# Relations in the tautological ring

*R. Pandharipande and A. Pixton*

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The notes below cover our series of three lectures at Humboldt University in Berlin for the October conference *Intersection theory on moduli space* (organized by G. Farkas). The topic concerns relations among the $\kappa$ classes in the tautological ring of the moduli space of curves $\mathcal{M}_g$. After a discussion of classical constructions ending in Theorem 1, we derive an explicit set of relations from the moduli space of stable quotients. In a series of steps, the stable quotient relations are transformed to simpler and simpler forms. The first step, Theorem 3, comes almost immediately from the virtual geometry of the moduli space of stable quotients. After a certain amount analysis, the simpler form of Proposition 3 is found. Our final result, Theorem 5, establishes a previously conjectural set of tautological relations proposed a decade ago by Faber-Zagier. A detailed presentation of the proof will appear in [7].

## A. Chern vanishing relations

Faber's original relations in *Conjectural description of the tautological ring* [1] are obtained from a very simple geometric construction. Let

$$\pi : \mathcal{C} \to \mathcal{M}_g$$

be the universal curve over the moduli space, and let

$$\pi^d : \mathcal{C}^d \to \mathcal{M}_g$$

be the map associated to the $d^{th}$ fiber product of the universal curve. For every point $[C, p_1, \ldots, p_d] \in \mathcal{C}^d$, we have the restriction map

$$H^0(C, \omega_C) \to H^0(C, \omega_C|_{p_1+\ldots+p_d}) \ . \tag{1}$$

Since the canonical bundle $\omega_C$ has degree $2g-2$, the map (1) is injective if $d > 2g-2$. Over the moduli space $\mathcal{C}^d$, we obtain the exact sequence

$$0 \to \mathbb{E} \to \Omega_d \to Q \to 0$$

where $\mathbb{E}$ is the rank $g$ Hodge bundle, $\Omega_d$ is the rank $d > 2g-2$ bundle with fiber $H^0(C, \omega_C|_{p_1+\ldots+p_d})$, and $Q$ is the quotient bundle of rank $d-g$. Hence,

$$c_k(Q) = 0 \in A^k(\mathcal{C}^d) \quad \text{for} \quad k > d-g\ .$$

After cutting such vanishing $c_k(Q)$ down with cotangent line and diagonal classes in $\mathcal{C}^d$ and pushing-forward via $\pi^d_*$ to $\mathcal{M}_g$, we arrive at Faber's relations in $R^*(\mathcal{M}_g)$.

From our point of view, at the center of Faber's relations in *Conjectural description of the tautological ring* [1] is the function

$$\Theta(t,x) = \sum_{d=0}^{\infty} \prod_{i=1}^{d} (1+it)\ \frac{(-1)^d}{d!}\frac{x^d}{t^d}\ .$$

The differential equation

$$t(x+1)\frac{d}{dx}\Theta + (t+1)\Theta = 0$$

is easily found. Hence, we obtain the following result.

**Lemma 1.** $\Theta = (1+x)^{-\frac{t+1}{t}}$ .

We introduce a variable set $\mathbf{z}$ indexed by pairs of integers

$$\mathbf{z} = \{\ z_{i,j} \mid i \geq 1, \quad j \geq i-1\ \}\ .$$

For monomials

$$\mathbf{z}^{\sigma} = \prod_{i,j} z_{i,j}^{\sigma_{i,j}}\ ,$$

we define

$$\ell(\sigma) = \sum_{i,j} i\sigma_{i,j}, \quad |\sigma| = \sum_{i,j} j\sigma_{i,j}\ .$$

Of course $|\mathrm{Aut}(\sigma)| = \prod_{i,j} \sigma_{i,j}!$ .

The variables $\mathbf{z}$ are used to define a differential operator

$$\mathcal{D} = \sum_{i,j} z_{i,j}\ t^j \left(x\frac{d}{dx}\right)^i\ .$$

After applying $\exp(\mathcal{D})$ to $\Theta$, we obtain

$$\begin{aligned}\Theta^{\mathcal{D}} &= \exp(\mathcal{D})\ \Theta \\ &= \sum_{\sigma}\sum_{d=0}^{\infty}\prod_{i=1}^{d}(1+it)\ \frac{(-1)^d}{d!}\frac{x^d}{t^d}\ \frac{d^{\ell(\sigma)}t^{|\sigma|}\mathbf{z}^{\sigma}}{|\mathrm{Aut}(\sigma)|}\end{aligned}$$

where $\sigma$ runs over all monomials in the variables $\mathbf{z}$. Define constants $C_d^r(\sigma)$ by the formula

$$\log(\Theta^{\mathcal{D}}) = \sum_{\sigma}\sum_{d=1}^{\infty}\sum_{r=-1}^{\infty} C_d^r(\sigma)\ t^r\frac{x^d}{d!}\mathbf{z}^{\sigma}\ .$$

By an elementary application of Wick, the $t$ dependence of $\log(\Theta^{\mathcal{D}})$ has at most simple poles.

Finally, we consider the following function,

$$\gamma = \sum_{i\geq 1}\frac{B_{2i}}{2i(2i-1)}\kappa_{2i-1}t^{2i-1} + \sum_{\sigma}\sum_{d=1}^{\infty}\sum_{r=-1}^{\infty} C_d^r(\sigma)\ \kappa_r t^r\frac{x^d}{d!}\mathbf{z}^{\sigma}\ .$$

Denote the $t^r x^d \mathbf{z}^{\sigma}$ coefficient of $\exp(-\gamma)$ by

$$\big[\exp(-\gamma)\big]_{t^r x^d \mathbf{z}^{\sigma}} \in \mathbb{Q}[\kappa_{-1},\kappa_0,\kappa_1,\kappa_2,\ldots]\ .$$

Our form of Faber's equations is the following result.

**Theorem 1.** *In $R^r(\mathcal{M}_g)$, the relation*

$$\big[\exp(-\gamma)\big]_{t^r x^d \mathbf{z}^{\sigma}} = 0$$

*holds when $r > -g + |\sigma|$ and $d > 2g-2$.*

In the tautological ring $R^*(\mathcal{M}_g)$, the conventions

$$\kappa_{-1} = 0, \qquad \kappa_0 = 2g-2$$

will always be followed. For fixed $g$ and $r$, Theorem 1 provides infinitely many relations by increasing $d$.

While the proof of Theorem 1 is appealingly simple, the relations do not seem to fit the other forms we will see later. The variables $z_{i,j}$ efficiently encode both the cotangent and diagonal operations studied in *Conjectural description of the tautological ring* [1]. In particular, the relations of Theorem 1 are equivalent to the mixing of all cotangent and diagonal operations studied there.

**B. Stable quotient relations**

I. *The function* $\Phi$.

The relations in the tautological ring $R^*(\mathcal{M}_g)$ obtained from *Moduli of stable quotients* [4] are based on the function

$$\Phi(t,x) = \sum_{d=0}^{\infty} \prod_{i=1}^{d} \frac{1}{1-it} \; \frac{(-1)^d}{d!} \frac{x^d}{t^d} \ .$$

Define the coefficients $C_d^r$ by the logarithm,

$$\log(\Phi) = \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} C_d^r t^r \frac{x^d}{d!} \ .$$

By an elementary application of Wick, the $t$ dependence has at most a simple pole. Let

$$\gamma = \sum_{i \geq 1} \frac{B_{2i}}{2i(2i-1)} \kappa_{2i-1} t^{2i-1} + \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} C_d^r \kappa_r t^r \frac{x^d}{d!} \ .$$

Denote the $t^r x^d$ coefficient of $\exp(-\gamma)$ by

$$\Big[\exp(-\gamma)\Big]_{t^r x^d} \in \mathbb{Q}[\kappa_{-1}, \kappa_0, \kappa_1, \kappa_2, \ldots] \ .$$

In fact, $[\exp(-\gamma)]_{t^r x^d}$ is homogeneous of degree $r$ in the $\kappa$ classes. The first tautological relations of *Moduli space of stable quotients* [4] are given by the following result.

