A MIRROR DUAL OF SINGLE HURWITZ NUMBERS

MOTOHICO MULASE

ABSTRACT. These are the lectures delivered at the *Summer School on Moduli of Curves and Gromov-Witten Theory* that took place at the Fourier Institute in Grenoble, France, in summer 2011. The main purpose of these lectures is to explain an idea that mirror symmetry is the Laplace transform for a certain class of mathematical problems, by going through a concrete example of single Hurwitz numbers. We construct a B-model mirror partner of the single Hurwitz theory. The key observation is that the Laplace transform of the combinatorial *cut-and-join* equation is equivalent to the Eynard-Orantin topological recursion that lives on the B-model side.

CONTENTS

1. INTRODUCTION

Mathematics thrives on mysteries. Mirror symmetry has been a great mystery for a long time, and has provided a driving force in many areas of mathematics. Even after more than two decades since its conception in physics, still it produces new mysteries for mathematicians to solve.

One of such new mysteries is the **remodeling conjecture** of B-model. This idea has been developed by Mariño [57], Bouchard-Klemm-Mariño-Pasquetti [5], and Bouchard-Mariño [6], based on the theory of topological recursion formulas of Eynard and Orantin [24, 27]. The remodeling conjecture states that the open and closed Gromov-Witten invariants of a toric Calabi-Yau threefold can be captured by the Eynard-Orantin topological recursion as a B-model that is constructed on the mirror curve.

The goal of these lectures is to present an idea that **mirror symmetry is the Laplace transform**. Instead of developing a general theory, we are focused on examining this idea

2000 *Mathematics Subject Classification.* Primary: 14H15, 14N35, 05C30, 11P21; Secondary: 81T30.

by going through a concrete example of single Hurwitz numbers here. Thus our main question is the following.

Question 1.1. *What is the mirror dual of the theory of single Hurwitz numbers?*

Here our usage of the terminology *mirror symmetry*, which is not conventional, requires an explanation. At least on surface our question does not seem to appeal to the idea of the *homological mirror symmetry* [49] directly. About a year ago, Boris Dubrovin and the author had the following conversation.

Mulase: *Hi Boris, good to see you! At last I think I am coming close to understanding what mirror symmetry is.*
Dubrovin: *Good to see you! And what do you think about mirror symmetry?*
Mulase: *It is the Laplace transform.*
Dubrovin: *Do you think so, too? But I have been saying so for the last 15 years!*
Mulase: *Oh, have you? But I'm not talking about the Fourier transform or the T-duality. It's the Laplace transform.*
Dubrovin: *I know.*
Mulase: *All right, then let's check if we have the same understanding. Question: What is the mirror dual of a point?*
Dubrovin: *It is the Lax operator of the KdV equations that was identified by Kontsevich.*
Mulase: *An operator is the mirror dual? Ah, I think you mean* $x = y^2$*, don't you?*
Dubrovin: *Yes, indeed! Now it is my turn to ask you a question. What is the mirror dual of the Weil-Petersson volume of the moduli space of bordered hyperbolic surfaces discovered by Mirzakhani?*
Mulase: *The sine function* $x = \sin y$*.*
Dubrovin: *Exactly!*
Mulase: *And in all these cases, the mirror symmetry is the Laplace transform.*
Dubrovin: *Of course it is.*
Mulase: *A, ha! Then we seem to have the same understanding of the mirror symmetry.*
Dubrovin: *Apparently we do!*

In the spirit of the above dialogue, the answer to our main question can be given as

Theorem 1.2 ([6, 25, 65])**.** *The mirror dual to single Hurwitz numbers is the Lambert function* $x = ye^{1-y}$*.*

A mathematical picture has emerged in the last few years since the discoveries of Eynard-Orantin [27], Mariño [57], Bouchard-Klemm-Mariño-Pasquetti [5] and Bouchard-Mariño [6] in physics, and many mathematical efforts including [11, 12, 25, 63, 54, 55, 64, 65, 88, 89, 90]. As a working hypothesis, we phrase it in the form of a principle.

Principle 1.3. *For a number of interesting cases, we have the following general structure.*

- *On the A-model side of topological string theory, we have a class of mathematical problems arising from combinatorics, geometry, and topology. The common feature of these problems is that they are somehow related to a lattice point counting of a collection of polytopes.*

- *On the B-model side, we have a universal theory due to Eynard and Orantin* [27]*. It is a framework of the recursion formula of a particular kind that is based on a* ***spectral curve*** *and two meromorphic functions on it that immerse an open part of the curve into a plane.*
- *The passage from A-model to B-model, i.e., the mirror symmetry operation of the class of problems that we are concerned, is given by the* ***Laplace transform****. The spectral curve on the B-models side is defined as the* ***Riemann surface*** *of the Laplace transform, which means that it is the domain of holomorphy of the Laplace transformed function.*

By now there are many examples of mathematical problems that fall in to this principle. Among them is the theory of single Hurwitz numbers that we are going to study in these lectures. Besides Hurwitz numbers, such examples as counting of Grothendieck's dessins d'enfants [11, 63, 67, 68, 69] and topological vertex [12, 88] have been mathematically established. Among somewhat more speculative examples we find HOMFLY polynomials of torus knots [9] and the Gromov-Witten invariants of $\mathbb{P}^1$ [70]. The study of single Hurwitz numbers exhibits all important ingredients found in these examples.

In Lecture 1, single Hurwitz numbers are defined, and a combinatorial equation that they satisfy (the cut-and-join equation) is proved. In Lecture 2, the Laplace transform of the Hurwitz numbers is computed. The Laplace transformed holomorphic functions live on the mirror B-side of the model, according to Princeple 1.3. The Lambert curve is defined as the domain of holomorphy of these holomorphic functions. In Lecture 3, we give the Laplace transform of the cut and join equation. The result is a simple polynomial recursion formula and is equivalent to the Eynard-Orantin formula for the Lambert curve. We also give a straightforward derivation of the Witten-Kontsevich theorem on the ψ-class intersection numbers [16, 48, 86], and the λ_g-formula of Faber and Pandharipande [29, 30], using the Eynard-Orantin topological recursion.

2. Single Hurwitz numbers

In this lecture we define the single Hurwitz numbers that we consider in these lectures, and derive a combinatorial equation that they satisfy, known as the *cut-and-join equation*. In all examples of Principle 1.3 we know so far, the A-model side always has a series of combinatorial equations that should uniquely determine the quantities in question, at least theoretically. But in practice solving these equations is quite complicated. As we develop in these lectures, the Laplace transform changes these equations to a *topological recursion* in the B-model side, which is an inductive formula based on the absolute value of the Euler characteristic of a punctured surface.

A *single Hurwitz number* counts the number of certain type of meromorphic functions defined on an algebraic curve C of genus g. Let $\mu = (\mu_1, \dots, \mu_\ell) \in \mathbb{Z}_+^\ell$ be a *partition* of a positive integer d of length ℓ. This means that $|\mu| \overset{\text{def}}{=} \mu_1 + \cdots + \mu_\ell = d$. Instead of ordering parts of μ in the decreasing order, we consider them as a vector consisting of ℓ positive integers. By a *Hurwitz covering* of type (g, μ) we mean a meromorphic function $h : C \to \mathbb{C}$ that has ℓ labelled poles $\{x_i, \dots, x_\ell\}$, such that the order of x_i is μ_i for every

$i = 1, \dots, \ell$, and that except for these poles, the holomorphic 1-form dh has simple zeros on $C \setminus \{x_i, \dots, x_\ell\}$ with distinct critical values of h. A meromorphic function of C is a holomorphic map of C onto $\mathbb{P}^1$. In algebraic geometry the situation described above is summarized as follows: $h : C \to \mathbb{P}^1$ is a *ramified covering* of $\mathbb{P}^1$, simply ramified except for $\infty \in \mathbb{P}^1$. We identify two Hurwitz coverings $h_1 : C_1 \to \mathbb{P}^1$ with poles at $\{x_1, \dots, x_\ell\}$ and $h_2 : C_2 \to \mathbb{P}^1$ with poles at $\{y_1, \dots, y_\ell\}$ if there is a biholomorphic map $\phi : C_1 \xrightarrow{\sim} C_2$ such that $\phi(x_i) = y_i$, $i = 1, \dots, \ell$, and

$$\begin{array}{ccc} C_1 & \xrightarrow[\sim]{\phi} & C_2 \\ & {\scriptstyle h_1}\searrow \quad \swarrow{\scriptstyle h_2} & \\ & \mathbb{P}^1 & \end{array} .$$

When $C_1 = C_2$, $x_i = y_i$, and $h_1 = h_2 = h$, such a biholomorphic map ϕ is called an *automorphism* of a Hurwitz covering h. Since biholomorphic Hurwitz coverings are identified, we need to count a Hurwitz covering with the automorphism factor $1/|\mathrm{Aut}(h)|$ by consistency. And when $\phi : C_1 \to C_2$ is merely a homeomorphism, we say h_1 and h_2 have the same *topological type*.

