Towards A Global Mirror Symmetry

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Plan of talk:

(I) what is "global" ?

(II) A conjectural physical package
BCOV

(III) An approach to hypersurface
LG-model

(IV) Un-expected bonus !
orbifold GW-Theory

1) WHAT IS "GLOBAL"?

"Local" Mirror symmetry $\neq$ Mirror symmetry of local Calabi-Yau

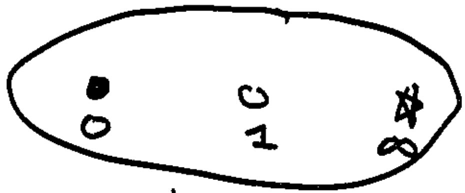
X - CY 3-fold $\xleftrightarrow{\text{MIRROR}}$ $X^\vee$ - another CY 3-fold

A-model	MIRROR	B-model
$\cup$		$\cup$
kahler str	$\longleftrightarrow$	complex str
GW-Theory $(g=0)$		periods

Famous Example: $X = \left\{ \sum_{i=1}^{5} x_i^5 = 0 \right\} \longleftrightarrow X^\vee = \left\{ \sum_{i=1}^{5} x_i^5 - 5\psi^{\frac{1}{5}} \prod_{i=1}^{5} x_i = 0 \right\} / \mathbb{Z}_5^3$

kahler str

$t \in$ kahler cone $\xleftrightarrow[\text{match}]{\text{NO}}$ complex str $\psi \in$



$\pi_1 = 1$

$\mathbb{P}^1 - \{0, 1, \infty\}$

$\pi_1 \neq 1$

A temporary solution at the time (20 years ago) $\Downarrow$ "local mirror" symmetry :

Kahler str $t \longleftrightarrow \psi \in \{$a neigh of $\psi = \infty\}$ "Local"

$\psi = \infty$ $\updownarrow$ large complex str limit

<u>Now</u>: Restore "Global" structure of Mirror symmetry

<u>Benefit</u>:
- (1) compute higher genus Gromov-Witten Theory
- (2) study modularity of "
- (3) Prove Landau-Ginzburg / Calabi-Yau correspondence.
- (4) More, ...

"GLOBAL" $=$ allow ψ to move around in the entire moduli space of complex str

II) A conjectural physical package

Background: B-model $(g=0)$

$\wr\wr$

periods

A toy model

elliptic curve

$$E(\psi) = \left\{ \sum_{i=1}^{3} x_i^3 - 3\psi^{\frac{1}{3}} \prod_{i=1}^{3} x_i = 0 \right\} \Big/ \mathbb{Z}_3^2$$

$\psi \in$

$\mathbb{P}^1 - \{0, 1, \infty\}$

CY-form:

$$H^{1,0}(E(\psi)) = \mathbb{C}$$

ω_ψ – holomorphic $(1,0)$ form

vary holomorphically as we vary ψ

another choice. $\omega(\psi) \longrightarrow f(\psi)\,\omega(\psi)$

holomorphic funct

$H_1(E|\psi|, \mathbb{Z})$ has a symplectic basis

A, B with $A^2 = B^2 = 0$ $A \cdot B = 1$
$B \cdot A = -1$

change basis $\begin{pmatrix} A \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} A' \\ B' \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\in\, SL_2\mathbb{Z}} \begin{pmatrix} A \\ B \end{pmatrix}$

<u>PERIOD</u>: $\underline{q} = \int_A \omega(\psi)$ $\quad p = \int_B \omega(\psi)$

vary ψ $\longrightarrow$ $\underline{q}(\psi), p(\psi)$ - multi-valued funct of $\mathbb{P}^1 - \{0, 1, \infty\}$

$\tau = \dfrac{p(\psi)}{\underline{q}(\psi)} \in \mathbb{H}_+$ - upper half-plane