**Theorem 2.** *In* $R^r(\mathcal{M}_g)$*, the relation*

$$\Big[\exp(-\gamma)\Big]_{t^r x^d} = 0$$

*holds when* $g - 2d - 1 < r$ *and* $g \equiv r + 1 \bmod 2$.

For fixed $r$ and $d$, if Theorem 2 applies in genus $g$, then Theorem 2 applies in genera $h = g - 2\delta$ for all natural numbers $\delta \in \mathbb{N}$. The genus shifting mod 2 property will also be present in the Faber-Zagier conjecture discussed later.

II. *Partitions, differential operators, and logs.*

We will write partitions $\sigma$ as $(1^{a_1}2^{a_2}3^{a_3}\dots)$ with

$$\ell(\sigma) = \sum_i a_i \quad \text{and} \quad |\sigma| = \sum_i i a_i \ .$$

The empty partition $\emptyset$ corresponding to $(1^0 2^0 3^0 \dots)$ is permitted. In all cases, we have

$$|\mathrm{Aut}(\sigma)| = a_1! a_2! a_3! \cdots \ .$$

Consider the infinite set of variables $p_1, p_2, p_3, \dots$ . Monomials in the $p_i$ correspond to partitions

$$p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots \quad \leftrightarrow \quad (1^{a_1}2^{a_2}3^{a_3}\dots) \ .$$

Given a partition $\sigma$, let $\mathbf{p}^\sigma$ denote the corresponding monomial. Let

$$\Phi^{\mathbf{p}}(t,x) = \sum_\sigma \sum_{d=0}^\infty \prod_{i=1}^d \frac{1}{1-it} \frac{(-1)^d}{d!} \frac{x^d}{t^d} \frac{d^{\ell(\sigma)} t^{|\sigma|} \mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}$$

where the first sum is over all partitions $\sigma$. The summand corresponding to the empty partition equals $\Phi(t,x)$.

The function $\Phi^{\mathbf{p}}$ is easily obtained from $\Phi$,

$$\Phi^{\mathbf{p}}(t,x) = \exp\left( \sum_{i=1}^\infty p_i t^i x \frac{d}{dx} \right) \ \Phi(t,x) \ .$$

Let $D$ denote the differential operator

$$D = \sum_{i=1}^\infty p_i t^i x \frac{d}{dx} \ .$$

Expanding the exponential of $D$, we obtain

$$\begin{aligned} \text{(2)} \qquad \Phi^{\mathbf{p}} &= \Phi + D\Phi + \frac{1}{2} D^2 \Phi + \frac{1}{6} D^3 \Phi + \dots \\ &= \Phi \left( 1 + \frac{D\Phi}{\Phi} + \frac{1}{2}\frac{D^2\Phi}{\Phi} + \frac{1}{6}\frac{D^3\Phi}{\Phi} + \dots \right) \ . \end{aligned}$$

Let $\gamma^* = \log(\Phi)$ be the logarithm,

$$D\gamma^* = \frac{D\Phi}{\Phi} \ .$$

After applying the logarithm to (2), we see

$$\begin{aligned}\log(\Phi^{\mathbf{P}}) &= \gamma^* + \log\left(1 + D\gamma^* + \frac{1}{2}(D^2\gamma^* + (D\gamma^*)^2) + \ \ldots\right) \\ &= \gamma^* + D\gamma^* + \frac{1}{2}D^2\gamma^* + \ldots\end{aligned}$$

where the dots stand for a universal expression in the $D^k\gamma^*$. In fact, a remarkable simplification occurs,

$$\log(\Phi^{\mathbf{P}}) = \exp\left(\sum_{i=1}^{\infty} p_i t^i x \frac{d}{dx}\right)\ \gamma^*\ .$$

The result follows from a general identity.

**Proposition 1.** *If $f$ is a function of $x$, then*

$$\log\left(\exp\left(\lambda x \frac{d}{dx}\right)\ f\right) = \exp\left(\lambda x \frac{d}{dx}\right)\ \log(f)\ .$$

*Proof.* A simple computation for monomials in $x$ shows

$$\exp\left(\lambda x \frac{d}{dx}\right)\ x^k = (e^{\lambda} x)^k\ .$$

Hence, since the differential operator is additive,

$$\exp\left(\lambda x \frac{d}{dx}\right)\ f(x) = f(e^{\lambda} x)\ .$$

The Proposition follows immediately. □

The coefficients of the logarithm may be written as

$$\begin{aligned}\log(\Phi^{\mathbf{P}}) &= \sum_{d=1}^{\infty}\sum_{r=-1}^{\infty} C_d^r(\mathbf{p})\ t^r \frac{x^d}{d!} \\ &= \sum_{d=1}^{\infty}\sum_{r=-1}^{\infty} C_d^r\ t^r \frac{x^d}{d!} \exp\left(\sum_{i=1}^{\infty} d p_i t^i\right) \\ &= \sum_{\sigma}\sum_{d=1}^{\infty}\sum_{r=-1}^{\infty} C_d^r\ t^r \frac{x^d}{d!}\ \frac{d^{\ell(\sigma)} t^{|\sigma|} \mathbf{p}^{\sigma}}{|\mathrm{Aut}(\sigma)|}\ .\end{aligned}$$

III. *Full system of tautological relations.*

Following Proposition 5 of *Moduli of stable quotients* [4], we can obtain a much larger set of relations in the tautological ring of $\mathcal{M}_g$ by including several factors of $\pi_*(s^{a_i}\omega^{b_i})$ in the integrand instead of just a single factor. We study the associated relations where the $a_i$ are always 1. The $b_i$ then form the parts of a partition $\sigma$.

To state the relations we obtain, we start by enriching the function $\gamma$ from Section B.I,

$$\begin{aligned}\gamma^{\mathbf{p}} &= \sum_{i\geq 1} \frac{B_{2i}}{2i(2i-1)}\kappa_{2i-1}t^{2i-1} \\ &\quad + \sum_{\sigma}\sum_{d=1}^{\infty}\sum_{r=-1}^{\infty} C_d^r \kappa_{r+|\sigma|}\; t^r \frac{x^d}{d!}\; \frac{d^{\ell(\sigma)}t^{|\sigma|}\mathbf{p}^{\sigma}}{|\mathrm{Aut}(\sigma)|}\ .\end{aligned}$$

Let $\widehat{\gamma}^{\mathbf{p}}$ be defined by a similar formula,

$$\begin{aligned}\widehat{\gamma}^{\mathbf{p}} &= \sum_{i\geq 1} \frac{B_{2i}}{2i(2i-1)}\kappa_{2i-1}(-t)^{2i-1} \\ &\quad + \sum_{\sigma}\sum_{d=1}^{\infty}\sum_{r=-1}^{\infty} C_d^r \kappa_{r+|\sigma|}\; (-t)^r \frac{x^d}{d!}\; \frac{d^{\ell(\sigma)}t^{|\sigma|}\mathbf{p}^{\sigma}}{|\mathrm{Aut}(\sigma)|}\ .\end{aligned}$$

The sign of $t$ in $t^{|\sigma|}$ does not change in $\widehat{\gamma}^{\mathbf{p}}$. The $\kappa_{-1}$ terms which appear will later be set to 0.

