LECTURES ON DONALDSON-THOMAS THEORY

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These are a very rough and skeletal set of notes for my lectures on Donaldson-Thomas theory for the Grenoble Summer School on moduli of curves and Gromov-Witten theory. The goal of these lectures is to give a basic introduction to the correspondence between GW theory and DT theory and discuss some techniques to study DT invariants. In the last lecture, I want to give an overview of the proof in the case of toric threefolds. For this, the main reference is the paper [8], arxiv:0809.3976. The current list of references is woefully incomplete, and I will try to fix them later.

1. Lecture 1

In these lectures, the focus will be on different approaches to counting algebraic curves satisfying various constraints on a smooth complex projective threefold X. Very roughly, the approaches correspond to thinking of curves either as parametrized objects or embedded objects; depending on which we choose, we have different limit points of smoothly embedded curves, leading to very different compactifications.

1.1. **Stable maps and Gromov-Witten theory.**

Fix a class $0 \neq \beta \in H_2(X, \mathbb{Z})$, and a genus g.

Definition 1.1. An n-pointed stable map to X of genus g and class β consists of data

$$(C, p_1, \ldots, p_n, f)$$

where C is a proper, connected curve of arithmetic genus g with at worst nodal singularities, $p_1, \ldots, p_n$ are smooth marked points of C and $f : C \to X$ is a map of degree

$$\beta = f_*([C]) \in H_2(X, \mathbb{Z}).$$

We further impose the condition that the data $(C, p_1, \ldots, p_n, f)$ has finite automorphism group.

Two such objects are identified if they differ by a reparametrization of the domain. Finiteness of the automorphism group concretely means that any irreducible component C_0 of C that is contracted by f must have at least three marked points or nodes if $g(C_0) = 0$ and at least one marked point or node if $g(C_0) = 1$.

It is easy to see how to define a family of stable maps over a base S; one can show that the corresponding moduli problem is representable by a proper separated Deligne-Mumford stack of finite type (since we have fixed β), which we denote $\overline{M}_{g,n}(X, \beta)$. In these lectures, it is more convenient to work with the variant of the above moduli space where we allow disconnected curves such that each connected component is not contracted, denoted $\overline{M}^{\bullet}_{g,n}(X, \beta)$.

It is useful to keep in mind how limits behave under certain kinds of degeneration.

Example 1.2. Consider the family of twisted cubics

$$f_t : \mathbb{P}^1 \to \mathbb{P}^3; [x, y] \mapsto [t \cdot x^3, x^2y, xy^2, y^3],$$

viewed as stable maps of genus 0 with $\beta = 3[\text{line}]$, where the family as $t \to 0$ is obtained by projecting away from the point $[1, 0, 0, 0]$. The limit as a cycle is a nodal curve contained in the plane; the limit as a stable map will be the normalization of the nodal curve.

Example 1.3. Consider the degeneration of a smooth conic $C \subset \mathbb{P}^3$ to a nonreduced line:

$$C_t = (x^2 - tyz = 0, w = 0) \subset \mathbb{P}^3, t \neq 0$$

The domain of the limiting stable map as $t \to 0$ will be a rational curve branched over the line with degree 2. Note that there is now a two-dimensional space of possible limits (depending on the specific degeneration); all these limits have a nontrivial automorphism group from the branched cover.

While the moduli space of stable maps is typically highly singular, its deformation theory is well-behaved in the following sense. We can model the deformation theory at a point of the moduli space in terms of cohomological data. For example, if we fix a stable map (C, f), the space of first-order deformations is given by

$$\mathrm{Def}(f) = H^0(C, f^*T_X);$$

there is a well-defined space of obstructions to extending infinitesimal deformations of maps, given by

$$\mathrm{Obs}(f) = H^1(C, f^*T_X).$$

If we consider the space of maps with fixed domain C, then locally this data gives a presentation of the mapping space $\mathrm{Mor}(C, X)$ as the subscheme of $\mathrm{Def}(f)$ determined by $\dim \mathrm{Obs}$ equations, so we get a natural lower bound for the dimension of the moduli space at this point.