We are calling a meromorphic function a *covering*. This is because if we remove the critical values of h (including ∞) from $\mathbb{P}^1$, then on this open set h becomes a topological covering. More precisely, let $B = \{z_1, \dots, z_r, \infty\}$ denote the set of distinct critical values of h. Then

$$h : C \setminus h^{-1}(B) \longrightarrow \mathbb{P}^1 \setminus B$$

is a topological covering of degree d. Each $z_k \in \mathbb{P}^1$ is a *branched point* of h, and a critical point (i.e., a zero of dh) is called a *ramification point* of h. Since dh has only simple zeros with distinct critical values of h, the number of ramification points of h, except for the poles, is equal to the number of branched points, which we denote by r. Therefore, $h^{-1}(z_k)$ consists of $d-1$ points. Then by comparing the Euler characteristic of the covering space and its base space, we obtain

$$d(2 - r - 1) = d \cdot \chi(\mathbb{P}^1 \setminus B) = \chi(C \setminus h^{-1}(B)) = 2 - 2g - r(d-1) - \ell,$$

or the Riemann-Hurwitz formula

$$r = 2g - 2 + |\mu| + \ell. \tag{2.1}$$

Let us define

Definition 2.1. The single Hurwitz number of type (g, μ) for $g \geq 0$ and $\mu \in \mathbb{Z}_+^\ell$ that we consider in these lectures is

$$H_g(\mu) = \frac{1}{r(g,\mu)!} \sum_{[h] \text{ type } (g,\mu)} \frac{1}{|\mathrm{Aut}(h)|}, \tag{2.2}$$

where the sum runs all topological equivalence classes of Hurwitz coverings of type (g, μ). Here

$$r = r(g, \mu) = 2g - 2 + (\mu_1 + \cdots + \mu_\ell) + \ell$$

is the number of simple ramification points of h.

Remark 2.2. Our definition of single Hurwitz numbers differ from the standard definition by two automorphism factors. The quantity $h_{g,\mu}$ of [20] and $H_g(\mu)$ are related by

$$H_g(\mu) = \frac{|\mathrm{Aut}(\mu)|}{r!}\, h_{g,\mu},$$

where $\mathrm{Aut}(\mu)$ is the group of permutations that permutes equal parts of μ considered as a partition. This is due to the convention that we label the poles of h and consider $\mu \in \mathbb{Z}_+^\ell$ as a vector, while we do not label simple ramification points.

Note that interchanging the entries of μ means permutation of the label of the poles $\{x_1, \dots, x_\ell\}$ of h. Thus it does not affect the count of single Hurwitz numbers. Therefore, as a function in $\mu \in \mathbb{Z}_+^\ell$, $H_g(\mu)$ is a symmetric function.

Single Hurwitz numbers satisfy a simple equation, known as the cut-and-join equation [33, 84]. Here we give it in the format used in [65].

Proposition 2.3 (Cut-and-join equation)**.** *Single Hurwitz numbers satisfy*

$$r(g,\mu)H_g(\mu) = \sum_{i<j}(\mu_i+\mu_j)H_g(\mu(\hat{i},\hat{j}),\mu_i+\mu_j) + \frac{1}{2}\sum_{i=1}^{\ell}\sum_{\alpha+\beta=\mu_i}\alpha\beta\left[H_{g-1}(\mu(\hat{i}),\alpha,\beta) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=\mu(\hat{i})}} H_{g_1}(I,\alpha)H_{g_2}(J,\beta)\right]. \tag{2.3}$$

Here we use the following notations.

- *$\mu(\hat{i})$ is the vector of $\ell-1$ entries obtained by deleting the i-th entry μ_i.*
- *$(\mu(\hat{i}),\alpha,\beta)$ is the vector of $\ell+1$ entries obtained by appending two new entries α and β to $\mu(\hat{i})$.*
- *$\mu(\hat{i},\hat{j})$ is the vector of $\ell-2$ entries obtained by deleting the i-th and the j-th entrs μ_i and μ_j.*
- *$(\mu(\hat{i},\hat{j}),\mu_i+\mu_j)$ is the vector of $\ell-1$ entries obtained by appending a new entry $\mu_i+\mu_j$ to $\mu(\hat{i},\hat{j})$.*

The final sum is over all partitions of g into non-negative integers g_1 and g_2, and a disjoint union decomposition of entries of $\mu(\hat{i})$ as a set, allowing the empty set.

Remark 2.4. Since $H_g(\mu)$ is a symmetric function, the way we append a new entry to a vector does not affect the function value.

The idea of the formula is to reduce the number r of simple ramification points. Note that

$$h : C \setminus h^{-1}(B) \longrightarrow \mathbb{P}^1 \setminus B$$

is a topological covering of degree d. Therefore, it is obtained by a representation

$$\rho : \pi_1(\mathbb{P}^1 \setminus B) \longrightarrow S_d,$$

where S_d is the permutation group of d letters. The covering space X_ρ of $\mathbb{P}^1 \setminus B$ is obtained by the quotient construction

$$X_\rho = \widetilde{X} \times_{\pi_1(\mathbb{P}^1\setminus B)} [d],$$

where $\widetilde{X}$ is the universal covering space of $\mathbb{P}^1 \setminus B$, and $[d] = \{1, 2, \dots, d\}$ is the index set on which $\pi_1(\mathbb{P}^1 \setminus B)$ acts via the representation ρ.

To make X_ρ a Hurwitz covering, we need to specify the monodromy of the representation at each branch point of B. Let $\{\gamma_1, \dots, \gamma_r, \gamma_\infty\}$ denote the collection of non-intersecting loops on $\mathbb{P}^1$, starting from $0 \in \mathbb{P}^1$, rotating z_k counter-clockwise, and coming back to 0, for each $k = 1, \dots, r$. The last loop γ_∞ does the same for $\infty \in \mathbb{P}^1$. Since $\mathbb{P}^1$ is simply connected, we have

$$\pi_1(\mathbb{P}^1 \setminus B) = \langle \gamma_1, \dots, \gamma_r, \gamma_\infty | \gamma_1 \cdots \gamma_r \cdot \gamma_\infty = 1\rangle.$$

Since X_ρ must have r simple ramification points over $\{z_1, \dots, z_r\}$, the monodromy at z_k is given by a transposition

$$\rho(\gamma_k) = (a_k b_k) \in S_d,$$

where $a_k, b_k \in [d]$ and all other indices are fixed by $\rho(\gamma_k)$. To impose the condition on poles $\{x_1, \dots, x_\ell\}$ of h, we need

$$\rho(\gamma_\infty) = c_1 c_2 \cdots c_\ell,$$

where $c_1, \dots, c_\ell$ are disjoint cycles of S_d of length $\mu_1, \dots, \mu_\ell$, respectively.

We want to reduce the number r by one. To do so, we simply merge z_r to ∞. The monodromy at ∞ then changes from $c_1 c_2 \cdots c_\ell$ to $(ab) \cdot c_1 c_2 \cdots c_\ell$, where $(ab) = (a_r b_r)$ is the transposition corresponding to γ_r. There are two cases we have now:

(1) Join case: a and b belong to two disjoint cycles, say $a \in c_i$ and $b \in c_j$;
(2) Cut case: both a and b belong to the same cycle, say c_i.

An elementary computation shows that $(ab)c_i c_j$ is a single cycle of length $\mu_i + \mu_j$. For the second case, the result depends on how far a and b are apart in cycle c_i. If b appears α slots after a with respect to the cyclic ordering, then

$$(ab)c_i = c_\alpha c_\beta, \tag{2.4}$$

where $\alpha + \beta = \mu_i$. The cycles c_α and c_β are disjoint of length α and β, respectively, and $a \in c_\alpha$, and $b \in c_\beta$. Note that everything is symmetric with respect to interchanging a and b. With this preparation, we can now give the proof of (2.3).

The right-hand side of (2.3) represents the set of all monodromy representations obtained by merging one of the branch points z_k to ∞. The factor r on the left-hand side represents the choice of z_k.

Since we are reducing $r = 2g - 2 + d + \ell$ by one without changing d, there are three different ways of reduction:

$$(g, \ell) \longmapsto (g, \ell - 1) \tag{2.5}$$

$$(g, \ell) \longmapsto (g - 1, \ell + 1) \tag{2.6}$$

$$(g,\ell) \longmapsto (g_1,\ell_1+1)+(g_2,\ell_2+1), \text{ where } \begin{cases} g_1+g_2=g \\ \ell_1+\ell_2=\ell-1. \end{cases} \tag{2.7}$$

The first reduction (2.5) is exactly the first line of the right-hand side of (2.3), which corresponds to the join case. Two cycles of length μ_i and μ_j are joined to form a longer cycle of length $\mu_i+\mu_j$. Note that the number a has to be recorded somewhere in this long cycle. This explains the factor $\mu_i+\mu_j$. Then the number b is automatically recorded, because it is the entry appearing exactly μ_i slots after a in this long cycle.

The second line of the right-hand side of (2.3) represents the cut cases. In (2.4), we have α choices for a and β choices for b. The symmetry of interchanging a and b explains the factor $\frac{1}{2}$. The first term of the second line of (2.3) corresponds to (2.6).

The second term of the second line corresponde to (2.7). Note that in this situation, merging a branched point to ∞ breaks the connectivity of the Hurwitz covering. We have two ramified coverings $h_1 : C_1 \to \mathbb{P}^1$ of degree d_1 and genus g_1 with ℓ_1+1 poles, and $h_2 : C_2 \to \mathbb{P}^1$ of degree d_2 and genus g_2 with ℓ_2+1 poles. If we denote by r_i the number of simple ramifications points of h_i for $i=1,2$, then we have

$$\begin{aligned} r_1 &= 2g_1-2+d_1+\ell_1+1 \\ +)\ \ r_2 &= 2g_2-2+d_2+\ell_2+1 \\ r-1 &= 2g-2+d+\ell-1. \end{aligned}$$

This completes the proof of (2.3).