The full system of relations are obtain from the coefficients of the functions

$$\exp(-\gamma^{\mathbf{p}}), \qquad \exp(-\sum_{r=0}^{\infty}\kappa_r t^r p_{r+1})\cdot \exp(-\widehat{\gamma}^{\mathbf{p}})$$

**Theorem 3.** *In $R^r(\mathcal{M}_g)$, the relation*

$$\Big[\exp(-\gamma^{\mathbf{p}})\Big]_{t^r x^d \mathbf{p}^\sigma} = (-1)^g \Big[\exp(-\sum_{r=0}^{\infty} \kappa_r t^r p_{r+1}) \cdot \exp(-\widehat{\gamma}^{\mathbf{p}})\Big]_{t^r x^d \mathbf{p}^\sigma}$$

*holds when $g - 2d - 1 + |\sigma| < r$.*

Again, we see the genus shifting mod 2 property. If the relation holds in genus $g$, then the *same* relation holds in genera $h = g - 2\delta$ for all natural numbers $\delta \in \mathbb{N}$.

In case $\sigma = \emptyset$, Theorem 3 specializes to the relation

$$\begin{aligned} \Big[\exp(-\gamma(t,x))\Big]_{t^r x^d} &= (-1)^g \Big[\exp(-\gamma(-t,x))\Big]_{t^r x^d} \\ &= (-1)^{g+r} \Big[\exp(-\gamma(t,x))\Big]_{t^r x^d}, \end{aligned}$$

nontrivial only if $g \equiv r + 1 \mod 2$. If the mod 2 condition holds, then we obtain the relations of Theorem 2.

Consider the case $\sigma = (1)$. The left side of the relation is then

$$\Big[\exp(-\gamma(t,x)) \cdot \left(-\sum_{d=1}^{\infty} \sum_{s=-1}^{\infty} C_d^s \ \kappa_{s+1} t^{s+1} \frac{dx^d}{d!}\right)\Big]_{t^r x^d}.$$

The right side is

$$(-1)^g \Big[\exp(-\gamma(-t,x)) \cdot \left(-\kappa_0 t^0 + \sum_{d=1}^{\infty} \sum_{s=-1}^{\infty} C_d^s \ \kappa_{s+1} (-t)^{s+1} \frac{dx^d}{d!}\right)\Big]_{t^r x^d}.$$

If $g \equiv r + 1 \mod 2$, then the large terms cancel and we obtain

$$-\kappa_0 \cdot \Big[\exp(-\gamma(t,x))\Big]_{t^r x^d} = 0\ .$$

Since $\kappa_0 = 2g - 2$ and

$$(g - 2d - 1 + 1 < r) \quad \Longrightarrow \quad (g - 2d - 1 < r),$$

we recover most (but not all) of the $\sigma = \emptyset$ equations.

If $g \equiv r \mod 2$, then the resulting equation is

$$\Big[\exp(-\gamma(t,x)) \cdot \left(\kappa_0 - 2\sum_{d=1}^{\infty} \sum_{s=-1}^{\infty} C_d^s \ \kappa_{s+1} t^{s+1} \frac{dx^d}{d!}\right)\Big]_{t^r x^d} = 0$$

when $g - 2d < r$.

IV. *Expanded form.*

Let $\sigma = (1^{a_1} 2^{a_2} 3^{a_3} \ldots)$ be a partition of length $\ell(\sigma)$ and size $|\sigma|$. We can directly write the corresponding relation in $R^*(\mathcal{M}_g)$ obtained from Theorem 3.

A *subpartition* $\sigma' \subset \sigma$ is obtained by selecting a nontrivial subset of the parts of $\sigma$. A *division* of $\sigma$ is a disjoint union

$$\sigma = \sigma^{(1)} \cup \sigma^{(2)} \cup \sigma^{(3)} \ldots \tag{3}$$

of subpartitions which exhausts $\sigma$. The subpartitions in (3) are unordered. Let $\mathcal{S}(\sigma)$ be the set of divisions of $\sigma$. For example,

$$\begin{aligned} \mathcal{S}(1^1 2^1) &= \{ \ (1^1 2^1), \ (1^1) \cup (2^1) \ \} \ , \\ \mathcal{S}(1^3) &= \{ \ (1^3), \ (1^2) \cup (1^1) \ \} \ . \end{aligned}$$

We will use the notation $\sigma^\bullet$ to denote a division of $\sigma$ with subpartitions $\sigma^{(i)}$. Let

$$m(\sigma^\bullet) = \frac{1}{|\mathrm{Aut}(\sigma^\bullet)|} \frac{|\mathrm{Aut}(\sigma)|}{\prod_{i=1}^{\ell(\sigma^\bullet)} |\mathrm{Aut}(\sigma^{(i)})|}.$$

Here, $\mathrm{Aut}(\sigma^\bullet)$ is the group permuting equal subpartitions. The factor $m(\sigma^\bullet)$ may be interpreted as counting the number of different ways the disjoint union can be made.

To write explicitly the $\mathbf{p}^\sigma$ coefficient of $\exp(\gamma^{\mathbf{p}})$, we introduce the functions

$$F_{n,m}(t,x) = -\sum_{d=1}^{\infty} \sum_{s=-1}^{\infty} C_d^s \ \kappa_{s+m} t^{s+m} \frac{d^n x^d}{d!}$$

for $n, m \geq 1$. Then,

$$|\mathrm{Aut}(\sigma)| \cdot \Big[ \exp(-\gamma^{\mathbf{p}}) \Big]_{t^r x^d \mathbf{p}^\sigma} =$$
$$\Bigg[ \exp(-\gamma(t,x)) \cdot \left( \sum_{\sigma^\bullet \in \mathcal{S}(\sigma)} m(\sigma^\bullet) \prod_{i=1}^{\ell(\sigma^\bullet)} F_{\ell(\sigma^{(i)}), |\sigma^{(i)}|} \right) \Bigg]_{t^r x^d} .$$

The length $\ell(\sigma^{*,\bullet})$ is the number of unmarked subpartitions.

Let $\sigma^{*,\bullet}$ be a division of $\sigma$ with a marked subpartition,

$$\sigma = \sigma^* \cup \sigma^{(1)} \cup \sigma^{(2)} \cup \sigma^{(3)} \ldots , \tag{4}$$

labelled by the superscript $*$. The marked subpartition is permitted to be empty. Let $\mathcal{S}^*(\sigma)$ denote the set of marked divisions of $\sigma$. Let

$$m(\sigma^{*,\bullet}) = \frac{1}{|\mathrm{Aut}(\sigma^\bullet)|}\frac{|\mathrm{Aut}(\sigma)|}{|\mathrm{Aut}(\sigma^*)|\prod_{i=1}^{\ell(\sigma^{*,\bullet})}|\mathrm{Aut}(\sigma^{(i)})|}.$$

Then, $|\mathrm{Aut}(\sigma)|$ times the right side of Theorem 3 may be written as

$$(-1)^{g+|\sigma|}|\mathrm{Aut}(\sigma)|\cdot\Bigg[\exp(-\gamma(-t,x))\cdot \\ \left(\sum_{\sigma^{*,\bullet}\in\mathcal{S}^*(\sigma)} m(\sigma^{*,\bullet})\prod_{j=1}^{\ell(\sigma^*)}\kappa_{\sigma^*_j-1}(-t)^{\sigma^*_j-1}\prod_{i=1}^{\ell(\sigma^{*,\bullet})}F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|}(-t,x)\right)\Bigg]_{t^r x^d}$$

To write Theorem 3 in the simplest form, the following definition with the Kronecker $\delta$ is useful,

$$m^{\pm}(\sigma^{*,\bullet}) = (1\pm\delta_{0,|\sigma^*|})\cdot m(\sigma^{*,\bullet}).$$