If we include deformations of the pointed curve $(C, p_1, \ldots, p_n)$, we obtain a formula for the expected dimension of $\overline{M}^{\bullet}_{g,n}(X, \beta)$:

$$\text{vdim} = 3g - 3 + n + \chi(C, f^*T_X) = -K_X \cdot \beta + (\dim X - 3)(1 - g) + n.$$

The fact that this expected dimension is independent of the stable map (C, f) is the first indication that the deformation theory is acting nicely; the more detailed concept here is that of a perfect two-term obstruction theory (corresponding to the fact that there aren't higher cohomology groups). See Jun Li's notes from this school for more information here.

While in general, the actual dimension of $\overline{M}^{\bullet}_{g,n}(X, \beta)$ is larger than the expected dimension, the main technical foundation of the subject is a construction of the virtual fundamental class

$$[\overline{M}^{\bullet}_{g,n}(X, \beta)]^{\text{vir}} \in H_{2\text{vdim}}(\overline{M}^{\bullet}_{g,n}(X, \beta)).$$

Here the degree is doubled because we are working with homology instead of Chow groups. We use the virtual class as a substitute for the role of the usual fundamental class in intersection theory.

Notice that when $\dim X = 3$, the expected dimension is

$$\text{vdim} = -K_X \cdot \beta + n,$$

which is independent of the genus g.

1.2. **Ideal sheaves and Donaldson-Thomas theory.** From now on, it will be important to only consider $\dim X = 3$. Fix a class $\beta \in H_2(X, \mathbb{Z})$ and an integer $\chi \in \mathbb{Z}$.

Definition 1.4. The Hilbert scheme $\text{Hilb}_\chi(X, \beta)$ of 1-dimensional subschemes of X with numerical invariants β, χ parametrizes subschemes $Z \subset X$, whose irreducible components are at most 1-dimensional such that

(1) $\chi(O_Z) = \chi$, and
(2) $[Z] = \beta \in H_2(X, \mathbb{Z})$.

Here, we use $[Z]$ to denote the cycle class of the 1-dimensional components of Z.

Example 1.5. Consider the limit of two skew lines in $\mathbb{P}^3$ (which is also a local model for the degeneration of the twisted cubic studied earlier).

$$C_t = (y = 0, z = t) \cup (x = 0, z = 0)$$

. The flat limit of ideals as $t \to 0$ is

$$(x, z) \cdot (y, z - t) = (xy, yz, x(z - t), z(z - t)) \to (xy, yz, xz, z^2) \subsetneq (xy, z),$$

so we have an embedded point.

This embedded point can be deformed off the curve, so there is no natural way to consider pure 1-dimensional objects. On the other hand, our subschemes are always embedded, so there are no automorphisms and we always work with proper schemes.

1.3. **Deformation theory.** The standard deformation theory of subschemes is not as well-behaved as stable maps. The deformation and obstruction space are given by

$$\mathrm{Hom}_X(\mathcal{I}_Z, \mathcal{O}_Z), \mathrm{Ext}^1_X(\mathcal{I}_Z, \mathcal{O}_Z)$$

respectively. Due to the existence of higher Ext groups, which do not typically vanish, there is not a well-defined expected dimension and the virtual class technology breaks down.

Instead, consider the moduli space $I_\chi(X, \beta)$ parametrizing pairs $(\mathcal{E}, \phi)$ such that

(1) $\mathcal{E}$ is a torsion-free sheaf X of rank 1,
(2) equipped with a trivialization $\phi : \det\mathcal{E} \cong \mathcal{O}_X$, and
(3) its numerical invariants are given by

$$c_2(\mathcal{E}) = -\beta, \chi(\mathcal{E}) = \chi(\mathcal{O}_X) - \chi.$$

The natural map

$$\mathrm{Hilb}_\chi(X, \beta) \to I_\chi(X, \beta); Z \mapsto \mathcal{I}_Z$$

is an isomorphism. On the level of closed points, we can see this as follows. The double-dual $\mathcal{E}^{\vee\vee}$ is a reflexive sheaf of rank 1 with trivial determinant, so must be $\mathcal{O}_X$; the natural inclusion

$$0 \to \mathcal{E} \to \mathcal{E}^{\vee\vee}$$

realizes $\mathcal{E}$ as the ideal sheaf of a subscheme.