$\downarrow$ $\qquad\qquad \downarrow$ - universal cover

$\psi \qquad \mathbb{P}^1 - \{0, 1, \infty\}$

$\qquad \qquad \| $

$\mathbb{H}_+ / \text{monodromy group} = \Gamma(3) \subset SL_2\mathbb{Z}$

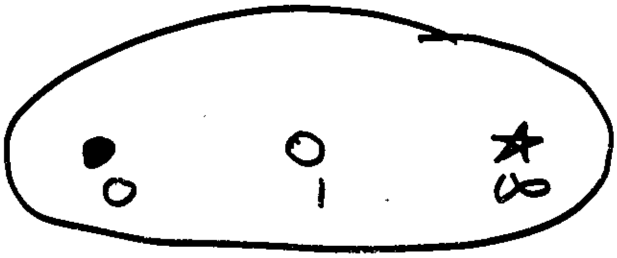
GW-Theory of kahler parameter t $\Longleftrightarrow$ Periods of ψ near $\left\{ \infty \right.$ $\left. \text{with a special choice of basis } A, B \right\}$

$\tau = i\infty$ "large complex str. limit"

Going Back to quintic 3-fold (B-model)

$\mathbb{H}_+ \ni \tau$ - modular coord

$\psi \in$



$= \mathbb{H}_+ / \Gamma \subset Sp(2, \mathbb{Z})$

$\mathbb{P}^1 - \{0, 1, \infty\}$

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$H_3(X^\vee, \mathbb{Z}) = 4$

GW - Theory of quintic $g=0$ $\quad \underset{\text{Mirror}}{\overset{\text{"local"}}{\longleftrightarrow}} \quad$ Periods near $\tau = i\infty$

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large complex limit

Remark :

(1) Above "local" mirror symmetry for $g=0$ has been verified by Givental, Liu-Liang-Yau in the middle of 90's.

Our interest

(2) Higher genus ($g>0$) GW - Theory

(3) "Global" properties of GW - function.

A conjectural physical package (BCOV) 7

I) GLOBAL PROPERTY:

1. exist a global $\mathcal{F}_g^B(\tau,\bar\tau)$ - genus g generating funct of "B-model GW theory"
   (i) defined for all τ
   (ii) non-holomorphic

2. Modular Invariance

$$\mathcal{F}_g^B(h\tau, h\bar\tau) = \mathcal{F}_g^B(\tau,\bar\tau)\, j(h,\tau)^k$$

$h \in \Gamma$ - monodromy group

3. $\partial_{\bar\tau} \mathcal{F}_g^B = \Gamma_g(\mathcal{F}_{g'<g}^B)$ - holomorphic anomaly equation

$\Uparrow$ BCOV, Klemm, ...

$$\mathcal{F}_g^B(\tau,\bar\tau) = \sum_j \mathcal{F}_{g,j}^B(\tau)\,(\operatorname{im}\tau)^{-j}$$

2 + 3 $\Rightarrow$ $\mathcal{F}_g^B(\tau)$ - quasi-modular form

(II) SPECIAL LIMIT (Mirror Conjectures)

B-model $\xleftrightarrow{\text{CY-to-CY}}$ A-model

① near $\tau = i\infty$ $\xleftrightarrow{\text{CY-to-CY}}$ Gromov-Witten Theory of X

original "local" mirror symmetry

Small / Large duality = Landau Ginzburg / CY correspondence

② near $\psi = 0$ (Gepner limit) $\xleftrightarrow{\text{CY-to-LG}}$ "conjectural Landau-Ginzburg model"

③ near $\psi = 1$ (conifold limit) $\xleftrightarrow{\text{CY-to-MA}}$ "conjectural matrix model"

④ Beyond quintic, other limits.

Klemm's group: Assume the existence of above package + general properties of special limits

known for mathematician only for $g = 0, 1$

$\Downarrow$

A STRIKING computation of GW-Theory of $g \leq 51$!

Goal of Remaining talk

- Describe an approach for a mathematical construction of above package for hypersurface.