There are two cases. If $g\equiv r+|\sigma| \bmod 2$, then Theorem 3 is equivalent to the vanishing of

$$\Bigg[\exp(-\gamma)\cdot\left(\sum_{\sigma^{*,\bullet}\in\mathcal{S}^*(\sigma)} m^-(\sigma^{*,\bullet})\prod_{j=1}^{\ell(\sigma^*)}\kappa_{\sigma^*_j-1}t^{\sigma^*_j-1}\prod_{i=1}^{\ell(\sigma^{*,\bullet})}F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|}\right)\Bigg]_{t^r x^d}.$$

If $g\equiv r+|\sigma|+1 \bmod 2$, then Theorem 3 is equivalent to the vanishing of

$$\Bigg[\exp(-\gamma)\cdot\left(\sum_{\sigma^{*,\bullet}\in\mathcal{S}^*(\sigma)} m^+(\sigma^{*,\bullet})\prod_{j=1}^{\ell(\sigma^*)}\kappa_{\sigma^*_j-1}t^{\sigma^*_j-1}\prod_{i=1}^{\ell(\sigma^{*,\bullet})}F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|}\right)\Bigg]_{t^r x^d}.$$

In either case, the relations are valid in the ring $R^*(\mathcal{M}_g)$ only if the condition $g-2d-1+|\sigma|<r$ holds.

V. *Further examples.*

If $\sigma = (k)$ has a single part, then the two cases of Theorem 3 are the following. If $g \equiv r + k \mod 2$, we have

$$\left[ \exp(-\gamma) \cdot \kappa_{k-1} t^{k-1} \right]_{t^r x^d} = 0$$

which is a consequence of Theorem 2. If $g \equiv r + k + 1 \mod 2$, we have

$$\left[ \exp(-\gamma) \cdot \left( \kappa_{k-1} t^{k-1} + 2F_{1,k} \right) \right]_{t^r x^d} = 0$$

If $\sigma = (k_1 k_2)$ has two distinct parts, then the two cases of Theorem 3 are as follows. If $g \equiv r + k_1 + k_2 \mod 2$, we have

$$\Big[ \exp(-\gamma) \cdot \big( \kappa_{k_1-1} \kappa_{k_2-1} t^{k_1+k_2-2} + \kappa_{k_1-1} t^{k_1-1} F_{1,k_2} + \kappa_{k_2-1} t^{k_2-1} F_{1,k_1} \big) \Big]_{t^r x^d} = 0 \ .$$

If $g \equiv r + k_1 + k_2 + 1 \mod 2$, we have

$$\Big[ \exp(-\gamma) \cdot \big( \kappa_{k_1-1} \kappa_{k_2-1} t^{k_1+k_2-2} + \kappa_{k_1-1} t^{k_1-1} F_{1,k_2} + \kappa_{k_2-1} t^{k_2-1} F_{1,k_1} + 2F_{2,k_1+k_2} + 2F_{1,k_1} F_{1,k_2} \big) \Big]_{t^r x^d} = 0 \ .$$

In fact, the $g \equiv r + k_1 + k_2 \mod 2$ equation above is not new. The genus $g$ and codimension $r_1 = r - k_2 + 1$ case of partition $(k_1)$ yields

$$\left[ \exp(-\gamma) \cdot \left( \kappa_{k_1-1} t^{k_1-1} + 2F_{1,k_1} \right) \right]_{t^{r_1} x^d} = 0 \ .$$

After multiplication with $\kappa_{k_2-1} t^{k_2-1}$, we obtain

$$\left[ \exp(-\gamma) \cdot \left( \kappa_{k_1-1} \kappa_{k_2-1} t^{k_1+k_2-2} + 2\kappa_{k_2-1} t^{k_2-1} F_{1,k_1} \right) \right]_{t^r x^d} = 0 \ .$$

Summed with the corresponding equation with $k_1$ and $k_2$ interchanged yields the above $g \equiv r + k_1 + k_2 \mod 2$ case.

VI. *Expanded form revisited.*

Consider the partition $\sigma = (k_1 k_2 \cdots k_\ell)$ with distinct parts. We obtain from Theorem 3, in the $g \equiv r + |\sigma| \mod 2$ case, the vanishing of

$$\left[\exp(-\gamma)\cdot \left( \sum_{\sigma^{*,\bullet} \in \mathcal{S}^*(\sigma)} (1-\delta_{0,|\sigma^*|}) \prod_{j=1}^{\ell(\sigma^*)} \kappa_{\sigma^*_j - 1} t^{\sigma^*_j - 1} \prod_{i=1}^{\ell(\sigma^{*,\bullet})} F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|} \right) \right]_{t^r x^d}$$

since all the factors $m(\sigma^{*,\bullet})$ are 1. In the $g \equiv r + |\sigma| + 1 \mod 2$ case, we obtain the vanishing of

$$\left[\exp(-\gamma)\cdot \left( \sum_{\sigma^{*,\bullet} \in \mathcal{S}^*(\sigma)} (1+\delta_{0,|\sigma^*|}) \prod_{j=1}^{\ell(\sigma^*)} \kappa_{\sigma^*_j - 1} t^{\sigma^*_j - 1} \prod_{i=1}^{\ell(\sigma^{*,\bullet})} F_{\ell(\sigma^{(i)}),|\sigma^{(i)}|} \right) \right]_{t^r x^d}$$

for the same reason.

**Proposition 2.** *The $g \equiv r + |\sigma|$ mod 2 case is a consequence of the $g \equiv r' + |\sigma'| + 1$ mod 2 cases of smaller partitions $\sigma'$.*

*Proof.* The strategy is identical to that employed in the special cases of the result proven in Section V. □

If $\sigma$ has repeated parts, the relations of Theorem 3 are obtained by viewing the parts are distinct and using the above formulas. For example, the two cases of Theorem 3 for $\sigma = (k^2)$ are as follows. If $g \equiv r + 2k \mod 2$, we have

$$\left[\exp(-\gamma)\cdot \left(\kappa_{k-1}\kappa_{k-1} t^{2k-2} + 2\kappa_{k-1} t^{k-1} F_{1,k}\right)\right]_{t^r x^d} = 0 \ .$$

If $g \equiv r + 2k + 1 \mod 2$, we have

$$\left[\exp(-\gamma)\cdot \left(\kappa_{k-1}\kappa_{k-1} t^{2k-2} + 2\kappa_{k-1} t^{k-1} F_{1,k} \right.\right. \\ \left.\left. + 2F_{2,2k} + 2F_{1,k}F_{1,k}\right)\right]_{t^r x^d} = 0 \ .$$

The factors occur via repetition of terms in the formulas for distinct parts.

VII. *Differential equations.*

The function $\Phi$ satisfies a basic differential equation obtained from the series definition,

$$\frac{d}{dx}(\Phi - tx\frac{d}{dx}\Phi) = -\frac{1}{t}\Phi \ .$$

After expanding and dividing by $\Phi$, we find

$$-tx\frac{\Phi_{xx}}{\Phi} - t\frac{\Phi_x}{\Phi} + \frac{\Phi_x}{\Phi} = -\frac{1}{t}$$

which can be written as

$$-t^2x\gamma^*_{xx} = t^2x(\gamma^*_x)^2 + t^2\gamma^*_x - t\gamma^*_x - 1 \tag{5}$$

where, as before, $\gamma^* = \log(\Phi)$. Equation (5) has been studied by Ionel in *Relations in the tautological ring* [3]. We present here results of hers which will be useful for us.