Remark 2.5. The reduction of the number r of simple ramification points by one is exactly reflecting the reduction of the Euler characteristic of the punctured surface $X_\rho = C \setminus h^{-1}(B)$ appearing in our consideration by one. Since we do not change the degree d of the covering, the reduction of r is simply reducing $2g-2+\ell$ by one.

3. The Laplace transform of the single Hurwitz numbers

In the second lecture, we compute the Laplace transform of the single Hurwitz number $H_g(\mu)$, considered as a function in $\mu \in \mathbb{Z}_+^\ell$. According to Principle 1.3, the result should give us the mirror dual of single Hurwitz numbers. We explain where the Lambert curve $x = ye^{1-y}$ comes from, and its essential role in computing the Laplace transform. The most surprising feature is that the result of the Laplace transform of $H_g(\mu)$ is a polynomial if $2g-2+\ell > 0$. This polynomiality produces powerful consequences, which is the main subject of the final lecture.

The most important reason why we are interested in Hurwitz numbers in a summer school on moduli of curves and Gromov-Witten invariants is because of the theorem due to Ekedahl, Lando, Shapiro and Vainshtein, that relates the single Hurwitz numbers with the intersection numbers of tautological classes on the moduli space of curves. Let us recall the necessary notations here. For more detailed explanation on these subjects, we refer to other talks given in this school.

Our main object is the moduli stack $\overline{\mathcal{M}}_{g,\ell}$ consisting of stable algebraic curves of genus $g \geq 0$ with $\ell \geq 1$ distinct smooth labeled points. Forgetting the last labeled point on a curve gives a canonical projection

$$\pi : \overline{\mathcal{M}}_{g,\ell+1} \longrightarrow \overline{\mathcal{M}}_{g,\ell}.$$

Since the last labeled point moves on the curve, the projection π can be considered as a universal family of ℓ-pointed curves. This is because for each point $[C, (x_1, \ldots, x_\ell)] \in \overline{\mathcal{M}}_{g,\ell}$, the fiber of π is indeed C itself.

If $x_{\ell+1} \in C$ is a smooth point of C other than $\{x_1, \ldots, x_\ell\}$, then this point represents an element $[C, (x_1, \ldots, x_{\ell+1})] \in \overline{\mathcal{M}}_{g,\ell+1}$. If $x_{\ell+1} = x_i$ for some $i = 1, \ldots, \ell$, then this point represents a stable curve obtained by attaching a rational curve $\mathbb{P}^1$ to C at the original location of x_i, while carrying three special points on it. One is the singular point at which C and $\mathbb{P}^1$ intersects. The other two points are labeled as x_i and $x_{\ell+1}$. And if $x_{\ell+1} \in C$ coincides with one of the nodal points of C, say $x \in C$, then this point represents another stable curve. This time, consider the local normalization $\widetilde{C} \to C$ about the singular point $x \in C$, and let x_+ and x_- be the two points in the fiber. The stable curve we have is the curve $\widetilde{C} \cup \mathbb{P}^1$, where the two curves intersect at x_+ and x_-. The labeled point $x_{\ell+1}$ is placed on the attached $\mathbb{P}^1$ different from these two singular points.

A universal family produces what we call the *tautological* bundles on the moduli space. The cotangent sheaf $T^*C = \omega_C$ of each fiber of π is glued together to form a *relative dualizing sheaf* ω on $\overline{\mathcal{M}}_{g,\ell+1}$. The push-forward

$$\mathbb{E} = \pi_* \omega$$

is such a tautological vector bundle on $\overline{\mathcal{M}}_{g,\ell}$ of fiber dimension

$$\dim H^0(C, \omega_C) = g,$$

and is called the *Hodge bundle* on $\overline{\mathcal{M}}_{g,\ell}$. By assigning $x_{\ell+1} = x_i$ to each $[C, (x_1, \ldots, x_\ell)] \in \overline{\mathcal{M}}_{g,\ell}$, we construct a section

$$\sigma_i : \overline{\mathcal{M}}_{g,\ell} \longrightarrow \overline{\mathcal{M}}_{g,\ell+1}.$$

It defines another tautological bundle

$$\mathbb{L}_i = \sigma^*(\omega)$$

on $\overline{\mathcal{M}}_{g,\ell}$. The fiber of $\mathbb{L}_i$ at $[C, (x_1, \ldots, x_\ell)]$ is identified with the cotangent line $T^*_{x_i} C$.

The *tautological classes* of $\overline{\mathcal{M}}_{g,\ell}$ are rational cohomology classes including

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,\ell}, \mathbb{Q}) \quad \text{and} \quad \lambda_j = c_j(\mathbb{E}) \in H^{2j}(\overline{\mathcal{M}}_{g,\ell}, \mathbb{Q}).$$

In these lectures we do not consider the other classes, such as the κ-classes.

With these notational preparations, we can now state an amazing theorem.

Theorem 3.1 (The ELSV formula [20])**.** *The single Hurwitz numbers are expressible as the intersection numbers of tautological classes on the moduli space* $\overline{\mathcal{M}}_{g,\ell}$ *as follows. Let*

$\mu \in \mathbb{Z}_+^\ell$ be a positive integer vector. Then we have

$$\begin{aligned} H_g(\mu) &= \prod_{i=1}^{\ell} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,\ell}} \frac{\sum_{j=0}^{g}(-1)^j \lambda_j}{\prod_{i=1}^{\ell}(1-\mu_i\psi_i)} \\ &= \sum_{n_1,\dots,n_\ell \geq 0} \sum_{j=0}^{g} (-1)^j \langle \tau_{n_1} \cdots \tau_{n_\ell} \lambda_j \rangle_{g,\ell} \prod_{i=1}^{\ell} \frac{\mu_i^{\mu_i+n_i}}{\mu_i!}. \end{aligned} \tag{3.1}$$

Here we use Witten's symbol

$$\langle \tau_{n_1} \cdots \tau_{n_\ell} \lambda_j \rangle_{g,\ell} = \int_{\overline{\mathcal{M}}_{g,\ell}} c_1(\mathbb{L}_1)^{n_1} \cdots c_1(\mathbb{L}_\ell)^{n_\ell} \cdot c_j(\mathbb{E}).$$

It is 0 unless $n_1 + \cdots + n_\ell + j = 3g - 3 + \ell$.

Remark 3.2. It is not our purpose to give a proof of the ELSV formula in these lectures. There are excellent articles by now about this remarkable formula. We refer to [37, 52, 73].

To explore the mirror partner to single Hurwitz numbers, we wish to compute the Laplace transform of the ELSV formula. Let us recall Stirling's formula

$$\frac{k^{k+n}}{k!} e^{-k} \sim \frac{1}{\sqrt{2\pi}}\, k^{n-\frac{1}{2}}, \qquad k >> 0 \tag{3.2}$$

for a fixed n.

Definition 3.3. For a complex parameter w with $Re(w) > 0$, we define

$$\xi_n(w) = \sum_{k=1}^{\infty} \frac{k^{k+n}}{k!} e^{-k} e^{-kw}. \tag{3.3}$$

Because of Stirling's formula (3.2), we expect that asymptotically near $w \sim 0$,

$$\xi_n(w) \sim \int_0^\infty \frac{1}{\sqrt{2\pi}} x^{n-\frac{1}{2}} e^{-xw} dx.$$

To illustrate our strategy of computing the Laplace transform, let us first compute

$$f_n(w) = \int_0^\infty x^n e^{-xw} dx.$$

We notice that

$$-\frac{d}{dw} f_n(w) = f_{n+1}(w).$$

Therefore, if we know $f_0(w)$, then we can calculate all $f_n(w)$ for $n > 0$. Of course we have

$$f_0(w) = \frac{1}{w}.$$

Therefore, we immediately conclude that

$$f_n(w) = \frac{\Gamma(n+1)}{w^{n+1}}, \tag{3.4}$$

which satisfies the initial condition and the differential recursion formula. The important fact in complex analysis is that when we derive a formula like (3.4), then it holds for an arbitrary n. In particular, we have

$$\xi_n(w) \sim \int_0^\infty \frac{1}{\sqrt{2\pi}} x^{n-\frac{1}{2}} e^{-xw} dx = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{2\pi}\; w^{n+\frac{1}{2}}}.$$

From this asymptotic expression, we learn that ξ_n has an expansion in $w^{-\frac{1}{2}}$. Thus to identify the domain of holomorphy, we wish to find a natural coordinate that behaves like $w^{-\frac{1}{2}}$.

Note that for every $n > 0$, the defining summation of $\xi_n(w)$ in (3.3) can be taken from $k = 0$ to ∞. For $n = 0$, the $k = 0$ term contributes 1 in the summation. So let us define

$$t - 1 = \xi_0(w) = \sum_{k=1}^\infty \frac{k^k}{k!} e^{-k} e^{-kw}. \tag{3.5}$$

Then the computation of the Laplace transform $\xi_n(w)$ is reduced to finding the inverse function $w = w(t)$ of (3.5), because all we need after identifying the inverse is to differentiate $\xi_0(w)$ n-times.