  Most of steps are conjectural at this moment
- Present some theorems in dimension one.

First advance: Gepner limit

Conjectural LG-model = Theory of Fan-Jarvis-Ruan-Witten (2007)

LG-model:

(i) $W: \mathbb{C}^N \to \mathbb{C}$ "non-degenerate" quasi-homogeneous poly

(ii) $G \subset \mathrm{Aut}(W)$ - finite abelian symmetry

Theory of Fan-Jarvis-Ruan-Witten:

- A complete A-model theory of LG-model based on solving Witten eqn

$$\bar\partial s_i + \overline{\partial_i W} = 0$$

- A GW-type curve counting theory
    - based on $\overline{\mathcal{M}}_{g,n}$
    - 2D TQFT
    - satisfies axioms of GW-theory

Much easier to calculate (ADE, elliptic singularity, g=0 quintic, expanding quickly what happen for (I): build a rigorous theory of $F_g^B(\tau, \bar{\tau})$ with expected properties

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- A hard problem
- Progress on $X = T^{2n}$ (Costello – Li)
- Our approach for hypersurfaces such as quintic 3-fold (Milanov – Krawitz – Shen)

<u>Key observation</u>:

CY-deformation: $x_1^3 + x_2^3 + x_3^3 - 3\psi^{\frac{1}{3}} x_1 x_2 x_3$

$$\cap \text{ subset}$$

$$x_1^3 + x_2^3 + x_3^3 - 3\psi^{\frac{1}{3}} x_1 x_2 x_3 + t_0 + t_1 x_1 + t_2 x_2 + t_3 x_3 + t_4 x_1 x_2 + t_5 x_1 x_3 + t_6 x_2 x_3$$

$$\parallel$$

miniversal deformation of singularity

$$W = x_1^3 + x_2^3 + x_3^3$$

A "baby" B-model of singularity / LG-model

$$W = x_1^3 + x_2^3 + x_3^3$$

Milnor ring $\quad Q_W = \dfrac{\mathbb{C}[x_1, x_2, x_3]}{\partial_{x_1} W,\ \partial_{x_2} W,\ \partial_{x_3} W}$ $\qquad$ 8-dim

$$= \{ 1,\ x_1,\ x_2,\ x_3,\ x_1x_2,\ x_1x_3,\ x_2x_3,\ x_1x_2x_3 \}$$

Miniversal deformation $\quad W_{(t_i, \sigma)} = x_1^3 + x_2^3 + x_3^3 + \sigma x_1 x_2 x_3$

$$+ t_0 + t_1 x_1 + t_2 x_2 + t_3 x_3 + t_4 x_1 x_2$$

$$+ t_5 x_1 x_3 + t_6 x_2 x_3$$

B-model parameter space $\quad = \{ (t_i, \sigma),\ |t_i| < \varepsilon,\ \sigma^3 \neq 27 \}$

For each $\{t_i, \sigma\}$.

$$Q_{W(t_i, \sigma)} = \mathbb{C}[x_1, x_2, x_3] \Big/ \partial_{x_1} W_{(t_i, \sigma)},\ \partial_{x_2} W_{(t_i, \sigma)},\ \partial_{x_3} W_{(t_i, \sigma)}$$

a family of Frobenius algebra

pairing: $\langle \phi_1, \phi_2 \rangle = \mathrm{Res} \dfrac{\phi_1 \phi_2\, dx_1 dx_2 dx_3}{dW_{(t_i, \sigma)}}$

Main properties

1. For generic $t_i \neq 0$, $W_{(t_i,\sigma)}$ – holomorphic Morse funct

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Frobenius algebra of such (t_i,σ) is Semi-Simple

2. pairing is NOT flat

Saito – Givental Theory:

(I) Saito – Theory:

Primitive form

replace $dx_1 dx_2 dx_3 \longrightarrow \frac{1}{\rho(\sigma)} dx_1 dx_2 dx_3$

$\Downarrow$

flat pairing $\Rightarrow$ Frobenius manifold str

$= \mathcal{F}_0^B(t_i,\sigma,\rho)$

$(t_i \neq 0)$ – semi simple

II) Givental Theory:

on semi-simple Frobenius ~~mfd~~, exist

$\mathcal{F}_g^B(t_i,\sigma,\rho) \qquad (t_i \neq 0)$

what we know about primitive form $\frac{1}{\rho} d\underline{x}$?

