zero section Δ of the vector bundle N. (As we mentioned, the characteristic classes are topological invariants of vector bundles. Therefore the characteristic classes of a sheaf supported on Δ depend only on its resolution in the neighborhood of Δ in $\overline{\mathcal{C}}_{g,n}$, which is topologically the same as the neighborhood of Δ in the total space of N.)

The Koszul resolution gives us the following exact sequence of sheaves:

$$0 \to \bigwedge^2 N^\vee \to N^\vee \to \mathcal{O}_N \to \mathcal{O}_\Delta \to 0,$$

where $\mathcal{O}_N$ is the sheaf of holomorphic functions on the total space of the vector bundle N. Hence we have

$$\mathrm{Td}(\mathcal{O}_\Delta) = \frac{\mathrm{Td}(N^\vee)}{\mathrm{Td}(\bigwedge^2 N^\vee)} = \frac{\nu_1}{1-e^{-\nu_1}} \cdot \frac{\nu_2}{1-e^{-\nu_2}} \cdot \frac{1-e^{-(\nu_1+\nu_2)}}{\nu_1+\nu_2}$$

$$= \frac{\nu_1\nu_2}{\nu_1+\nu_2} \cdot \frac{1-e^{-(\nu_1+\nu_2)}}{(1-e^{-\nu_1})(1-e^{-\nu_2})} = \frac{\nu_1\nu_2}{\nu_1+\nu_2} \cdot \frac{1-e^{-\nu_1}+e^{-\nu_1}-e^{-(\nu_1+\nu_2)}}{(1-e^{-\nu_2})(1-e^{-\nu_2})}$$

$$= \frac{\nu_1\nu_2}{\nu_1+\nu_2} \cdot \left[\frac{1}{1-e^{-\nu_2}} + \frac{e^{-\nu_1}}{1-e^{-\nu_1}}\right] = \frac{\nu_1\nu_2}{\nu_1+\nu_2} \cdot \left[\frac{1}{1-e^{-\nu_2}} + \frac{1}{1-e^{-\nu_1}} - 1\right]$$

$$= 1 + \Delta \cdot \sum_{k\geq 1} \frac{B_{2k}}{(2k)!} \, \frac{\nu_1^{2k-1}+\nu_2^{2k-1}}{\nu_1+\nu_2},$$

where B_{2n} are the Bernoulli numbers (see Remark 3.5).

3.2.2 The right-hand side of GRR Now we can assemble the right-hand side of GRR from the results of our computations.

We have

$$\mathrm{Td}(p^\vee) = \frac{\omega}{1-e^{-\omega}} \left[1 + \Delta \cdot \sum_{k\geq 1} \frac{B_{2k}}{(2k)!} \, \frac{\nu_1^{2k-1}+\nu_2^{2k-1}}{\nu_1+\nu_2}\right].$$

We must change the signs of the odd degree terms to obtain $\mathrm{Td}(p)$. But the part in brackets is even, while the only odd term of $\omega/(1-e^{-\omega})$ is $\omega/2$. Thus

$$\mathrm{Td}(p) = \frac{\omega}{e^{\omega}-1} \left[1 + \Delta \cdot \sum_{k\geq 1} \frac{B_{2k}}{(2k)!} \, \frac{\nu_1^{2k-1}+\nu_2^{2k-1}}{\nu_1+\nu_2}\right].$$

Further,

$$\mathrm{ch}(\mathcal{L})\mathrm{Td}(p) = \frac{e^\omega \omega}{e^\omega - 1}\left[1 + \Delta \cdot \sum_{k\geq 1} \frac{B_{2k}}{(2k)!}\,\frac{\nu_1^{2k-1}+\nu_2^{2k-1}}{\nu_1+\nu_2}\right]$$

$$= \frac{\omega}{1-e^{-\omega}}\left[1 + \Delta \cdot \sum_{k\geq 1} \frac{B_{2k}}{(2k)!}\,\frac{\nu_1^{2k-1}+\nu_2^{2k-1}}{\nu_1+\nu_2}\right].$$

(By a coincidence, this is also equal to $\mathrm{Td}(p^\vee)$, but there is no reason to be confused by that.) Now, taking into account that $\omega\Delta = 0$, we obtain

$$\mathrm{ch}(\mathcal{L})\mathrm{Td}(p) = 1 + \frac{\omega}{2} + \sum_{k\geq 1} \frac{B_{2k}}{(2k)!}\left[\omega^{2k} + \Delta \cdot \frac{\nu_1^{2k-1}+\nu_2^{2k-1}}{\nu_1+\nu_2}\right]$$

$$= 1 + \frac{\omega}{2} + \sum_{k\geq 1} \frac{B_{2k}}{(2k)!}\left[K^{2k} - \sum_{i=1}^{n} D_i^{2k} + \Delta \cdot \frac{\nu_1^{2k-1}+\nu_2^{2k-1}}{\nu_1+\nu_2}\right].$$

Finally, according to the GRR formula, the push-forward p_* of this class is equal to the class $\mathrm{ch}(\Lambda) - 1$. For clarity, we separate this equality into homogeneous parts.