To kill the pole and match the required constant term, we will consider the function

$$\Gamma = -t\left(\sum_{i\geq 1}\frac{B_{2i}}{2i(2i-1)}t^{2i-1} + \gamma^*\right) \ .$$

The differential equation (5) becomes

$$tx\Gamma_{xx} = x(\Gamma_x)^2 + (1-t)\Gamma_x - 1 \ .$$

The differential equation is easily seen to uniquely determine $\Gamma$ once the initial conditions

$$\Gamma(t,0) = -\sum_{i\geq 1}\frac{B_{2i}}{2i(2i-1)}t^{2i}$$

are specified. By Ionel's first result,

$$\Gamma_x = \frac{-1+\sqrt{1+4x}}{2x} + \frac{t}{1+4x} + \sum_{k=1}^{\infty}\sum_{j=0}^{k} t^{k+1}q_{k,j}(-x)^j(1+4x)^{-j-\frac{k}{2}-1}$$

where the postive integers $q_{k,j}$ (defined to vanish unless $k \geq j \geq 0$) are defined via the recursion

$$q_{k,j} = (2k+4j-2)q_{k-1,j-1} + (j+1)q_{k-1,j} + \sum_{m=0}^{k-1}\sum_{l=0}^{j-1} q_{m,l}q_{k-1-m,j-1-l}$$

from the initial value $q_{0,0} = 1$.

Ionel's second result is obtained by integrating $\Gamma_x$ with respect to $x$. She finds

$$\Gamma = \Gamma(0,x) + \frac{t}{4}\log(1+4x) - \sum_{k=1}^{\infty}\sum_{j=0}^{k} t^{k+1} c_{k,j}(-x)^j(1+4x)^{-j-\frac{k}{2}}$$

where the coefficients $c_{k,j}$ are determined by

$$q_{k,j} = (2k+4j)c_{k,j} + (j+1)c_{k,j+1}$$

for $k \geq 1$ and $k \geq j \geq 0$.

While the derivation of the formula for $\Gamma_x$ is straightforward, the formula for $\Gamma$ is quite subtle as the intial conditions (given by the Bernoulli numbers) are used to show the vanishing of constants of integration. Said differently, the recusions for $q_{k,j}$ and $c_{k,j}$ must be shown to imply the formula

$$c_{k,0} = \frac{B_k}{k(k-1)} \ .$$

A third result of Ionel's is the determination of the extremal $c_{k,k}$,

$$\sum_{k=1}^{\infty} c_{k,k} z^k = \log\left(\sum_{k=1}^{\infty} \frac{(6k)!}{(2k)!(3k)!}\left(\frac{z}{72}\right)^k\right) \ .$$

The formula for $\Gamma$ becomes simpler after the following very natural change of variables,

$$u = \frac{t}{\sqrt{1+4x}} \quad \text{and} \quad y = \frac{-x}{1+4x} \ . \tag{6}$$

The change of variables defines a new function

$$\widehat{\Gamma}(u,y) = \Gamma(t,x) \ .$$

The formula for $\Gamma$ implies

$$\frac{1}{t}\Gamma(u,y) = \frac{1}{t}\Gamma(0,y) - \frac{1}{4}\log(1+4y) - \sum_{k=1}^{\infty}\sum_{j=0}^{k} c_{k,j}u^k y^j \ .$$

Ionel's fourth result relates coefficients of series after the change of variables (6). Given any series

$$P(t,x) \in \mathbb{Q}[[t,x]],$$

let $\widehat{P}(u,y)$ be the series obtained from the change of variables (6). Ionel proves coefficient relation

$$\big[P(t,x)\big]_{t^r x^d} = (-1)^d\big[(1+4y)^{\frac{r+2d-2}{2}} \cdot \widehat{P}(u,y)\big]_{u^r y^d} \ .$$

VII. *Analysis of the relations of Theorem 2*

We now study in detail the simple relations of Theorem 2,

$$\big[\exp(-\gamma)\big]_{t^r x^d} = 0 \ \in R^r(\mathcal{M}_g)$$

when $g-2d-1 < r$ and $g \equiv r+1 \bmod 2$. Let

$$\widehat{\gamma}(u,y) = \gamma(t,x)$$

be obtained from the variable change (6),

$$\widehat{\gamma}(u,y) = \frac{\kappa_0}{4}\log(1+4y) + \sum_{k=1}^{\infty}\sum_{j=0}^{k} \kappa_k c_{k,j} u^k y^j$$

modulo $\kappa_{-1}$ terms which we set to 0. Applying Ionel's coefficient result,

$$\begin{aligned}
\big[\exp(-\gamma)\big]_{t^r x^d} &= \Big[(1+4y)^{\frac{r+2d-2}{2}} \cdot \exp(-\widehat{\gamma})\Big]_{u^r y^d} \\
&= \left[(1+4y)^{\frac{r+2d-2}{2}-\frac{\kappa_0}{4}} \cdot \exp(-\sum_{k=1}^{\infty}\sum_{j=0}^{k} \kappa_k c_{k,j} u^k y^j)\right]_{u^r y^d} \\
&= \left[(1+4y)^{\frac{r-g+2d-1}{2}} \cdot \exp(-\sum_{k=1}^{\infty}\sum_{j=0}^{k} \kappa_k c_{k,j} u^k y^j)\right]_{u^r y^d} .
\end{aligned}$$

In the last line, the substitution $\kappa_0 = 2g-2$ has been made.

Consider first the exponent of $1+4y$. By the assumptions on $g$ and $r$ in Theorem 2,

$$\frac{r-g+2d-1}{2} \geq 0$$

and the fraction is integral. Hence, the $y$ degree of the prefactor

$$(1+4y)^{\frac{r-g+2d-1}{2}}$$

is exactly $\frac{r-g+2d-1}{2}$. The $y$ degree of the exponential factor is bounded from above by the $u$ degree. We conclude

$$\left[(1+4y)^{\frac{r-g+2d-1}{2}} \cdot \exp(-\sum_{k=1}^{\infty}\sum_{j=0}^{k} \kappa_k c_{k,j} u^k y^j)\right]_{u^r y^d} = 0$$

is the *trivial* relation unless

$$r \geq d - \frac{r-g+2d-1}{2} = -\frac{r}{2} + \frac{g+1}{2} .$$

Rewriting the inequality, we obtain $3r \geq g+1$ which is equivalent to $r > \lfloor \frac{g}{3} \rfloor$. The conclusion is in agreement with the proven freeness of $R^*(\mathcal{M}_g)$ up to (and including) degree $\lfloor \frac{g}{3} \rfloor$.

A similar connection between Theorem 2 and Ionel's relations in [3] has also been found by Shengmao Zhu [8].

VIII. *Analysis of the relations of Theorem 3*

For the relations of Theorem 3, we will require additional notation. To start, let

$$\gamma^c(u,y) = \sum_{k=1}^{\infty} \sum_{j=0}^{k} \kappa_k c_{k,j} u^k y^j \ .$$

By Ionel's second result,

$$\frac{1}{t}\Gamma = \frac{1}{t}\Gamma(0,x) + \frac{1}{4}\log(1+4x) - \sum_{k=1}^{\infty}\sum_{j=0}^{k} t^k c_{k,j}(-x)^j(1+4x)^{-j-\frac{k}{2}} \ .$$

Let $c^0_{k,j} = c_{k,j}$. We define the constants $c^n_{k,j}$ for $n \geq 1$ by

$$\left(x\frac{d}{dx}\right)^n \frac{1}{t}\Gamma = \left(x\frac{d}{dx}\right)^{n-1}\left(\frac{-1}{2t} + \frac{1}{2t}\sqrt{1+4x}\right) - \sum_{k=0}^{\infty}\sum_{j=0}^{k+n} t^k c^n_{k,j}(-x)^j(1+4x)^{-j-\frac{k}{2}} \ .$$