Here we utilize the *Lagrange Inversion Formula.*

Theorem 3.4 (The Lagrange Inversion Formula)**.** *Let $f(y)$ be a holomorphic function defined near $y = 0$ such that $f(0) = 0$ and $f'(0) \neq 0$. Then the inverse of the function*

$$x = \frac{y}{f(y)}$$

is given by

$$y = \sum_{k=1}^\infty \left[\frac{d^{k-1}}{dy^{k-1}} (f(y))^k \right]_{y=0} \frac{x^k}{k!}.$$

We give a proof of this formula in Appendix. For our purpose, let us consider the case $f(y) = e^{y-1}$. The function

$$x = ye^{1-y} \tag{3.6}$$

is called the *Lambert function.* Then the Lagrange Inversion Formula immediately tells us that its inverse function is given by

$$y = \sum_{k=1}^\infty \frac{k^{k-1}}{k!} e^{-k} x^k.$$

So if we substitute

$$x = e^{-w}, \tag{3.7}$$

then we have

$$y = \sum_{k=1}^\infty \frac{k^{k-1}}{k!} e^{-k} e^{-kw} = \xi_{-1}(w). \tag{3.8}$$

The differential of the Lambert function gives

$$dx = (1-y)e^{1-y}dy.$$

Therefore, we have

$$-\frac{d}{dw} = x\frac{d}{dx} = \frac{y}{1-y}\frac{d}{dy}. \tag{3.9}$$

Since

$$t-1 = \xi_0(w) = -\frac{d}{dw}\xi_{-1}(w) = \frac{y}{1-y},$$

we conclude that

$$y = \frac{t-1}{t}. \tag{3.10}$$

As a consequence, we complete

$$-\frac{d}{dw} = x\frac{d}{dx} = \frac{y}{1-y}\frac{d}{dy} = t^2(t-1)\frac{d}{dt}. \tag{3.11}$$

We also obtain a formula for w in terms of t, since $e^{-w} = ye^{1-y}$.

$$w = -\frac{1}{t} - \log\left(1-\frac{1}{t}\right) = \sum_{m=2}^{\infty}\frac{1}{m}\,\frac{1}{t^m}. \tag{3.12}$$

Notice that near $w=0$, we have $t \sim \sqrt{2w}$, as we wished. Now we can calculate $\xi_n(w)$ in terms of t for every $n \geq 0$.

Definition 3.5. As a function in t, we denote

$$\hat{\xi}_n(t) = \xi_n(w(t)). \tag{3.13}$$

Proposition 3.6. *For every $n \geq 0$, $\hat{\xi}_n(t)$ is a polynomial in t of degree $2n+1$. For $n > 0$ it has an expansion*

$$\hat{\xi}_n(t) = (2n-1)!!t^{2n+1} - \frac{(2n+1)!!}{3}\,t^{2n} + \cdots + a_n t^{n+2} + (-1)^n n!\,t^{n+1}, \tag{3.14}$$

where a_n is defined by

$$a_n = -\big[(n+1)a_{n-1} + (-1)^n n!\big]$$

and is identified as the sequence A001705 or A081047 of the On-Line Encyclopedia of Integer Sequences.

Proof. It is a straightforward calculation of

$$\hat{\xi}_n(t) = t^2(t-1)\frac{d}{dt}\hat{\xi}_{n-1}(t) = \left(t^2(t-1)\frac{d}{dt}\right)^n (t-1).$$

□

Remark 3.7. Other than those identified in (3.14), we do not know the general coefficients of $\hat{\xi}_n(t)$.

Theorem 3.8 (Laplace transform of single Hurwitz numbers)**.** *The Laplace transform of single Hurwitz numbers is given by*

$$
\begin{aligned}
F_{g,\ell}(t) = F_{g,\ell}(t_1, t_2, \dots, t_\ell) &= \sum_{\mu \in \mathbb{Z}_+^\ell} H_g(\mu) e^{-|\mu|} e^{-(\mu_1 w_1 + \cdots + \mu_\ell w_\ell)} \\
&= \sum_{n_1,\dots,n_\ell \geq 0} \sum_{j=0}^{g} (-1)^j \langle \tau_{n_1} \cdots \tau_{n_\ell} \lambda_j \rangle_{g,\ell} \prod_{i=1}^{\ell} \hat{\xi}_{n_i}(t_i).
\end{aligned}
\tag{3.15}
$$

This is a polynomial of degree $3(2g-2+\ell)$*. Its highest degree terms form a homogeneous polynomial*

$$
F_{g,\ell}^{top}(t) = \sum_{n_1+\cdots+n_\ell = 3g-3+\ell} \langle \tau_{n_1} \cdots \tau_{n_\ell} \rangle_{g,\ell} \prod_{i=1}^{\ell} (2n_i - 1)!!\, t_i^{2n_i+1}, \tag{3.16}
$$

and the lowest degree terms also form a homogeneous polynomial

$$
F_{g,\ell}^{lowest}(t) = \sum_{n_1+\cdots+n_\ell = 2g-3+\ell} (-1)^{3g-3+\ell} \langle \tau_{n_1} \cdots \tau_{n_\ell} \lambda_g \rangle_{g,\ell} \prod_{i=1}^{\ell} n_i!\, t_i^{n_i+1}. \tag{3.17}
$$

Remark 3.9. We note that there is no a priori reason for the Laplace transform of $H_g(\mu)$ to be a polynomial. Because it is a polynomial, we obtain a polynomial generating function of linear Hodge integrals $\langle \tau_{n_1} \cdots \tau_{n_\ell} \lambda_j \rangle_{g,\ell}$. We utilize this polynomiality in the final lecture.

Remark 3.10. The existence of the polynomials $\hat{\xi}_n(t)$ in (3.15) is significant, because it reflects the ELSV formula (3.1). Indeed, Eynard [24] predicts that this is the general structure of the Eynard-Orantin formalism.

4. The power of the remodeled B-model

In this third lecture, we will first compute the Laplace transform of the cut-and-join equation. The result turns out to be a simple polynomial recursion formula. Here again there is no a priori reason for the result to be a polynomial relation, because the cut-and-join equation (2.3) contains unstable geometries, and they contribute non-polynomial terms after the Laplace transform.

We then remark that the Laplace transform of the cut-and-join equation is equivalent to the Eynard-Orantin topological recursion formula [27, 24] based on the Lambert curve (3.6) as the spectral curve of the theory. This fact solves the Bouchard-Mariño conjecture [6] of Hurwitz numbers [25, 65], and establishes the Lambert curve as the remodeled B-model corresponding to single Hurwitz numbers through mirror symmetry.

The unexpected power [65] of the topological recursion formula appearing in our context is the following.

(1) It restricts to the top degree terms, and recovers the Dijkgraaf-Verlinde-Verlinde formula, or the Virasoro constraint condition, for the *psi*-class intersection numbers on $\overline{\mathcal{M}}_{g,\ell}$.

(2) It also restricts to the lowest degree terms, and recovers the λ_g-conjecture of Faber that was proved in [29, 30].

In other words, we obtain a straightforward, simple proofs of the Witten conjecture and Faber's λ_g-conjecture from the Laplace transform of the cut-and-join equation. We note that the Laplace transform contains the information of the large μ asymptotics. Therefore, our proof [65] of the Witten conjecture uses the same idea of Okounkov and Pandharipande [73], yet it is much simpler because we do not have to use any of the asymptotic analyses of matrix integrals, Hurwitz numbers, and graph enumeration.

The proof of the λ_g-conjecture using the topological recursion is still somewhat mysterious. Here again the complicated combinatorics is wiped out and we have a transparent proof.

Let us first state the Laplace transform of the cut-and-join equation.

Theorem 4.1. *The polynomial generating functions of the linear Hodge integrals $F_{g,\ell}(t)$ satisfy the following topological recursion formula*

$$
\begin{aligned}
(4.1)\quad & \left(2g-2+\ell+\sum_{i=1}^{\ell}\frac{1}{t_i}D_i\right)F_{g,\ell}(t_1,t_2,\dots,t_\ell)\\
&= \sum_{i<j}\frac{t_i^2(t_j-1)D_iF_{g,\ell-1}\big(t_{[\ell;\hat{j}]}\big)-t_j^2(t_i-1)D_jF_{g,\ell-1}\big(t_{[\ell;\hat{i}]}\big)}{t_i-t_j}\\
&\quad+\sum_{i=1}^{\ell}\Big[D_{u_1}D_{u_2}F_{g-1,\ell+1}\big(u_1,u_2,t_{[\ell;\hat{i}]}\big)\Big]_{u_1=u_2=t_i}\\
&\quad+\frac{1}{2}\sum_{i=1}^{\ell}\sum_{\substack{g_1+g_2=g\\ J\sqcup K=[\ell;\hat{i}]}}^{\text{stable}} D_iF_{g_1,|J|+1}(t_i,t_J)\cdot D_iF_{g_2,|K|+1}(t_i,t_K),
\end{aligned}
$$

where $D_i = t_i^2(t_i-1)\frac{\partial}{\partial t_i}$. As before, $[\ell]=\{1,\dots,\ell\}$ is the index set, and $[\ell;\hat{i}]$ is the index set obtained by deleting i from $[\ell]$. The last summation is taken over all partitions $g=g_1+g_2$ of the genus g and disjoint union decompositions $J\sqcup K=[\ell;\hat{i}]$ satisfying the stability conditions $2g_1-1+|J|>0$ and $2g_2-1+|K|>0$. For a subset $I\subset[\ell]$ we write $t_I=(t_i)_{i\in I}$.