Theorem 3.16. *We have*

$$\begin{aligned} \mathrm{ch}_0(\Lambda) - 1 &= g - 1; \\ \mathrm{ch}_{2k}(\Lambda) &= 0; \\ \mathrm{ch}_{2k-1}(\Lambda) &= \frac{B_{2k}}{(2k)!}\left[\kappa_{2k-1} - \sum_{i=1}^{n} \psi_i^{2k-1} + \delta_{2k-1}^{\Lambda}\right], \end{aligned}$$

where δ_{2k-1}^{Λ} *is the push-forward* p_* *of the class* $\Delta \cdot (\nu_1^{2k-1} + \nu_2^{2k-1})/(\nu_1+\nu_2)$.

A curious reader can check that

$$\nu_1^p + \nu_2^p = \sum_{l=0}^{[p/2]} (-1)^l \frac{p}{p-l}\binom{p-l}{l}(\nu_1\nu_2)^l(\nu_1+\nu_2)^{p-2l},$$

and hence the class δ_{2k-1}^{Λ} is expressed via the standard classes $\delta_{k,l}$ as

$$\delta_{2k-1}^{\Lambda} = \sum_{l=0}^{k-1} (-1)^l \frac{2k-1}{2k-1-l}\binom{2k-1-l}{l}\delta_{2k-2-2l,l}.$$

Theorem 3.16 expresses the Chern characters of the Hodge bundle via the ψ-, κ-, and δ-classes. To conclude this section we show how to obtain similar expressions for the classes $\lambda_i = c_i(\Lambda)$.

Proposition 3.17. *We have*

$$1 + \lambda_1 + \lambda_2 + \cdots + \lambda_g = \exp\left(\mathrm{ch}_1 - \mathrm{ch}_2 + \frac{\mathrm{ch}_3}{2!} - \frac{\mathrm{ch}_4}{3!} + \dots\right)$$

$$= \exp\left(\mathrm{ch}_1 + \frac{\mathrm{ch}_3}{2!} + \frac{\mathrm{ch}_5}{4!} + \dots\right),$$

where $\mathrm{ch}_k = \mathrm{ch}_k(\Lambda)$.

We leave the proof as an exercise to the reader.

Corollary 3.18. *Every class λ_i is a polynomial in the ψ-, κ-, and δ-classes.*

Corollary 3.19. *We have $c(\Lambda)c(\Lambda^\vee) = 1$.*

Example 3.20. Applying Theorem 3.16 and Proposition 3.17, we obtain

$$\lambda_1 = \frac{1}{12}(\kappa_1 - \sum_{i=1}^{n} \psi_i + \delta_{0,0});$$

$$\lambda_2 = \frac{1}{2}\lambda_1^2;$$

$$\lambda_3 = \frac{1}{6}\lambda_1^3 - \frac{1}{360}(\kappa_3 - \sum_{i=1}^{n} \psi_i^3 + \delta_{2,0} - 3\delta_{0,1}).$$

3.3 Eliminating κ- and δ-classes

In the previous section we expressed the λ-classes in terms of ψ-, κ-, and δ-classes.

Now we will reduce any integral over $\overline{\mathcal{M}}_{g,n}$ involving ψ-, κ-, and δ-classes to a combination of integrals involving only ψ-classes. This does not mean that the κ- and δ-classes can be expressed in terms of ψ-classes. Indeed, the integrals that we will obtain involve moduli spaces $\overline{\mathcal{M}}_{g',n'}$ with g' and n' not necessarily equal to g and n. To do that, we will use computations with attaching and forgetful maps.

Before going into our computations, recall the following basic property that we will use.