**Lemma 2.** *For $n > 0$, there are constants $b^n_j$ satisfying*

$$\left(x\frac{d}{dx}\right)^{n-1}\left(\frac{1}{2t}\sqrt{1+4x}\right) = \sum_{j=0}^{n-1} b^n_j u^{-1} y^j \ .$$

*Moreover, $b^n_{n-1} = -2^{n-2}\cdot(2n-5)!!$ where $(-1)!! = 1$ and $(-3)!! = -1$.*

*Proof.* The result is obtained by simple induction. The negative evaluations $(-1)!! = 1$ and $(-3)!! = -1$ arise from the $\Gamma$-regularization. □

**Lemma 3.** *For $n > 0$, we have $c^n_{0,n} = 4^{n-1}(n-1)!$.*

**Lemma 4.** *For $n > 0$ and $k > 0$, we have*

$$c^n_{k,k+n} = (6k)(6k+4)\cdots(6k+4(n-1))\ c_{k,k}.$$

Consider next the full set of equations given by Theorem 3 in the expanded form of Section VI. The function $F_{n,m}$ may be rewritten as

$$\begin{aligned} F_{n,m}(t,x) &= -\sum_{d=1}^{\infty}\sum_{s=-1}^{\infty} C_d^s\ \kappa_{s+m} t^{s+m} \frac{d^n x^d}{d!} \\ &= -t^m \left(x\frac{d}{dx}\right)^n \sum_{d=1}^{\infty}\sum_{s=-1}^{\infty} C_d^s\ \kappa_{s+m} t^s \frac{x^d}{d!}. \end{aligned}$$

We may write the result in terms of the constants $b_j^n$ and $c_{k,j}^n$,

$$\begin{aligned} t^{-(m-n)}F_{n,m} &= -\delta_{n,1}\frac{\kappa_{m-1}}{2} \\ &\quad + (1+4y)^{-\frac{n}{2}}\Big(\sum_{j=0}^{n-1}\kappa_{m-1}b_j^n u^{n-1}y^j - \sum_{k=0}^{\infty}\sum_{j=0}^{k+n}\kappa_{k+m}c_{k,j}^n u^{k+n}y^j\Big) \end{aligned}$$

Define the functions $G_{n,m}(u,y)$ by

$$G_{n,m}(u,y) = \sum_{j=0}^{n-1}\kappa_{m-1}b_j^n u^{n-1}y^j - \sum_{k=0}^{\infty}\sum_{j=0}^{k+n}\kappa_{k+m}c_{k,j}^n u^{k+n}y^j\ .$$

Let $\sigma = (1^{a_1}2^{a_2}3^{a_3}\ldots)$ be a partition of length $\ell(\sigma)$ and size $|\sigma|$. We assume the parity condition

$$g \equiv r + |\sigma| + 1\ . \tag{7}$$

Let $G_\sigma^{\pm}(u,y)$ be the following function associated to $\sigma$,

$$G_\sigma^{\pm}(u,y) = \sum_{\sigma^\bullet\in\mathcal{S}(\sigma)}\prod_{i=1}^{\ell(\sigma^\bullet)}\left(G_{\ell(\sigma^{(i)}),|\sigma^{(i)}|} \pm \frac{\delta_{\ell(\sigma^{(i)}),1}}{2}\sqrt{1+4y}\ \kappa_{|\sigma^{(i)}|-1}\right)\ .$$

The relations of Theorem 3 written in the variables $u$ and $y$ is

$$\Big[(1+4y)^{\frac{r-|\sigma|-g+2d-1}{2}}\exp(-\gamma^c)\left(G_\sigma^+ + G_\sigma^-\right)\Big]_{u^{r-|\sigma|+\ell(\sigma)}y^d} = 0$$

In fact, the relations of Theorem 3 can be written in a much more efficient form when the strategy of Proposition 2 is used to take out lower equations.

**Theorem 4.** *In $R^r(\mathcal{M}_g)$, the relation*

$$\Big[(1+4y)^{\frac{r-|\sigma|-g+2d-1}{2}}\exp\left(-\gamma^c + \sum_{\sigma\neq\emptyset} G_{\ell(\sigma),|\sigma|}\frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}\right)\Big]_{u^{r-|\sigma|+\ell(\sigma)}y^d\mathbf{p}^\sigma} = 0$$

*holds when $g-2d-1+|\sigma|<r$ and $g\equiv r+|\sigma|+1$ mod 2.*

Consider the exponent of $1+4y$. By the inequality and the parity condition (7),

$$\frac{r-|\sigma|-g+2d-1}{2} \geq 0$$

and the fraction is integral. Hence, the $y$ degree of the prefactor

$$(1+4y)^{\frac{r-|\sigma|-g+2d-1}{2}}$$

is exactly $\frac{r-|\sigma|-g+2d-1}{2}$. The $y$ degree of the exponential factor is bounded from above by the $u$ degree. We conclude the relation of Theorem 4 is *trivial* unless

$$r-|\sigma|+\ell(\sigma) \geq d - \frac{r-|\sigma|-g+2d-1}{2} = -\frac{r-|\sigma|}{2} + \frac{g+1}{2} \ .$$

Rewriting the inequality, we obtain

$$3r \geq g+1+3|\sigma|-2\ell(\sigma)$$

which is consistent with the proven freeness of $R^*(\mathcal{M}_g)$ up to (and including) degree $\lfloor \frac{g}{3} \rfloor$.

X. *Another form*

A subset of the equations of Theorem 4 admits an especially simple description. Consider the function

$$H_{n,m}(u) = 2^{n-2}(2n-5)!!\ \kappa_{m-1}u^{n-1} + 4^{n-1}(n-1)!\ \kappa_m u^n + \sum_{k=1}^{\infty}(6k)(6k+4)\cdots(6k+4(n-1))c_{k,k}\ \kappa_{k+m}u^{k+n}\ .$$

**Proposition 3.** *In $R^r(\mathcal{M}_g)$, the relation*

$$\left[\exp\left(-\sum_{k=1}^{\infty}c_{k,k}\kappa_k u^k - \sum_{\sigma\neq\emptyset}H_{\ell(\sigma),|\sigma|}\frac{\mathbf{p}^\sigma}{|\mathrm{Aut}(\sigma)|}\right)\right]_{u^{r-|\sigma|+\ell(\sigma)}\mathbf{p}^\sigma} = 0$$

*holds when $3r \geq g+1+3|\sigma|-2\ell(\sigma)$ and $g \equiv r+|\sigma|+1$ mod 2.*

The main advantage of Proposition 3 is the dependence on only the function

$$\sum_{k=1}^{\infty}c_{k,k}z^k = \log\left(\sum_{k=1}^{\infty}\frac{(6k)!}{(2k)!(3k)!}\left(\frac{z}{72}\right)^k\right)\ . \qquad (8)$$

Proposition 3 only provides finitely many relations for fixed $g$ and $r$. In Theorem 5 of Section C.I below, a more elegant set of relations in $R^r(M_g)$ conjectured by Faber-Zagier is presented. In fact, we show Proposition 3 is equivalent to the Faber-Zagier conjecture.