The biggest difference between the cut-and-join equation (2.3) and the Laplace transformed formula (4.1) is the restriction to stable geometries in the latter. In the case of the cut-and-join equation, the cut case contains $g_1=0$ and $I=\emptyset$. Then $H_{g_2}(J,\beta)$ has the same complexity of $H_g(\mu)$. Thus the cut-and-join equation is simply a relation among Hurwitz numbers.

The new feature of our (4.1) is that it is a genuine recursion formula about linear Hodge integrals. Indeed, we can re-write the formula as follows.

$$(4.2)\quad \sum_{n_{[\ell]}} \langle \tau_{n_{[\ell]}} \Lambda_g^\vee(1) \rangle_{g,\ell} \left((2g-2+\ell)\hat{\xi}_{n_{[\ell]}}(t_{[\ell]}) + \sum_{i=1}^{\ell} \frac{1}{t_i}\ \hat{\xi}_{n_i+1}(t_i)\hat{\xi}_{[\ell;\hat{i}]}(t_{[\ell;\hat{i}]}) \right)$$

$$= \sum_{i<j} \sum_{m,n_{[\ell;\hat{i}\hat{j}]}} \langle \tau_m \tau_{n_{[\ell;\hat{i}\hat{j}]}} \Lambda_g^\vee(1) \rangle_{g,\ell-1} \hat{\xi}_{n_{[\ell;\hat{i}\hat{j}]}}(t_{[\ell;\hat{i}\hat{j}]}) \frac{\hat{\xi}_{m+1}(t_i)\hat{\xi}_0(t_j)t_i^2 - \hat{\xi}_{m+1}(t_j)\hat{\xi}_0(t_i)t_j^2}{t_i - t_j}$$

$$+ \frac{1}{2} \sum_{i=1}^{\ell} \sum_{n_{[\ell;\hat{i}]}} \sum_{a,b} \Bigg(\langle \tau_a \tau_b \tau_{n_{[\ell;\hat{i}]}} \Lambda_{g-1}^\vee(1) \rangle_{g-1,\ell+1}$$

$$+ \sum_{\substack{g_1+g_2=g \\ I \coprod J = [\ell;\hat{i}]}}^{\text{stable}} \langle \tau_a \tau_{n_I} \Lambda_{g_1}^\vee(1) \rangle_{g_1,|I|+1} \langle \tau_b \tau_{n_J} \Lambda_{g_2}^\vee(1) \rangle_{g_2,|J|+1} \Bigg) \hat{\xi}_{a+1}(t_i)\hat{\xi}_{b+1}(t_i)\hat{\xi}_{n_{[\ell;\hat{i}]}}(t_{[\ell;\hat{i}]}),$$

where $[\ell] = \{1, 2 \ldots, \ell\}$ is the index set, and for a subset $I \subset [\ell]$, we denote

$$t_I = (t_i)_{i \in I}, \quad n_I = \{\, n_i \mid i \in I \,\}, \quad \tau_{n_I} = \prod_{i \in I} \tau_{n_i}, \quad \hat{\xi}_{n_I}(t_I) = \prod_{i \in I} \hat{\xi}_{n_i}(t_i).$$

We also use Zhou's symbol

$$\Lambda_g^\vee(1) = 1 - \lambda_1 + \lambda_2 - \cdots + (-1)^g \lambda_g.$$

It is now obvious that in (4.2), the complexity $2g-2+\ell$ is reduced exactly by 1 on the right-hand side. Thus we can compute linear Hodge integrals one by one using this formula.

The Deligne-Mumford stack $\overline{\mathcal{M}}_{g,\ell}$ is defined as the moduli space of *stable* curves satisfying the stability condition $2-2g-\ell < 0$. However, Hurwitz numbers are well defined for *unstable* geometries $(g,\ell) = (0,1)$ and $(0,2)$. It is an elementary exercise to show that

$$H_0\big((d)\big) = \frac{d^{d-1}}{d!}.$$

The ELSV formula remains true for unstable cases by *defining*

$$(4.3)\quad \int_{\overline{\mathcal{M}}_{0,1}} \frac{1}{1-k\psi} = \frac{1}{k^2},$$

$$(4.4)\quad \int_{\overline{\mathcal{M}}_{0,2}} \frac{1}{(1-\mu_1\psi_1)(1-\mu_2\psi_2)} = \frac{1}{\mu_1+\mu_2}.$$

In terms of single Hurwitz numbers, we have

$$H_0\big((\mu_1,\mu_2)\big) = \frac{\mu_1^{\mu_1}}{\mu_1!} \cdot \frac{\mu_2^{\mu_2}}{\mu_2!} \cdot \frac{1}{\mu_1+\mu_2}.$$

From these expressions we can actually compute $F_{0,1}(t)$ and $F_{0,2}(t_1,t_2)$. Since these computations are quite involved, we refer to [25, 65]. What happens often in mathematics is what we call a *miraculous cancellation*. In our situation, when we honestly compute all terms appearing in the Laplace transform in the cut-and-join equation (2.3), somewhat miraculously, all non-polynomial terms cancel out, and the rest becomes an effective recursion formula (4.2).

Now let us move to proving the Witten conjecture and the λ_g-formula using our recursion (4.2). Although these important formulas have been proved a long time ago, we present a new proofs here just to illustrate the use of the topological recursion.

The DVV formula for the Virasoro constraint condition on the ψ-class intersections reads

$$\begin{aligned}(4.5)\quad \langle\tau_{n_{[\ell]}}\rangle_{g,\ell} &= \sum_{j\geq 2}\frac{(2n_1+2n_j-1)!!}{(2n_1+1)!!(2n_j-1)!!}\langle\tau_{n_1+n_j-1}\tau_{n_{[\ell;\hat{1}\hat{j}]}}\rangle_{g,\ell-1}\\ &+\frac{1}{2}\sum_{a+b=n_1-2}\left(\langle\tau_a\tau_b\tau_{n_{[\ell;\hat{1}]}}\rangle_{g-1,\ell+1}+\sum_{\substack{g_1+g_2=g\\ J\sqcup K=[\ell;\hat{1}]}}^{\text{stable}}\langle\tau_a\tau_{n_J}\rangle_{g_1,|J|+1}\cdot\langle\tau_b\tau_{n_K}\rangle_{g_2,|K|+1}\right)\\ &\qquad\times\frac{(2a+1)!!(2b+1)!!}{(2n_1+1)!!}.\end{aligned}$$

Here $[\ell;\hat{1}\hat{j}]=\{2,3,\dots,\hat{j},\dots,\ell\}$, and for a subset $I\subset[\ell]$ we write

$$n_I=(n_i)_{i\in I}\qquad\text{and}\qquad \tau_{n_I}=\prod_{i\in I}\tau_{n_i}.$$

Proposition 4.2. *The DVV formula* (4.5) *is exactly the relation among the top degree coefficients of the recursion* (4.1)*.*

Proof. Choose $n_{[\ell]}$ so that $|n_{[\ell]}|=n_1+n_2+\cdots+n_\ell=3g-3+\ell$. The degree of the left-hand side of (4.1) is $3(2g-2+\ell)+1$. So we compare the coefficients of $t_1^{2n_1+2}\prod_{j\geq 2}t_j^{2n_j+1}$ in the recursion formula. The contribution from the left-hand side of (4.1) is

$$\langle\tau_{n_{[\ell]}}\rangle_{g,\ell}(2n_1+1)!!\prod_{j\geq 2}(2n_j-1)!!.$$

The contribution from the first line of the right-hand side comes from

$$\begin{aligned}&\sum_{j\geq 2}\langle\tau_m\tau_{n_{[\ell;\hat{1}\hat{j}]}}\rangle_{g,\ell-1}(2m+1)!!\frac{t_1^2t_jt_1^{2m+3}-t_j^2t_1t_j^{2m+3}}{t_1-t_j}\\ &\qquad=\sum_{j\geq 2}\langle\tau_m\tau_{n_{[\ell;\hat{1}\hat{j}]}}\rangle_{g,\ell-1}(2m+1)!!t_1t_j\frac{t_1^{2m+4}-t_j^{2m+4}}{t_1-t_j}\\ &\qquad\qquad=\sum_{j\geq 2}\langle\tau_m\tau_{n_{[\ell;\hat{1}\hat{j}]}}\rangle_{g,\ell-1}(2m+1)!!\sum_{a+b=2m+3}t_1^{a+1}t_j^{b+1},\end{aligned}$$

where $m = n_1 + n_j - 1$. The matching term in this formula is $a = 2n_1 + 1$ and $b = 2n_j$. Thus we extract as the coefficient of $t_1^{2n_1+2}\prod_{j\geq 2} t_j^{2n_j+1}$

$$\sum_{j\geq 2} \langle \tau_{n_1+n_j-1}\tau_{n_{[\ell;\hat{1}\hat{j}]}}\rangle_{g,\ell-1}(2n_1+2n_j-1)!!\prod_{k\neq 1,j}(2n_k-1)!!.$$