Let $f : X \to Y$ be a morphism of smooth compact manifolds. Then f induces two maps in cohomology: the pull-back $f^* : H^*(Y, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ and the fiberwise integration $f_* : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$. If the cohomology classes are represented by generic Poincaré dual cycles, then f^* and f_* are the geometric preimage and image of cycles. Let $\alpha \in H^*(X, \mathbb{Q})$ and $\beta \in H^*(Y, \mathbb{Q})$. In general $f^*(f_*(\alpha)) \neq \alpha$ (and these classes have different degrees). Similarly, in general $f_*(f^*(\beta)) \neq \beta$ (and these classes also have different degrees). Instead, the following equality is true:

$$f_*(\alpha f^*(\beta)) = f_*(\alpha)\beta$$

and therefore

$$\int_X \alpha f^*(\beta) = \int_Y f_*(\alpha)\beta.$$

This property is called the *projection formula.*

3.3.1 Equivalence between $\overline{\mathcal{M}}_{g,n+1}$ and $\overline{\mathcal{C}}_{g,n}$ Recall that $\overline{\mathcal{M}}_{g,n+1}$ and $\overline{\mathcal{C}}_{g,n}$ are naturally isomorphic as families over $\overline{\mathcal{M}}_{g,n}$ (see Proposition 2.2). Let us make a small dictionary between the tautological classes defined on this family viewed as a moduli space and as a universal curve.

Consider the set of stable curves that contain a spherical component with exactly three special points: a node and the marked points number i and $n+1$.

The points encoding such curves form a divisor $\delta_{(i,n+1)}$ of $\overline{\mathcal{M}}_{g,n+1}$.

Further, consider the set of stable curves that contain a spherical component with exactly three special points: two nodes and the marked point number $n+1$.

The points encoding such curves form a codimension 2 subvariety $\delta_{(n+1)}$ of $\overline{\mathcal{M}}_{g,n+1}$.

$\diamondsuit$

Lemma 3.21. *Under the identification of $\overline{\mathcal{M}}_{g,n+1}$ with $\overline{\mathcal{C}}_{g,n}$, the divisor $\delta_{(i,n+1)}$ is identified with the divisor D_i, the subvariety $\delta_{(n+1)}$ is identified with Δ, while the class ψ_{n+1} is identified with K.*

Proof. The first two identifications follow immediately from the proof of Proposition 2.2: they correspond to cases (ii) and (iii).

The identification of ψ_{n+1} with K is more delicate. Recall that K is the first Chern class of $\mathcal{L}(D)$. In case (i), the fiber of $\mathcal{L}_{n+1}$ over $\overline{\mathcal{M}}_{g,n+1}$ is naturally identified with the fiber of $\mathcal{L}$ over $\overline{\mathcal{C}}_{g,n}$.

This identification fails in cases (ii) and (iii). However, case (iii) is not important, since it only occurs on a codimension 2 subvariety, hence does not influence the first Chern class. Case (ii), on the other hand, must be inspected more closely.

A holomorphic section of $\mathcal{L}$ at the neighborhood of $\hat{x}_i$ in the figure is a holomorphic 1-form on the fibers of $\overline{\mathcal{C}}_{g,n}$. It is naturally extended by 0 on the genus 0 component containing x_i and x_{n+1} (cf Proposition 2.17). Thus the sections of $\mathcal{L}_{n+1}$ acquire an additional zero whenever we are in case (ii). Thus $\mathcal{L}_{n+1}$ is identified not with $\mathcal{L}$ but with $\mathcal{L}(D)$. ♢

3.3.2 Eliminating κ-classes: the forgetful map The contents of this and the next section are a reformulation of certain results of [1].

Let $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map. Our aim is to compare the tautological classes on $\overline{\mathcal{M}}_{g,n+1}$ and the pull-backs of analogous tautological class from $\overline{\mathcal{M}}_{g,n}$. It turns out that the difference is easily expressed if we consider $\overline{\mathcal{M}}_{g,n+1}$ as the universal curve over $\overline{\mathcal{M}}_{g,n}$.

Theorem 3.22. *We have*

$$\begin{aligned}
\psi_{n+1} \quad &\overset{1}{=} K, \\
\psi_i^d - \ p^*\psi_i^d &\overset{2}{=} (-D_i)^{d-1} D_i, \\
\kappa_m - \ p^*\kappa_m &\overset{3}{=} K^m, \\
\delta_{k,0} - p^*\delta_{k,0} &\overset{4}{=} (-D)^k D + (\nu_1+\nu_2)^{k+1} - \nu_1^{k+1} - \nu_2^{k+1}, \\
\delta_{k,l} - \ p^*\delta_{k,l} &\overset{5}{=} (\nu_1+\nu_2)^{k+1}\Delta^l \qquad \text{for } l \neq 0.
\end{aligned}$$

Proof. We are going to prove Equalities (1), (2), and (3), leaving the last two equalities as an exercise[1].