- ρ always exist locally $\Rightarrow$ local Saito-Givental Theory
- explicit formula for ADE, elliptic singularities
- difficult to get explicit formula in general
- along CY-direction (maginal deformation), related to periods.

$\Downarrow$ leads to

<u>Global Saito-Givental Theory</u>

(under developed by Milanov, —)

Global Saito-Givental Theory in dim = 1
(Milanov, –)

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starting point: $\rho(\sigma)$ – period, i.e. $\rho(\sigma) = \int_A \omega(\sigma)$

Recall $\tau = \dfrac{\int_B \omega(\sigma)}{\int_A \omega(\sigma)} \in \mathbb{H}_+$

A, B – symplectic basis

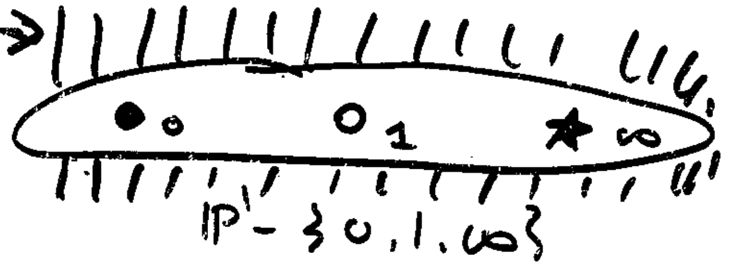
flat coord along σ-direction

B-model parameter space $= \{(t_i, \tau),\ |t_i| < \varepsilon,\ \tau \in \mathbb{H}_+\}$.

$\downarrow$ / monodromy group

$\{(t_i, \sigma),\ |t_i| < \varepsilon,\ \sigma^3 \neq 27\}$

$t_i \longrightarrow$



— CY-deformat.

$(t_i, \tau) \longrightarrow$ Frobenius mfd str $\longrightarrow \mathcal{F}_0^{B}(t_i, \tau)$

$(t_i \neq 0, \tau) \longrightarrow \mathcal{F}_g^{B}(t_i, \tau) \longrightarrow$

semi-simple

holomorphic with respect to t_i, τ

Non-Modular Invariante (Milanov, –)

Let $\mathcal{D}^{B}_{SG}(t_i;\tau) = \exp\left(\sum_{g\geq 0} \hbar^{2g-2} \mathcal{F}^{B}_{g}\right)$

$h \in \Gamma$

$$\mathcal{D}^{B}_{SG}(t_i; h\tau) = \widehat{X}_h \, \mathcal{D}^{B}_{SG}$$

where $\widehat{X}_h$ – differential operator defined out of $h \in \Gamma(3)$

Corollary: $\mathcal{F}^{B}_{g}(t_i,\tau)$ is not modular

A magic trick: Anti-holomorphic completion

We found an explicit way to complete

$$\mathcal{F}^{B}_{g}(t_i;\tau) \longrightarrow \mathcal{F}^{B}_{g}(t_i,\tau,\bar{\tau})$$

$$\mathcal{F}^{B}_{g}(t_i,\tau,\bar{\tau}) = \sum_{j=1}^{k} \mathcal{F}^{B}_{g,j}(\tau)(\operatorname{im}\tau)^{-j}$$

quasi-modular form

$\mathcal{F}^{B}_{g,j}$ defined via Feymann diagram expansion

motivated by Aganagic-Bouchard-Klemm

Easy Fact: $\mathcal{F}^{B}_{g}(t_i,\tau,\bar{\tau})$ satisfies holomorphic anomaly equation