The contributions of the second and the third lines of the right-hand side of (4.1) are

$$\frac{1}{2}\sum_{a+b=n_1-2}\left(\langle \tau_a\tau_b\tau_{L\setminus\{1\}}\rangle_{g-1,\ell+1} + \frac{1}{2}\sum_{\substack{g_1+g_2=g\\ J\sqcup K=[\ell;\hat{1}]}}^{\text{stable}} \langle \tau_a\tau_{n_J}\rangle_{g_1,|J|+1}\cdot\langle \tau_b\tau_{n_K}\rangle_{g_2,|K|+1}\right)$$
$$\times (2a+1)!!(2b+1)!!\prod_{j\geq 2}(2n_j-1)!!.$$

We have thus recovered the Witten-Kontsevich theorem [16, 48, 86]. □

The λ_g formula [29, 30, **?**, 53] is

$$\langle \tau_{n_{[\ell]}}\lambda_g\rangle_{g,\ell} = \binom{2g-3+\ell}{n_{[\ell]}} b_g, \tag{4.6}$$

where

$$\binom{2g-3+\ell}{n_{[\ell]}} = \binom{2g-3+\ell}{n_1,\dots,n_\ell} \tag{4.7}$$

is the multinomial coefficient, and

$$b_g = \frac{2^{2g-1}-1}{2^{2g-1}}\,\frac{|B_{2g}|}{(2g)!}$$

is a coefficient of the series

$$\sum_{j=0}^{\infty} b_j s^{2j} = \frac{s/2}{\sin(s/2)}.$$

Proposition 4.3. *The lowest degree terms of the topological recursion* (4.1) *proves the combinatorial factor of the* λ_g *formula*

$$\langle \tau_{n_{[\ell]}}\lambda_g\rangle_{g,\ell} = \binom{2g-3+\ell}{n_{[\ell]}}\langle \tau_{2g-1}\lambda_g\rangle_{g,1}. \tag{4.8}$$

Proof. Choose $n_{[\ell]}$ subject to $|n_{[\ell]}| = 2g-3+\ell$. We compare the coefficient of the terms of $\prod_{i\geq 1} t_i^{n_i+1}$ in (4.1), which has degree $|n_{[\ell]}|+\ell = 2g-3+2\ell$. The left-hand side contributes

$$(-1)^{2g-3+\ell}(-1)^g\langle \tau_{n_{[\ell]}}\lambda_g\rangle_{g,\ell}\prod_{i\geq 1} n_i!\left(2g-2+\ell-\sum_{i=1}^{\ell}(n_i+1)\right)$$

$$= (-1)^{\ell}(-1)^g \langle \tau_{n_{[\ell]}} \lambda_g \rangle_{g,\ell} (\ell - 1) \prod_{i \geq 1} n_i!.$$

The lowest degree terms of the first line of the right-hand side are

$$(-1)^g \sum_{i<j} \sum_m \langle \tau_m \tau_{n_{[\ell;\hat{i}\hat{j}]}} \lambda_g \rangle_{g,\ell-1} (-1)^m (m+1)! \frac{t_i^{m+4} - t_j^{m+4}}{t_i - t_j} (-1)^{2g-3+\ell-n_i-n_j} \prod_{k \neq i,j} n_k! t_k^{n_k+1}.$$

Since $m = n_i + n_j - 1$, the coefficient of $\prod_{i \geq 1} t_i^{n_i+1}$ is

$$-(-1)^g (-1)^{2g-3+\ell} \sum_{i<j} \langle \tau_{n_i+n_j-1} \tau_{n_{[\ell;\hat{i}\hat{j}]}} \lambda_g \rangle_{g,\ell-1} \binom{n_i + n_j}{n_i} \prod_{i \geq 1} n_i!.$$

Note that the lowest degree coming from the second and the third lines of the right-hand side of (4.1) is $|n_{[\ell]}| + \ell + 2$, which is higher than the lowest degree of the left-hand side. Therefore, we have obtained a recursion equation with respect to ℓ

$$(\ell - 1) \langle \tau_{n_{[\ell]}} \lambda_g \rangle_{g,\ell} = \sum_{i<j} \langle \tau_{n_i+n_j-1} \tau_{n_{[\ell;\hat{i}\hat{j}]}} \lambda_g \rangle_{g,\ell-1} \binom{n_i + n_j}{n_i}. \tag{4.9}$$

The solution of the recursion equation (4.9) is the multinomial coefficient (4.7). □

Remark 4.4. Although the topological recursion (4.1) determines all linear Hodge integrals, the closed formula

$$b_g = \langle \tau_{2g-2} \lambda_g \rangle_{g,1} \qquad g \geq 1$$

does not directly follow from it.

Appendix A. The Eynard-Orantin topological recursion on a genus 0 curve

We are not in the place to formally present the Eynard-Orantin formalism in an axiomatic way. Instead of giving the full account, we are satisfied in Appendix A to give an explanation of a limited case when the *spectral curve* of the theory is $\mathbb{P}^1$. The word "spectral curve" was used in [27] because of the analogy of the spectral curves appearing in the Lax formalism of integrable systems.

We start with the spectral curve $C = \mathbb{P}^1 \setminus S$, where $S \subset \mathbb{P}^1$ is a finite set. We also need two generic elements x and y of $H^0(C, \mathcal{O}_C)$, where $\mathcal{O}_C$ denotes the sheaf of holomorphic functions on C. The condition we impose on x and y is that the holomorphic maps

$$x : C \longrightarrow \mathbb{C} \qquad \text{and} \qquad y : C \longrightarrow \mathbb{C} \tag{A.1}$$

have only simple ramification points, i.e., their derivatives dx and dy have simple zeros, and that

$$(x, y) : C \ni t \longmapsto \big(x(t), y(t)\big) \in \mathbb{C}^2 \tag{A.2}$$

is an immersion. Let $\Lambda^1(C)$ denote the sheaf of meromorphic 1-forms on C, and

$$H^n = H^0\big(C^n, \mathrm{Sym}^n(\Lambda^1(C))\big) \tag{A.3}$$

the space of meromorphic symmetric differentials of degree n. The Cauchy differentiation kernel is an example of such differentials:

$$W_{0,2}(t_1,t_2)=\frac{dt_1\otimes dt_2}{(t_1-t_2)^2}\in H^2. \tag{A.4}$$

In the literatures starting from [27], the Cauchy differentiation kernel has been called the *Bergman kernel*, even thought it has nothing to do with the Bergman kernel in complex analysis. A bilinear operator

$$K: H\otimes H\longrightarrow H \tag{A.5}$$

naturally extends to

$$K: H^{n_1+1}\otimes H^{n_2+1}\ni (f_0,f_1,\dots,f_{n_1})\otimes(h_0,h_1,\dots,h_{n_2}) \longmapsto (K(f_0,h_0),f_1,\dots,f_{n_1},h_1,\dots,h_{n_2})\in H^{n_1+n_2+1}$$

$$K: H^{n+1}\ni (f_0,f_1,\dots,f_{n_1})\longmapsto (K(f_0,f_1),f_2,\dots,f_{n_1})\in H^n.$$

Suppose we are given an infinite sequence $\{W_{g,n}\}$ of differentials $W_{g,n}\in H^n$ for all (g,n) subject to the stability condition $2g-2+n>0$. We say *this sequence satisfies a topological recursion with respect to the kernel* K if

$$W_{g,n}=K(W_{g,n-1},W_{0,2})+K(W_{g-1,n+1})+\frac{1}{2}\sum_{\substack{g_1+g_2=g\\ I\sqcup J=[n;\hat{1}]}}^{\text{stable}} K\big(W_{g_1,|I|+1},W_{g_2,|J|+1}\big). \tag{A.6}$$

The characteristic of the Eynard-Orantin theory lies in the particular choice of the *Eynard kernel* that reflects the parametrization (A.2) and the ramified coverings (A.1). Let $\mathcal{A}=\{a_1,\dots,a_r\}\subset C$ be the set of simple ramification points of the x-projection map. Since locally at each a_λ the x-projection is a double-sheeted covering, we can choose the deck transformation map

$$s_\lambda: U_\lambda \overset{\sim}{\longrightarrow} U_\lambda, \tag{A.7}$$

where $U_\lambda\subset C$ is an appropriately chosen simply connected neighborhood of a_λ.

Definition A.1. The *Eynard kernel* is the linear map $H\otimes H\to H$ defined by

$$\begin{aligned} &K\big(f_1(t_1)dt_1, f_2(t_2)dt_2\big)\\ &=\frac{1}{2\pi i}\sum_{\lambda=1}^{r}\oint_{|t-a_\lambda|<\epsilon} K_\lambda(t,t_1)\Big(f_1(t)dt\otimes f_2\big(s_\lambda(t)\big)ds_\lambda(t)+f_2(t)dt\otimes f_1\big(s_\lambda(t)\big)ds_\lambda(t)\Big), \end{aligned} \tag{A.8}$$

where

$$K_\lambda(t,t_1)=\frac{1}{2}\left(\int_t^{s_\lambda(t)} W_{0,2}(t,t_1)\,dt\right)\otimes dt_1\cdot\frac{1}{\Big(y(t)-y\big(s_\lambda(t)\big)\Big)dx(t)}, \tag{A.9}$$

and $\frac{1}{dx(t)}$ is the contraction operator with respect to the vector field

$$\left(\frac{dx}{dt}\right)^{-1}\frac{\partial}{\partial t}.$$

The integration is taken with respect to the t-variable along a small loop around a_λ that contains no singularities other than $t = a_\lambda$. A topological recursion with respect to the Eynard kernel is what we call the *Eynard-Orantin recursion* in these lectures.