1. According to Lemma 3.21, we have $\psi_{n+1} = K$.

2. According to Proposition 2.17, we have $\psi_i - p^*\psi_i = \delta_{(i,n+1)} \iff \psi_i - \delta_{(i,n+1)} = p^*\psi_i$. On the other hand, the line bundle $\mathcal{L}_i$ restricted to $\delta_{(i,n+1)}$ is trivial, hence $\psi_i \delta_{(i,n+1)} = 0$. Thus

$$(\psi_i - \delta_{(i,n+1)})^d = p^*\psi_i^d \iff \psi_i^d + (-\delta_{(i,n+1)})^d = p^*\psi_i^d$$

$$\iff \psi_i^d - p^*\psi_i^d = -(-\delta_{(i,n+1)})^d.$$

It remains to note that, according to Lemma 3.21, $\delta_{(i,n+1)}$ is identified with D_i.

3. If $\tilde{p} : \overline{\mathcal{C}}_{g,n+1} \to \overline{\mathcal{C}}_{g,n}$ is the forgetful map for universal curves, we have $\tilde{p}^* K = K - D_{n+1}$. On the other hand, $K D_{n+1} = 0$. Hence

$$(K - D_{n+1})^{m+1} = \tilde{p}^* K^{m+1} \iff K^{m+1} - D_{n+1}^{m+1} = \tilde{p}^* K^{m+1}.$$

The push-forward of this equality to the moduli spaces gives

$$\kappa_m - p^*\kappa_m = \psi_{n+1}^m = K^m.$$

♢

[1] The last two equalities are not more complicated than then first three, but require some juggling between $\Delta \subset \overline{\mathcal{C}}_{g,n}$, $\Delta \subset \overline{\mathcal{C}}_{g,n+1}$, and $\overline{\mathcal{M}}_{g,n+1}$ identified with $\overline{\mathcal{C}}_{g,n}$, which makes the text of the proof rather ugly.

Corollary 3.23. *Let Q be a polynomial in variables κ_m, $\delta_{k,l}$, $\psi_1, \dots, \psi_n$. Let $\widetilde{Q}$ be the polynomial obtained from Q by the substitution $\kappa_i \mapsto \kappa_i - \psi_{n+1}^i$. Then we have*

$$\int_{\overline{\mathcal{M}}_{g,n}} \kappa_m \, Q = \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_{n+1}^{m+1} \, \widetilde{Q}.$$

Proof. By definition, $\kappa_m = p_*(K^{m+1})$. By the projection formula, we obtain

$$\int_{\overline{\mathcal{M}}_{g,n}} p_*(K^{m+1}) \cdot Q = \int_{\overline{\mathcal{C}}_{g,n} = \overline{\mathcal{M}}_{g,n+1}} K^{m+1} \cdot p^* Q.$$

According to Theorem 3.22, the pull-back p^* modifies each term of the polynomial Q. However most of these modifications play no role, because they vanish when we multiply them by K^{m+1}.

Indeed, the difference between ψ_i^d and $p^*\psi_i^d$ is a multiple of D_i, and we know that $D_i K = 0$. Similarly, the difference between $\delta_{k,l}$ and $p^*\delta_{k,l}$ is either a multiple of Δ (if $l \neq 0$), or a sum of a multiple of Δ and a multiple of D (if $l = 0$). But we know that $K\Delta = KD = 0$.

The only remaining terms are κ_i, and for these the difference is important. According to Theorem 3.22, we have $p^*\kappa_i = \kappa_i - K^{i+1}$. Recalling that K (on $\overline{\mathcal{C}}_{g,n}$) is the same as ψ_{n+1} (on $\overline{\mathcal{M}}_{g,n+1}$), we obtain that we must replace every κ_i in Q by $\kappa_i - \psi_{n+1}^{i+1}$. Thus we obtain exactly the polynomial $\widetilde{Q}$ and the assertion of the corollary. ♢

Corollary 3.23 allows us to express an integral involving at least one κ-class as a combination of integrals with fewer κ-classes.