Modular Invariance (Milanov, –) : $\mathcal{F}_g^B(t_i, \tau, \bar{\tau})$ is modular invariance

<u>Assume</u>: $\mathcal{F}_g^B(t_i \neq 0, \tau, \bar{\tau})$ extends to $t_i = 0$

$$\mathcal{F}_g^B(\tau, \bar{\tau}) = \mathcal{F}_g^B(0, \tau, \bar{\tau})$$

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somehow a difficult problem!

<u>Corollary (a)</u>:

$$\mathcal{F}_g^B(t_i, \tau\, \bar{\tau}) = \sum_I a_I(\tau, \bar{\tau})\, t_I$$

then $a_I(\tau, \bar{\tau})$ are classical almost holo modular form

or $\mathcal{F}_g^B(t_i, \tau) = \sum_I a_I(\tau)\, t_I$

classical quasi-modular form

Corollary (b)

$\mathcal{F}_g^B(\tau, \bar{\tau})$ satisfies holomorphic anomaly equ

$\parallel$

desired B-model theory

V) An un-expected Bonus !

$$\| $$

$\mathcal{F}_g^B(t_i, \tau)$ has a mirror of its own !

A-model Mirror : $\{x_1^3 + x_2^3 + x_3^3 = 0\} / \mathbb{Z}_3^2$

$\| $

$\mathbb{P}^1_{3,3,3}$ = 

orbifold $\mathbb{P}^1$

$\updownarrow$

LG-dual ; $(W = x_1^3 + x_2^3 + x_3^3,\ \mathbb{Z}_3^3)$

Theorem : (Krawitz - Shen)

① Near $\tau = 0$, $\mathcal{F}_g^B(t_i, \tau) = \mathcal{F}_g^{FJRW}(t_i', \tau')$

$\Downarrow$ extends to $t_i = 0$ $\Leftarrow$ $\Downarrow$ extends to $t_i' = 0$

② Near $\tau = i\infty$, $\mathcal{F}_g^B(t_i, \tau) = \mathcal{F}_g^{GW}(t_i', q = e^{\frac{2\pi i \tau}{3}})$

③ Same holds for $X_9 = x_1^2 + x_2^4 + x_3^4 \Leftrightarrow \mathbb{P}^1_{2,4,4}$

$J_{10} = x_1^2 + x_2^3 + x_3^6 \Leftrightarrow \mathbb{P}^1_{2,3,6}$

Two Bonuses :

(I) GW-Theories of $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{2,4,4}$, $\mathbb{P}^1_{2,3,6}$ are quasi-modular

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wanted very much by mathematician

(II) LG/CY - correspondence holds for all $\boxed{\text{genera}}$ for these examples.

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First example of all genera

A less important Result :

Restrict to $t_i = 0$ $\Longrightarrow$ recover elliptic curve

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Known already by Okounkov Pandharipande

Global Mirror Symmetry in dim one

(Milanov, –, Shen)

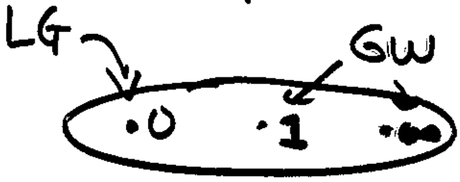
a <u>complete classification</u>:

2 <u>interesting pheomenon</u>:

(I) Absence of conifold limit

Ex: $W = x_1^3 + x_2^3 + x_3^3 - 3\gamma^{\frac{1}{3}} x_1 x_2 x_3$

$\gamma = 1$ is a Large complex str limit / GW - point



II) $\gamma = \infty$ is not always large complex structure limit

Ex: $W = x_1^2 + x_2^3 + x_3^6 + \gamma x_2 x_3^4$

$\gamma = \infty$ – Gepner limit