Appendix B. The Lagrange Inversion Formula

In Appendix B we give a brief proof of the Lagrange Inversion Formula. For more detail, we refer to [85].

Theorem B.1. *Let $x = f(y)$ be a holomorphic function in y defined on a neighborhood of $y = b$. Let $f(b) = a$, and suppose $f'(b) \neq 0$. Then the inverse function $y = y(x)$ is given by the following expansion near $x = a$:*

$$y - b = \sum_{k=1}^{\infty} \frac{d^{k-1}}{dy^{k-1}}\left(\frac{y-b}{f(y)-a}\right)^k\Bigg|_{y=b}\frac{(x-a)^k}{k!}. \tag{B.1}$$

Proof. Let us recall the Cauchy integration formula

$$\phi(s) = \frac{1}{2\pi i}\oint\frac{\phi(t)dt}{t-s},$$

where $\phi(t)$ is a holomorphic function defined on a neighborhood of $t = s$, and the integration contour is a small simple loop inside this neighborhood counterclockwisely rotating around the point s. Since $x = f(y)$ is one-to-one near $y = b$, for a point s close to b, we have

$$\begin{aligned}
\frac{1}{f'(s)} &= \frac{1}{f'\big(f^{-1}(f(s)\big)} \\
&= \frac{1}{2\pi i}\oint\frac{df(t)}{f'\big(f^{-1}(f(t)\big)\big(f(t)-f(s)\big)} \\
&= \frac{1}{2\pi i}\oint\frac{f'(t)dt}{f'(t)\big(f(t)-f(s)\big)} \\
&= \frac{1}{2\pi i}\oint\frac{dt}{f(t)-f(s)}.
\end{aligned}$$

Therefore, assuming that s is close enough to b, we compute

$$\begin{aligned}
y - b &= \int_b^y 1\cdot ds = \int_b^y\left(\frac{1}{2\pi i}\oint\frac{f'(s)dt}{f(t)-f(s)}\right)ds \\
&= \int_b^y\left(\frac{1}{2\pi i}\oint\frac{f'(s)dt}{\big(f(t)-a\big)-\big(f(s)-a\big)}\right)ds
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_b^y \oint \frac{\frac{f'(s)}{f(t)-a}}{1-\frac{f(s)-a}{f(t)-a}} dtds \\
&= \frac{1}{2\pi i}\int_b^y \sum_{n=0}^{\infty}\oint \frac{f'(s)}{f(t)-a}\left(\frac{f(s)-a}{f(t)-a}\right)^n dsdt \\
&= \frac{1}{2\pi i}\int_{f(b)}^{f(y)} \sum_{n=0}^{\infty}\oint \frac{1}{f(t)-a}\left(\frac{f(s)-a}{f(t)-a}\right)^n df(s)dt \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\oint \frac{1}{\big(f(t)-a\big)^{n+1}}\cdot\frac{\big(f(y)-a\big)^{n+1}}{n+1}dt \\
&= \frac{1}{2\pi i}\sum_{k=1}^{\infty}\oint \frac{dt}{\big(f(t)-a\big)^{k}}\cdot\frac{(x-a)^k}{k} \\
&= \frac{1}{2\pi i}\sum_{k=1}^{\infty}\oint \frac{1}{(y-b)^k}\left(\frac{y-b}{f(y)-a}\right)^k dy\cdot\frac{(x-a)^k}{k} \\
&= \sum_{k=1}^{\infty}\frac{d^{k-1}}{dy^{k-1}}\left(\frac{y-b}{f(y)-a}\right)^k\Bigg|_{y=b}\cdot\frac{(x-a)^k}{k(k-1)!}.
\end{aligned}
$$

□

The following formula is a straightforward application of the above Lagrange Inversion Theorem.

Corollary B.2. *If*

$$x=\frac{y}{f(y)},\quad f(0)=0,\quad \textit{and}\quad f'(0)\neq 0,$$

then the inverse function is given by

$$y=\sum_{k=1}^{\infty}\frac{d^{k-1}}{dy^{k-1}}\big(f(y)\big)^k\Big|_{y=0}\frac{x^k}{k!}. \tag{B.2}$$

Acknowledgement. The author thanks the organizers of the Summer School for inviting him to present these lectures. He also thanks Vincent Bouchard, Bertrand Eynard, and Jian Zhou for tireless explanations of their works to him. This work is supported by NSF grants DMS-0905981, DMS-1104734, and DMS-1104751, and Université de Genève.

References

[1] G. V. Belyi, *On galois extensions of a maximal cyclotomic fields*, Math. U.S.S.R. Izvestija **14**, 247–256 (1980).

[2] J. Bennett, D. Cochran, B. Safnuk, and K. Woskoff, *Topological recursion for symplectic volumes of moduli spaces of curves*, in preparation.

[3] A. Bertram, R. Cavalieri, and G. Todorov, *Evaluating tautological classes using only Hurwitz numbers*, Trans. Amer. Math. Soc. **360**, 6103–6111 (2008).

[4] G. Borot, B. Eynard, M. Mulase and B. Safnuk, Hurwitz numbers, matrix models and topological recursion, arXiv:0906.1206 [math.Ph] (2009).

[5] V. Bouchard, A. Klemm, M. Mariño, and S. Pasquetti, *Remodeling the B-model*, Commun. Math. Phys. **287**, 117–178 (2008).

[6] V. Bouchard and M. Mariño, *Hurwitz numbers, matrix models and enumerative geometry*, Proc. Symposia Pure Math. **78**, 263–283 (2008).

[7] A. Brini, *The local Gromov-Witten theory of $\mathbb{CP}^1$ and integrable hierarchies*, arXiv:1002.0582v1 [math-ph].

[8] A. Brini and R. Cavalieri, *Open orbifold Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_n$: localization and mirror symmetry*, arXiv:1007.0934 (2010).

[9] A. Brini, B. Eynard, and M. Mariño, *Torus knots and mirror symmetry* arXiv:1105.2012.

[10] R. Cavalieri, P. Johnson, and H. Markwig, *Chamber Structure of Double Hurwitz numbers*, arXiv:1003.1805 (2010).

[11] K. Chapman, M. Mulase, and B. Safnuk, *Topological recursion and the Kontsevich constants for the volume of the moduli of curves*, Preprint (2010).

[12] L. Chen, *Bouchard-Klemm-Marino-Pasquetti Conjecture for $\mathbb{C}^3$*, arXiv:0910.3739 (2009).

[13] L. Chen, Y. Li, and K. Liu, *Localization, Hurwitz numbers and the Witten conjecture*, arXiv:math.AG/0609263 [math.AG] (2006).

[14] D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs **68**, AMS, 1999.

[15] R. Dijkgraaf and C. Vafa, *Two Dimensional Kodaira-Spencer Theory and Three Dimensional Chern-Simons Gravity*, arXiv:0711.1932 [hep-th].

[16] R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Loop equations and Virasoro constraints in non-perturbative two-dimensional quantum gravity*, Nucl. Phys. **B348**, 435–456 (1991).

[17] B. Dubrovin, *Geometry of 2D topological field theories*, in "Integrable systems and quantum groups," Lecture Notes in Math. **1620**, 120–348 (1994).

[18] B. Dubrovin and Y. Zhang, *Frobenius manifolds and Virasoro constraints*, Selecta Mathematica, New Ser. **5**, 423–466 (1999).

[19] T. Eguchi, K. Hori, and C. Xiong, *Quantum cohomology and Virasoro algebra*, Phys. Lett. **402**, 71–80 (1997).

[20] T. Ekedahl, S. Lando, M. Shapiro, A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. **146**, 297–327 (2001).

[21] B. Eynard, *Topological expansion for the 1-hermitian matrix model correlation functions*, arXiv:hep-th/0407261.

[22] B. Eynard, *Recursion between volumes of moduli spaces*, arXiv:0706.4403 [math-ph].

[23] B. Eynard, *All order asymptotic expansion of large partitions*, arXiv:0804.0381 [math-ph].

[24] B. Eynard, *Intersection numbers of spectral curves*, arXiv:1104.0176.

[25] B. Eynard, M. Mulase and B. Safnuk, *The Laplace transform of the cut-and-join equation and the Bouchard-Mariño conjecture on Hurwitz numbers*, arXiv:0907.5224 math.AG (2009).

[26] B. Eynard, M. Mariño and N. Orantin, *Holomorphic anomaly and matrix models*, Journal of High Energy Physics **06** 058, (2007) [arXiv:hep-th/0702110].

[27] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, Communications in Number Theory and Physics **1**, 347–452 (2007).

[28] B. Eynard and N. Orantin, *Weil-Petersson volume of moduli spaces, Mirzakhani's recursion and matrix models*, arXiv:0705.3600 [math-ph] (2007).

[29] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139**, 173–199 (2000).

[30] C. Faber and R. Pandharipande, *Hodge integrals, partition matrices, and the λ_g conjecture*, Ann. of Math. **157**, 97–124 (2003).

[31] E. Getzler, *The Virasoro conjecture for Gromov-Witten invariants*, In "Algebraic Geometry: Hirzebruch 70," Contemp. Math. **241**, 147–176 (1999).