Example 3.24. We have

$$\int_{\overline{\mathcal{M}}_{0,5}} \kappa_1^2 = \int_{\overline{\mathcal{M}}_{0,6}} \psi_6^2(\kappa_1 - \psi_6) = \int_{\overline{\mathcal{M}}_{0,7}} \psi_7^2 \psi_6^2 - \int_{\overline{\mathcal{M}}_{0,6}} \psi_6^3 = 6 - 1 = 5.$$

3.3.3 Eliminating δ-classes: the attaching map Recall that $\Delta \subset \overline{\mathcal{C}}_{g,n}$ is the set of nodes of the singular fibers. Take a singular stable curve, choose a node, unglue the branches of the curve at the node and number the two preimages of the node (there are two ways of doing that). We obtain a new stable curve that has two new marked points. It can either be connected of genus $g - 1$, or composed of two connected components of genera g_1 and g_2,

$g_1 + g_2 = g$. Denote by $\overline{\mathcal{M}}_{\text{split}}$ the disjoint union

$$\overline{\mathcal{M}}_{\text{split}} = \bigsqcup_{\substack{I_1 \cup I_2 = \{1,\dots,n\} \\ g_1+g_2=g}} \overline{\mathcal{M}}_{g_1, I_1 \cup \{n+1\}} \times \overline{\mathcal{M}}_{g_2, I_2 \cup \{n+2\}} \bigsqcup \overline{\mathcal{M}}_{g-1,n+2}.$$

We have actually proved the following statement.

Lemma 3.25. $\overline{\mathcal{M}}_{\text{split}}$ *is a* 2*-sheeted covering of* Δ.

The same 2-sheeted covering was previously denoted by $\widetilde{\Delta}$.

Denote by $j : \overline{\mathcal{M}}_{\text{split}} \to \overline{\mathcal{M}}_{g,n}$ the composition $\overline{\mathcal{M}}_{\text{split}} \xrightarrow{c} \Delta \xrightarrow{p} \overline{\mathcal{M}}_{g,n}$. The image of j is the boundary $\delta_{0,0} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ of the moduli space.

A tautological class on $\overline{\mathcal{M}}_{\text{split}}$ is defined by taking the same tautological class on every component of $\overline{\mathcal{M}}_{\text{split}}$.

Our aim is now to study the difference between a tautological class on $\overline{\mathcal{M}}_{\text{split}}$ and the pull-back of the analogous class from $\overline{\mathcal{M}}_{g,n}$.

Theorem 3.26. *We have*

$$\begin{aligned} \kappa_m - j^*\kappa_m &\overset{1}{=} 0, \\ \psi_i^d - j^*\psi_i^d &\overset{2}{=} 0, \\ \delta_{k,l} - j^*\delta_{k,l} &\overset{3}{=} (\psi_1 + \psi_2)^{k+1}(\psi_1\psi_2)^l. \end{aligned}$$

Proof. Equalities (1) and (2) follow from the obvious fact that $\tilde{\jmath}^* K = K$ and $\tilde{\jmath}^* D_i = D_i$, where $\tilde{\jmath}$ is the map of universal curves associated with j.

For Equality (3) we only sketch the argument.

First consider the case $k = l = 0$. Rewrite the equality as

$$j^*\delta_{0,0} = \delta_{0,0} - (\psi_{n+1} + \psi_{n+2}).$$

The class $j^*\delta_{0,0}$ is almost the same as the self-intersection of the boundary $\delta_{0,0}$ of the moduli space. More precisely, different connected components of $\overline{\mathcal{M}}_{\text{split}}$ are degree 2 coverings of irreducible components of $\delta_{0,0}$, and we are interested in the pull-backs of the intersections of these components with $\delta_{0,0}$.

Some of the intersection points encode stable curves with two nodes. Such intersections are transversal and they give rise to the term $\delta_{0,0}$ in the right-hand side.

But when we try to intersect a component of $\delta_{0,0}$ with itself we obtain a nontransversal intersection. To determine the intersection in cohomology we need to know the first Chern class of the normal line bundle to $\delta_{0,0}$ in $\overline{\mathcal{M}}_{g,n}$. It turns out that this first Chern class equals $-j_*(\psi_{n+1} + \psi_{n+2})$. Thus we obtain the term $-(\psi_{n+1} + \psi_{n+2})$ in the right-hand side.