## C. The conjecture of Faber-Zagier

I. *The function* $\Psi$

A third set of relations is defined as follows Let

$$\mathbf{p} = \{ \; p_1, p_3, p_4, p_6, p_7, p_9, p_{10}, \ldots \; \}$$

be a variable set indexed by integers not congruent to 2 mod 3. Let

$$\Psi(t, \mathbf{p}) = (1 + tp_3 + t^2 p_6 + t^3 p_9 + \ldots) \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} t^i$$
$$+ (p_1 + tp_4 + t^2 p_7 + \ldots) \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{6i+1}{6i-1} t^i$$

Define the constants $C^r(\sigma)$ by the formula

$$\log(\Psi) = \sum_{\sigma} \sum_{r=0}^{\infty} C^r(\sigma) \; t^r \mathbf{p}^{\sigma} \; .$$

Here and below, $\sigma$ denotes a partition which avoids all parts congruent to 2 mod 3. Let

$$\gamma = \sum_{\sigma} \sum_{r=0}^{\infty} C^r(\sigma) \; \kappa_r t^r \mathbf{p}^{\sigma} \; .$$

Our main result, starting from the stable quotient relations, is the following final form.

**Theorem 5.** *In* $R^r(\mathcal{M}_g)$*, the relation*

$$\big[\exp(-\gamma)\big]_{t^r \mathbf{p}^{\sigma}} = 0$$

*holds when* $g - 1 + |\sigma| < 3r$ *and* $g \equiv r + |\sigma| + 1 \mod 2$.

The relations of Theorem 5 were conjectured earlier by Faber and Zagier from data and a study of the Gorenstein quotient of $R^*(\mathcal{M}_g)$. To the best of our knowledge, a relation in $R^*(\mathcal{M}_g)$ which is not in the span of the relations of Theorem 5 has not yet been found. In particular, all relations obtained from Theorem 1 to date are in the span of Theorem 5 (and conversely). It is very reasonable to expect the spans of the relations in Theorem 1 and Theorem 5 exactly coincide. Whether Theorem 5 exhausts all relations in $R^*(\mathcal{M}_g)$ is a very interesting question.

Theorem 5 is much more efficient than Theorem 1 for several reasons. Theorem 5 only provides finitely many relations in $R^r(\mathcal{M}_g)$ for fixed $g$ and $r$, and thus may be calculated completely. When the relations yield a Gorenstein ring with socle in $R^{g-2}(\mathcal{M}_g)$, no further relations are possible. However, the relations of Theorem 5 do not always yield such a Gorenstein ring (failing first in genus 24 as checked by Faber). For $g < 24$, Faber's calculations show Theorem 5 does provide all relations in $R^*(\mathcal{M}_g)$. For higher genus $g \geq 24$, either Theorem 5 fails to provide all the relations in $R^*(\mathcal{M}_g)$ *or* $R^*(\mathcal{M}_g)$ is not Gorenstein.

II. *Connection to the stable quotient relations*

Theorem 5 is derived from Proposition 3. In fact, Proposition 3 is equivalent to Theorem 5. The derivation is obtained by a triangular transformation among distinguished generators. A certain amount of differential algebra is required.

Consider the relation obtained from the partition $\sigma = (1)$ in Proposition 3 and the Conjecture. For convenience, let

$$\begin{aligned} A(z) &= \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \left(\frac{z}{72}\right)^i , \\ B(z) &= \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{6i+1}{6i-1} \left(\frac{z}{72}\right)^i \end{aligned}$$

The conjectures predict no room for different relations in $R^*(\mathcal{M}_g)$ for $\sigma = (1)$, so we must have

$$-\frac{1}{2} + z + 6z \left(z\frac{d}{dz}\right) \log(A)$$

proportional to $B/A$. We find the equation

$$-\frac{1}{2} + z + 6z \left(z\frac{d}{dz}\right) \log(A) = \frac{1}{2}\frac{B}{A}$$

holds. Equivalently,

$$-\frac{1}{2}A + zA + 6z^2 \, \frac{dA}{dz} = \frac{1}{2}B$$

More interesting is the partition $\sigma = (11)$. Here we predict, once the definitions are unwound, that

$$z + 4z^2 + 36z^2 \left(z\frac{d}{dz}\right)^2 \log(A) + 24z^2 \left(z\frac{d}{dz}\right) \log A$$

is a linear combination of 1 and $B^2/A^2$. We find the equation

$$z + 4z^2 + 36z^2 \left(z\frac{d}{dz}\right)^2 \log(A) + 24z^2 \left(z\frac{d}{dz}\right) \log A = \frac{1}{4} - \frac{1}{4}\frac{B^2}{A^2}$$

holds.

In fact, the main hypergeometric differential equation satisfied by the function $A$ is

$$36z^2\frac{d^2}{dz^2}A + (72z-6)\frac{d}{dz}A + 5A = 0\ .$$

In Section D below, further details describing the use of such differential equations to prove Theorem 5 from Proposition 3 are presented.

III. *Functions*

While the functions $A(z)$ and $B(z)$ of Section II have radius of convergence 0, an additional double factorial in the denominator yields convergent classical series,

$$\begin{aligned} \frac{3}{2t}\sin\left(\frac{2}{3}\sin^{-1}(t)\right) &= \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!(2i+1)!!}\left(\frac{t^2}{216}\right)^i, \\ -\frac{3}{4t}\sin\left(\frac{4}{3}\sin^{-1}(t)\right) &= \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!(2i+1)!!}\frac{6i+1}{6i-1}\left(\frac{t^2}{216}\right)^i. \end{aligned}$$

IV. *Remarks*

Stable quotients relations in $R^*(\mathcal{M}_g)$ have several advantages over other geometric constructions. We have already seen here the possibility of exact evaluation. Another advantage we have not explored in these lectures is the extension of the stable quotients relations over $\overline{\mathcal{M}}_g$. The boundary terms of the stable quotients relations are tautological. A study of the relations among the $\kappa$ classes in the tautological ring of the moduli space of curves of compact type $\mathcal{M}^c_{g,n}$ has been undertaken in [5, 6]. For example, the Gorenstein predictions for $\kappa$ classes are proven there if $n \geq 1$. But even for $\mathcal{M}_g$, the extension of the stable quotients relations over $\overline{\mathcal{M}}_g$ has significant consequences. Since we know the stable quotients relations are all relations in $R^*(\mathcal{M}_g)$ for $g < 24$, the following result holds.

**Proposition 4.** *For $g < 24$, we have a right exact sequence*

$$R^*(\partial\overline{\mathcal{M}}_g) \to R^*(\overline{\mathcal{M}}_g) \to R^*(\mathcal{M}_g) \to 0 \ .$$

Speculations about such right exactness for tautological rings were advanced in [2].

## D. The equivalence

I. *Notation*

The relations in Theorem 5 and Proposition 3 have a similar flavor. We start with formal series related to

$$A(z) = \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \left(\frac{z}{72}\right)^i ,$$

we insert classes $\kappa_r$, we exponentiate, and we extract coefficients to obtain relations among the $\kappa$ classes. In order to make the similarities clearer, we will introduce additional notation.

If $F$ is a formal power series in $z$,

$$F = \sum_{r=0}^{\infty} c_r z^r$$

with coefficients in a ring $R$, let

$$\{F\}_\kappa = \sum_{r=0}^{\infty} c_r \kappa_r z^r$$

be the series with $\kappa$-classes inserted.

Let $A$ be as above, and let $B$ be the function defined in C.II. Let

$$C = \frac{B}{A} ,$$

and let

$$E = \exp(-\{\log(A)\}_\kappa) = \exp\left(-\sum_{k=1}^{\infty} c_{k,k}\kappa_k z^k\right) .$$

We will rewrite the relations of Theorem 5 and Proposition 3 in terms of $C$ and $E$. The equivalence between the two will rely on properties of the differential equations satisfied by $C$.

II. *Rewriting the relations*

The relations of Theorem 5 conjectured by Faber-Zagier are straightforward to rewrite using the above notation:

$$(9)\quad \Big[E \cdot \exp\Big( - \Big\{ \log(1 + p_3 z + p_6 z^2 + \cdots \\ + C(p_1 + p_4 z + p_7 z^2 + \cdots)) \Big\}_\kappa \Big) \Big]_{z^r p^\sigma} = 0$$

for $3r \geq g + |\sigma| + 1$ and $3r \equiv g + |\sigma| + 1 \mod 2$. We call the above relations $\mathsf{FZ}$.