[32] A. Givental, *Semisimple Frobenius structures at higher genus*, Intern. Math. Res. Notices **23**, 1265–1286 (2001).

[33] I.P. Goulden and D.M. Jackson, *Transitive factorisations into transpositions and holomorphic mappings on the sphere*, Proc. A.M.S., **125**, 51–60 (1997).

[34] I.P. Goulden, D.M. Jackson and A. Vainshtein, *The number of ramified coverings of the sphere by the torus and surfaces of higher genera*, Ann. of Comb. **4**, 27–46 (2000).

[35] I.P. Goulden, D.M. Jackson and R. Vakil, *The Gromov-Witten potential of a point, Hurwitz numbers, and Hodge integrals*, Proc. London Math. Soc. **83:3**, 563–581 (2001).

[36] I.P. Goulden, D.M. Jackson and R. Vakil, *A short proof of the λ_g-conjecture without Gromov-Witten theory: Hurwitz theory and the moduli of curves*, arXiv:math/0604297v1 [math.AG] (2006).

[37] T. Graber, R. Vakil, *Hodge integrals and Hurwitz numbers via virtual localization*, Compositio Math. **135**, 25–36 (2003).
[38] T. Graber, R. Vakil, *Relative virtual localization and vanishing of tautological classes on moduli spaces of curves*, Duke Math. J. **130**, 1–37 (2005).
[39] G. Harder and M. S. Narasimhan, *On the cohomology groups of moduli spaces of vector bundles on curves*, Math. Ann. **212**, 215–248 (1975).
[40] J. L. Harer, *The cohomology of the moduli space of curves*, in Theory of Moduli, Montecatini Terme, 1985 (Edoardo Sernesi, ed.), Springer-Verlag, 1988, pp. 138–221.
[41] J. L. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, Inventiones Mathematicae **85**, 457–485 (1986).
[42] Tamás Hausel and Michael Thaddeus, Mirror symmetry, Langlands duality, and the Hitchin system, Inventiones Math. **153**, 197–229 (2003).
[43] A. Hodge and M. Mulase, *Hitchin integrable systems, deformations of spectral curves, and KP-type equations*, Advanced Studies in Pure Mathematics **59**, 31–77 (2010).
[44] A. Hurwitz, *Über Riemann'sche Flächen mit gegebene Verzweigungspunkten*, Mathematische Annalen **39**, 1–66 (1891).
[45] M. Kazarian, *KP hierarchy for Hodge integrals*, arXiv:0809.3263 (2008).
[46] M. Kazarian, S. Lando, *An algebro-geometric proof of Witten's conjecture*, J. Amer. Math. Soc. **20**, 1079–1089 (2007).
[47] Y.S. Kim and K. Liu, *A simple proof of Witten conjecture through localization*, preprint arXiv:math/0508384 [math.AG] (2005).
[48] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Communications in Mathematical Physics **147**, 1–23 (1992).
[49] M. Kontsevich, *Homological algebra of mirror symmetry*, arXiv:alg-geom/9411018 (1994).
[50] J. Li, C.-C. M. Liu, K. Liu, and J. Zhou, *A mathematical theory of the topological vertex*, Geom. Topol. **13**, 527–621 (2009).
[51] C.-C. M. Liu, *Formulae of one-partition and two-partition Hodge Integrals*, Geom. & Top. Monographs **8**, 105–128 (2006).
[52] C.-C. M. Liu, *Lectures on the ELSV formula*, arXiv:1004.0853. In "Transformation Groups and Moduli Spaces of Curves," Adv. Lect. Math. (ALM) **16**, 195–216, Higher Education Press and International Press, Beijing-Boston, 2010.
[53] C.-C. M. Liu, K. Liu, J. Zhou, *A proof of a conjecture of Mariño-Vafa on Hodge Integrals*, J. Differential Geom. **65**, no. 2, 289–340 (2003).
[54] K. Liu and H. Xu, *A simple proof of Mirzakhani's recursion formula of Weil–Petersson volumes*, arXiv:0705.2086 [math.AG].
[55] K. Liu and H. Xu, *Recursion formulae of Higher Weil–Petersson volumes*, Intern. Math. Res. Notices, Vol. 2009, No. 5, 835–859 (2009).
[56] M. Mariño, *Chern-Simons theory, matrix models, and topological strings*, Oxford University Press, 2005.
[57] M. Mariño, *Open string amplitudes and large order behavior in topological string theory*, JHEP **0803**, 060 (2008).
[58] A. Mironov and A. Morozov, *Virasoro constraints for Kontsevich-Hurwitz partition function*, arXiv:0807.2843 [hep-th] (2008).
[59] M. Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, Invent. Math. **167**, 179–222 (2007).
[60] M. Mirzakhani, *Weil-Petersson volumes and intersection theory on the moduli space of curves*, J. Amer. Math. Soc. **20**, 1–23 (2007).
[61] M. Mulase, *Asymptotic analysis of a Hermitian matrix integral*, International Journal of Mathematics **6**, 881–892 (1995).
[62] M. Mulase and M. Penkava, *Ribbon graphs, quadratic differentials on Riemann surfaces, and algebraic curves defined over* $\overline{\mathbb{Q}}$, The Asian Journal of Mathematics **2** (4), 875–920 (1998).
[63] M. Mulase and M. Penkava, *Topological recursion for the Poincaré polynomial of the combinatorial moduli space of curves*, Preprint arXiv:1009.2135 math.AG (2010).
[64] M. Mulase and B. Safnuk, *Mirzakhani's recursion relations, Virasoro constraints and the KdV hierarchy*, Indian J. Math. **50**, 189–228 (2008).

[65] M. Mulase and N. Zhang, *Polynomial recursion formula for linear Hodge integrals*, Communications in Number Theory and Physics **4**, (2010).
[66] D. Mumford, *Towards an enumerative geometry of the moduli space of curves* (1983), in "Selected Papers of David Mumford," 235–292 (2004).
[67] P. Norbury, *Counting lattice points in the moduli space of curves*, arXiv:0801.4590 (2008).
[68] P. Norbury, *String and dilaton equations for counting lattice points in the moduli space of curves*, arXiv:0905.4141 (2009).
[69] P. Norbury and N. Scott, *Polynomials representing Eynard-Orantin invariants*, arXiv:1001.0449 (2010).
[70] P. Norbury and N. Scott, *Gromov-Witten invariants of* $\mathbb{P}^1$ *and Eynard-Orantin invariants*, arXiv:1106.1337 (2011).
[71] A. Okounkov, *Random matrices and random perputations*, International Mathematics Research Notices **2000**, 1043–1095 (2000).
[72] A. Okounkov, *Toda equations for Hurwitz numbers*, Math. Res. Lett. **7**, 447 (2000) [arXiv:math.AG/0004128].
[73] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers, and matrix models, I*, Proc. Symposia Pure Math. **80**, 325–414 (2009).
[74] A. Okounkov and R. Pandharipande, *The equivariant Gromov-Witten theory of* $\mathbb{P}^1$, arXiv:math/0207233 [math.AG] (2002).
[75] A. Okounkov and R. Pandharipande, *Virasoro constraints for target curves*, arXiv:math/0308097 [math.AG] (2003).
[76] A. Okounkov and R. Pandharipande, *Hodge integrals and invariants of the unknot*, Geom. Topol. **8**, 675–699 (2004).
[77] H. Ooguri, P. Sulkowski, M. Yamazaki, *Wall Crossing As Seen By Matrix Models*, arXiv:1005.1293 (2010).
[78] R. Penner, *Perturbation series and the moduli space of Riemann surfaces*, J. Differ. Geom. **27**, 35–53 (1988).
[79] L. Schneps and P. Lochak, *Geometric Galois actions*, London Mathematical Society Lecture Notes Series **242**, 1997.
[80] D. D. Sleator, R. E. Tarjan, and W. P. Thurston, *Rotation distance, triangulations, and hyperbolic geometry*, Journal of the American Mathematical Society **1**, 647–681 (1988).
[81] K. Strebel, *Quadratic differentials*, Springer-Verlag, 1984.
[82] A. Strominger, S. T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. **B 479**, 243–259 (1996).
[83] G. 't Hooft, *A planer diagram theory for strong interactions*, Nuclear Physics **B 72**, 461–473 (1974).
[84] R. Vakil, Harvard Thesis 1997.
[85] E.T. Whittaker and G.A. Watson, *A course of modern analysis*, Fourth edition, Cambridge University Press, 1927.
[86] E. Witten, *Two dimensional gravity and intersection theory on moduli space*, Surveys in Differential Geometry **1**, 243–310 (1991).
[87] J. Zhou, *Hodge integrals, Hurwitz numbers, and symmetric groups*, preprint, arXiv:math/0308024 [math.AG] (2003).
[88] J. Zhou, *Local Mirror Symmetry for One-Legged Topological Vertex*, arXiv:0910.4320 (2009).
[89] J. Zhou, *Local Mirror Symmetry for the Topological Vertex* arXiv:0911.2343 (2009).
[90] S. Zhu, *The Laplace transform of the cut-and-join equation of Mariño-Vafa formula and its applications* arXiv:1001.0618.

Department of Mathematics, University of California, Davis, CA 95616–8633, U.S.A.
E-mail address: `mulase@math.ucdavis.edu`