The case of general k, l is similar, since $\delta_{k,l}$ is the intersection of $\delta_{0,0}$ with $(\nu_1+\nu_2)^k(\nu_1\nu_2)^l$, which is transformed into $(\psi_1+\psi_2)^k(\psi_1\psi_2)^l$ on $\overline{\mathcal{M}}_{\rm split}$. ♢

Corollary 3.27. *Let Q be a polynomial in variables κ_m, $\delta_{k,l}$, $\psi_1, \ldots, \psi_n$. Let $\widetilde{Q}$ be the polynomial obtained from Q by the substitution*

$$\delta_{i,j} \mapsto \delta_{i,j} - (\psi_{n+1}+\psi_{n+1})^{i+1}(\psi_{n+1}\psi_{n+1})^j.$$

Then we have

$$\int\limits_{\overline{\mathcal{M}}_{g,n}} \delta_{k,l}\, Q = \frac{1}{2} \int\limits_{\overline{\mathcal{M}}_{\rm split}} (\psi_{n+1}+\psi_{n+1})^k(\psi_{n+1}\psi_{n+1})^l\, \widetilde{Q}.$$

Proof. By definition,

$$\delta_{k,l} = \frac{1}{2} j_* \left((\psi_{n+1}+\psi_{n+2})^k(\psi_{n+1}\psi_{n+2})^l\right),$$

where the factor 1/2 appears because $\overline{\mathcal{M}}_{\rm split}$ is a double covering of Δ. The projection formula implies that

$$\frac{1}{2}\int\limits_{\overline{\mathcal{M}}_{g,n}} j_*\left((\psi_{n+1}+\psi_{n+2})^k(\psi_{n+1}\psi_{n+2})^l\right)\cdot Q =$$

$$\frac{1}{2}\int\limits_{\overline{\mathcal{M}}_{\rm split}} (\psi_{n+1}+\psi_{n+1})^k(\psi_{n+1}\psi_{n+1})^l \cdot j^* Q.$$

According to Theorem 3.26, every term of Q remains unchanged after a pull-back by j, except for the terms $\delta_{k,l}$, that are transformed according to the rule

$$j^*\delta_{k,l} = \delta_{k,l} - (\psi_{n+1}+\psi_{n+2})^{k+1}(\psi_{n+1}\psi_{n+2})^l.$$

In other words, $j^*Q = \widetilde{Q}$, so we obtain the claim of the corollary. ♢

Corollary 3.27 allows us to express an integral involving at least one δ-class as a combination of integrals with fewer δ-classes.

Applying Corollaries 3.23 and 3.27 several times allow us to reduce any integral involving ψ-, κ-, and δ-classes to a combination of integrals involving only ψ-classes. These integrals are the subject of the next section.

4 Around Witten's conjecture

Now our aim is to compute the integrals

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

4.1 The string and dilaton equations

Proposition 4.1. *For $2-2g-n<0$ we have*

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \ldots \psi_n^{d_n} = \sum_{i=1}^{n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_i^{d_i-1} \ldots \psi_n^{d_n};$$

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \ldots \psi_n^{d_n}\, \psi_{n+1} = (2g-2+n) \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}.$$

These equations are called the *string* and the *dilaton* equations.

Proof. We will need to consider the line bundles $\mathcal{L}_i$ both on $\overline{\mathcal{M}}_{g,n}$ and on $\overline{\mathcal{M}}_{g,n+1}$. We momentarily denote the former by $\mathcal{L}'_i$, $1 \le i \le n$, the latter retaining the notation $\mathcal{L}_i$, $1 \le i \le n+1$. Further, denote by ψ'_i the first Chern class of the line bundle $\mathcal{L}'_i$ on $\overline{\mathcal{M}}_{g,n}$ and, by abuse of notation, its pull-back to $\overline{\mathcal{M}}_{g,n+1}$ by the map forgetting the $(n+1)$st marked point. Let ψ_i be the first Chern class of $\mathcal{L}_i$ on $\overline{\mathcal{M}}_{g,n+1}$.

According to Proposition 2.17, we have

$$\psi_i = \psi'_i + \delta_{(i,n+1)}. \tag{1}$$

Moreover, we have

$$\psi_i \cdot \delta_{(i,n+1)} = \psi_{n+1} \cdot \delta_{(i,n+1)} = 0 \tag{2}$$

(because the line bundles $\mathcal{L}_i$ and $\mathcal{L}_{n+1}$ are trivial over $\delta_{(i,n+1)}$) and

$$\delta_{(i,n+1)} \cdot \delta_{(j,n+1)} = 0 \quad \text{for } i \neq j \tag{3}$$

(because the divisors have an empty geometric intersection).