The stable quotient relations of Proposition 3 are a bit more complicated to rewrite in terms of $C$ and $E$. Let

$$2^{-n} C_n = 2^{n-2}(2n-5)!! z^{n-1} + 4^{n-1}(n-1)! z^n \\ + \sum_{k=1}^{\infty} (6k)(6k+4)\cdots(6k+4(n-1)) c_{k,k} z^{k+n}.$$

We see

$$H_{n,m}(z) = 2^{-n} z^{n-m} \{z^{m-n} C_n\}_\kappa.$$

The series $C_n$ satisfy

$$C_1 = C, \qquad C_{i+1} = \left(12z^2 \frac{d}{dz} - 4iz\right) C_i.$$

Since $C$ satisfies the differential equation

$$12z^2 \frac{dC}{dz} = 1 + 4zC - C^2,$$

each $C_n$ can be expressed as a polynomial in $C$ and $z$:

$$C_1 = C, \quad C_2 = 1 - C^2, \quad C_3 = -8z - 2C + 2C^3, \ldots, \ .$$

Proposition 3 can then be rewritten as follows (after an appropriate change of variables):

$$(10)\qquad \left[E\cdot \exp\left(-\sum_{\sigma\neq\emptyset}\{z^{|\sigma|-\ell(\sigma)}C_{\ell(\sigma)}\}_\kappa \frac{p^\sigma}{|\operatorname{Aut}(\sigma)|}\right)\right]_{z^r p^\sigma} = 0$$

for $3r \geq g+3|\sigma|-2\ell(\sigma)+1$ and $3r \equiv g+3|\sigma|-2\ell(\sigma)+1 \bmod 2$. We call the stable quotients relations $\mathsf{SQ}$.

The $\mathsf{FZ}$ and $\mathsf{SQ}$ relations now look much more similar, but the relations in (9) are indexed by partitions with no parts of size 2 mod 3 and satisfy a slightly different inequality. The indexing differences can be erased by noting the variables $p_{3k}$ are actually not necessary in (9) if we are just interested in the *ideal* generated by a set of relations (rather than the linear span). If we remove the variables $p_{3k}$ and reindex the others, we obtain the following equivalent form of the $\mathsf{FZ}$ relations:

$$(11)\qquad \Big[E\cdot \exp\big(-\big\{\log(1+C(p_1+p_2z+p_3z^2+\cdots))\big\}_\kappa\big)\Big]_{z^r p^\sigma} = 0$$

for $3r \geq g+3|\sigma|-2\ell(\sigma)+1$ and $3r \equiv g+3|\sigma|-2\ell(\sigma)+1 \bmod 2$.

II. *Comparing the relations*

We now explain how to write the $\mathsf{SQ}$ relations (10) as linear combinations of the $\mathsf{FZ}$ relations (11) with coefficients in $\mathbb{Q}[\kappa_0,\kappa_1,\kappa_2,\ldots]$. In fact, the associated matrix will be triangular with diagonal entries equal to 1.

We start with further notation. For a partition $\sigma$, let

$$\mathsf{FZ}_\sigma = \big[\exp\big(-\big\{\log(1+C(p_1+p_2z+p_3z^2+\cdots))\big\}_\kappa\big)\big]_{p^\sigma}$$

and

$$\mathsf{SQ}_\sigma = \left[\exp\left(-\sum_{\sigma\neq\emptyset}\{z^{|\sigma|-\ell(\sigma)}C_{\ell(\sigma)}\}_\kappa \frac{p^\sigma}{|\operatorname{Aut}(\sigma)|}\right)\right]_{p^\sigma}$$

be power series in $z$ with coefficients that are polynomials in the $\kappa$ classes. The relations themselves are given by $[E\cdot\mathsf{SQ}_\sigma]_{z^r}$ and $[E\cdot\mathsf{FZ}_\sigma]_{z^r}$.

For each $\sigma$, we can write $\mathsf{SQ}_\sigma$ in terms of the $\mathsf{FZ}_\sigma$. For example,

$$\begin{aligned}
\mathsf{SQ}_{(111)} &= -\frac{1}{6}\{C_3\}_\kappa + \frac{1}{2}\{C_2\}_\kappa\{C_1\}_\kappa - \frac{1}{6}\{C_1\}_\kappa^3 \\
&= \frac{4}{3}\kappa_1 z + \frac{1}{3}\{C\}_\kappa - \frac{1}{3}\{C^3\}_\kappa + \frac{1}{2}(\kappa_0 - \{C^2\}_\kappa)\{C\}_\kappa - \frac{1}{6}\{C\}_\kappa^3 \\
&= \left(\frac{4}{3}\kappa_1 z\right) + \left(\left(\frac{1}{3} + \frac{\kappa_0}{2}\right)\{C\}_\kappa\right) \\
&\qquad + \left(-\frac{1}{3}\{C^3\}_\kappa - \frac{1}{2}\{C^2\}_\kappa\{C\}_\kappa - \frac{1}{6}\{C\}_\kappa^3\right) \\
&= \frac{4}{3}\kappa_1 z\,\mathsf{FZ}_\emptyset + \left(-\frac{1}{3} - \frac{\kappa_0}{2}\right)\mathsf{FZ}_{(1)} + \mathsf{FZ}_{(111)}\,.
\end{aligned}$$

We then obtain a corresponding linear relation between the relations themselves:

$$[E\cdot\mathsf{SQ}_{(111)}]_{z^r} = \frac{4}{3}\kappa_1[E\cdot\mathsf{FZ}_\emptyset]_{z^{r-1}} + \left(-\frac{1}{3} - \frac{\kappa_0}{2}\right)[E\cdot\mathsf{FZ}_{(1)}]_{z^r} + [E\cdot\mathsf{FZ}_{(111)}]_{z^r}.$$

Constructing such linear combinations in general is not hard. When expanded in terms of $C$ as in the above example, $\mathsf{FZ}_\sigma$ always contains exactly one term of the form

$$\{z^{a_1}C\}_\kappa\{z^{a_2}C\}_\kappa\cdots\{z^{a_m}C\}_\kappa\,. \tag{12}$$

All the other terms involve higher powers of $C$. If we expand $\mathsf{SQ}_\sigma$ in terms of $C$, we can look at the terms of the form (12) which appear to determine how to write the $\mathsf{SQ}_\sigma$ as a linear combination of the $\mathsf{FZ}_{\widehat{\sigma}}$.

We must check the terms involving higher powers of $C$ also match up. The matching amounts to proving an identity between the coefficients of $C_i$ when expressed a polynomial in $C$. Define polynomials $f_{ij} \in \mathbb{Z}[z]$ by

$$C_i = \sum_{j=0}^{i} f_{ij} C^j,$$

and let

$$f = 1 + \sum_{i,j\geq 1} \frac{(-1)^{j-1} f_{ij}}{i!(j-1)!} x^i y^j.$$

**Lemma 5.** *There exists a power series $g \in \mathbb{Q}[z][[x]]$ such that $f = e^{yg}$.*

The Lemma (which can be proven in straightforward fashion using the differential equation satisfied by $f$) is the precise consistency statement needed to express the $\mathsf{SQ}$ relations as linear combinations of the

FZ relations. The associated matrix is triangular with respect to the partial ordering of partitions by size, and the diagonal entries are easily computed to be equal to 1. Hence, the matrix is invertible. We conclude the SQ relations are equivalent to the FZ relations.

## References


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