Let us first prove the string relation. We have

$$\psi_i^d - (\psi'_i)^d \overset{(1)}{=} \delta_{(i,n+1)} \left(\psi_i^{d-1} + \cdots + (\psi'_i)^{d-1}\right) \overset{(2)}{=} \delta_{(i,n+1)}\, (\psi'_i)^{d-1},$$

where we set by convention $(\psi'_i)^{-1} = 0$. Thus

$$\psi_i^d \;=\; (\psi'_i)^d + \delta_{(i,n+1)}\, (\psi'_i)^{d-1}.$$

It follows that

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \ \dots \ \psi_n^{d_n} =$$

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \left[(\psi_1')^{d_1} + \delta_{(1,n+1)} \ (\psi_1')^{d_1-1}\right] \ \dots \left[(\psi_n')^{d_n} + \delta_{(n,n+1)} \ (\psi_n')^{d_n-1}\right] \stackrel{(3)}{=}$$

$$\int_{\overline{\mathcal{M}}_{g,n+1}} (\psi_1')^{d_1} \ \dots \ (\psi_n')^{d_n} \ + \sum_{i=1}^{n} \int_{\overline{\mathcal{M}}_{g,n+1}} (\psi_1')^{d_1} \ \dots \ \delta_{(i,n+1)} \ (\psi_i')^{d_i-1} \ \dots \ (\psi_n')^{d_n} \ .$$

The first integral is equal to 0, because the integrand is a pull-back from $\overline{\mathcal{M}}_{g,n}$. As for the integrals composing the sum, we integrate the class $\delta_{(i,n+1)}$ over the fibers of the projection $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. This is equivalent to restricting the integral to the divisor $\delta_{(i,n+1)}$, which is naturally isomorphic to $\overline{\mathcal{M}}_{g,n}$. Finally, we obtain

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \ \dots \ \psi_n^{d_n} = \sum_{i=1}^{n} \int_{\overline{\mathcal{M}}_{g,n}} (\psi_1')^{d_1} \ \dots \ (\psi_i')^{d_i-1} \ \dots \ (\psi_n')^{d_n} \ .$$

This proves the string relation.

Now let us prove the dilaton relation. We have

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \ \dots \ \psi_n^{d_n} \ \psi_{n+1} \stackrel{(1)}{=}$$

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \Big(\psi_1' + \delta_{(1,n+1)}\Big)^{d_1} \ \dots \ \Big(\psi_n' + \Delta_{(n,n+1)}\Big)^{d_n} \ \psi_{n+1} \stackrel{(2)}{=}$$

$$\int_{\overline{\mathcal{M}}_{g,n+1}} (\psi_1')^{d_1} \ \dots \ (\psi_n')^{d_n} \ \psi_{n+1} =$$

$$(2g-2+n) \ \int_{\overline{\mathcal{M}}_{g,n}} (\psi_1')^{d_1} \ \dots \ (\psi_n')^{d_n}.$$

The last equality is obtained by integrating the factor ψ_{n+1} over the fibers of the projection $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$.

This proves the dilaton relation. ◊

Proposition 4.2. *The string and the dilaton equations, together with the initial conditions*

$$\int_{\overline{\mathcal{M}}_{0,3}} 1 = 1, \qquad \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24},$$

allow one to compute all integrals

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$$

for $g = 0, 1$.

We leave the proof as an exercise to the reader.

Example 4.3. Recall the first equality of Example 3.20:

$$\lambda_1 = \frac{1}{12}(\kappa_1 - \sum \psi_i + \delta_{0,0}).$$

This equality is true for all g and n, but let us consider the case $g = n = 1$. We have (*Cf* Example 2.26)

$$\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = \frac{1}{24}; \qquad \int_{\overline{\mathcal{M}}_{1,1}} \kappa_1 = \int_{\overline{\mathcal{M}}_{1,2}} \psi_2^2 = \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}; \qquad \int_{\overline{\mathcal{M}}_{1,1}} \delta_{0,0} = \frac{1}{2}.$$

Thus we have checked that the expression for λ_1 over $\overline{\mathcal{M}}_{1,1}$ is indeed correct: $1/24 = 1/12 \cdot (1/24 - 1/24 + 1/2)$.

4.2 KdV and Virasoro

It looks like we are very close to our goal of computing all intersection numbers of ψ-classes and thus of all tautological classes. However the last step is actually quite hard; therefore we only formulate the theorems that make it possible to compute the remaining integrals, but do not give the proofs.

Introduce the following generating series:

$$F(t_0, t_1, \dots) = \sum_{\substack{g \geq 0, n \geq 1 \\ 2-2g-n<0}} \sum_{d_1, \dots, d_n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} \frac{t_{d_1} \dots t_{d_n}}{n!}.$$

The coefficients of this series encode all possible integrals involving ψ-classes.

Theorem 4.4. *The series F satisfies the following partial differential equation:*

$$\frac{\partial^2 F}{\partial t_0 \partial t_1} = \frac{1}{2}\left(\frac{\partial^2 F}{\partial t_0^2}\right)^2 + \frac{1}{12}\frac{\partial^4 F}{\partial t_0^4}.$$

This partial differential equation is one of the forms of the *Korteweg - de Vries* or *KdV* equation[2]. The result of Theorem 4.4 was conjectured by E. Witten in [22] and proved by M. Kontsevich [14]. Today several other proofs exist, [13] being, probably the simplest one.

Let $G(p_1, p_3, p_5, \dots)$ be the power series obtained from F by the substitution $t_d = (2d-1)!!p_{2d+1}$. For $k \geq 1$, denote by a_k the operator of multiplication by p_k and by a_{-k} the operator $k\frac{\partial}{\partial p_k}$.

Theorem 4.5. *For $m \geq -1$ we have*

$$\left(a_{-(2m+3)} - \sum_{\substack{i+j=-2m \\ i>j}} a_i a_j - \frac{1}{8}\delta_{0,m} \right) e^G = 0.$$

These equations are called the *Virasoro constraints*.

Example 4.6. The Virasoro constraints corresponding to $m = -1$ and $m = 0$ are equivalent to the string and the dilaton equations.

It is straightforward to see that the Virasoro constraints together with the initial condition $F = t_0^3/6 + \dots$ determine the series F completely. Indeed, suppose that in the integrand $\prod \psi_i^{d_i}$ the value $m+1$ is the biggest value of the d_i's and appears exactly k times. Then the application of the mth Virasoro constraint leads to integrals where the value $d_i = m + 1$ appears less than k times.

A slightly more ingenious argument shows that the KdV and the string equations, together with the initial condition $F = t_0^3/6 + \dots$, determine F uniquely. (We leave this as an exercise to the reader.) Thus Theorems 4.4 and 4.5 describe the same series F in two different ways.

A modern proof of both theorems can be found in [12]. We do not give it here; instead we will say a few words about the origins of Witten's conjecture in 2-dimensional quantum gravity.

In general relativity, the gravitational field is a quadratic form of signature $(1, 3)$ on the 4-dimensional space-time. Since the problem of constructing a quantum theory of gravity is extremely difficult, physicists started with a simpler model of 2-dimensional gravity. In this model, the space-time is a surface, while the gravitational field is a Riemannian metric on this surface.

The aim of a quantum theory of gravity is then to define and compute certain integrals over the space of all possible Riemannian metrics on all possible surfaces. The space of metrics is infinite-dimensional and does not carry a natural measure, therefore the definition of such integrals is problematic.

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[2] The usual form of the KdV equation is $\partial U/\partial t_1 = U\partial U/\partial t_0 + \frac{1}{12}\partial^3 U/\partial t_0^3$ obtained by differentiating our equation once with respect to t_0 and letting $U = \partial^2 F/\partial t_0^2$.

Physicists found two ways to give a meaning to integrals over the space of metrics.

The first way is to replace Riemannian metrics by a discrete approximation, namely, surfaces obtained by glueing together very small equilateral triangles. In this method, integrals over the space of metrics are replaced by sums over triangulations. This leads to combinatorial problems of enumerating triangulations. These problems, although difficult, can be solved, and the KdV equation appeared in the works devoted to the enumeration of triangulations.

The second way to define infinite-dimensional integrals is to first perform the integration over the space of conformally equivalent metrics.

Two Riemannian metrics are *conformally equivalent* if one is obtained from the other by multiplication by a positive function. An equivalence class of conformally equivalent metrics is exactly the same thing as a Riemann surface. Indeed, one of the ways to introduce a complex structure on a surface is to define a linear operator J acting on its tangent bundle such that $J^2 = -1$. The action of the operator is then interpreted as multiplication of the tangent vectors by i. In the case where the surface is endowed with a Riemannian metric, the operator J is simply the rotation by $90°$, and this does not change if we multiply the metric by a positive function.

The space of positive functions is still infinite-dimensional, but it turns out that one can perform the integration over this space by a formal trick. (It would be more precise to say that the integral is *defined* in a meaningful way and the trick serves as a motivation.) After that, there remains an integral over the moduli space of Riemann surfaces. It so happens that this integral is

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n}.$$

What Witten's conjecture actually says is that we obtain the same answer using the two methods of defining infinite-dimensional integrals.

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