While the underlying schemes are identical, the natural deformation and obstruction spaces for torsion free sheaves with trivialized determinant are given by

$$\mathrm{Ext}^1_0(\mathcal{E}, \mathcal{E}), \mathrm{Ext}^2_0(\mathcal{E}, \mathcal{E}).$$

Here, Ext^k_0 denotes the traceless Ext groups, given by the kernel of the trace map

$$\mathrm{Ext}^k(\mathcal{E}, \mathcal{E}) \to H^k(\mathcal{O}_X).$$

By Serre duality, the traceless Ext-groups vanish in dimension 0 and 3, so this approach gives a perfect obstruction theory, with expected dimension

$$\mathrm{vdim} = \chi(\mathcal{E}, \mathcal{E}) - \chi(\mathcal{O}_X) = -K_X \cdot \beta.$$

The observation that moduli of ideal sheaves is better behaved than the Hilbert scheme goes back to Richard Thomas's thesis [12]. As in Gromov-Witten theory, we again have a virtual class with this dimension. Notice that the virtual dimension is independent of n and equal to the virtual dimension of $\overline{M}_g, 0(X, \beta)$.

1.4. **Primary invariants.** Let $\gamma_1, \ldots, \gamma_n \in H^*(X)$ be cohomology classes (dual to a collection of topological cycles on X); by the expected dimension formula, if

$$\sum(\frac{1}{2}\deg\gamma_i - 1) = -K_X \cdot \beta,$$

we expect a finite number of curves intersecting these cycles.

In Gromov-Witten theory, we can model this enumerative problem using the evaluation maps

$$\mathrm{ev}_k : \overline{M}_{g,n}(X, \beta) \to X$$

and consider the primary invariant

$$\langle \gamma_1, \ldots, \gamma_n \rangle^{GW}_{\beta,g} = \int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod \mathrm{ev}_k^* \gamma_k \in \mathbb{Q}.$$

In Donaldson-Thomas theory, we have no evaluation maps; however, we can use Chern classes of the tautological sheaf

$$\mathcal{I} \to \mathrm{Hilb}(X) \times X.$$

Given $\gamma \in H^*(X)$, we can define an operator

$$\sigma_0(\gamma) : \pi_{\mathrm{Hilb},*}\left(-\mathrm{ch}_2(\mathcal{I}) \cup \pi_X^*(\gamma) \cap \pi_{\mathrm{Hilb}}^*(-)\right) : H_*(\mathrm{Hilb}) \to H_*(\mathrm{Hilb}).$$

Primary Donaldson-Thomas invariants are defined by

$$\langle \gamma_1, \ldots, \gamma_n \rangle^{DT}_{\beta,m} = \deg \prod \sigma_0(\gamma_k) [I_\chi(X, \beta)]^{\mathrm{vir}} \in \mathbb{Z},$$

where by the dimension count, the expression on the right is a degree 0 class so has a well-defined degree. The motivation here is that $-\mathrm{ch}_2(I_Z) = \beta$ is represented by the cycle $[Z]$.

1.5. **Correspondence.** Despite the heuristic similarity, there is no natural geometric mechanism for these two sets of invariants to agree; in fact they clearly don't agree (e.g. GW invariants are typically rational while DT invariants are always integral). Nevertheless, for primary invariants we expect a simple correspondence relating the two (see [5]).

Set

$$\mathsf{Z}'_{GW}(\gamma_1, \ldots, \gamma_n; u)_\beta = \sum_g u^{2g-2} \langle \gamma_1, \ldots, \gamma_n \rangle^{GW}_{\beta,g}$$

and

$$\mathsf{Z}_{DT}(\gamma_1, \dots, \gamma_n; q)_\beta = \sum_\chi \langle \gamma_1, \dots, \gamma_n \rangle^{DT}_{\beta,\chi} q^\chi.$$

It is convenient to normalize the DT series by the contribution of $\beta = 0$:

$$\mathsf{Z}'_{DT}(\gamma; q)_\beta = \frac{\mathsf{Z}_{DT}(\gamma; q)_\beta}{\mathsf{Z}_{DT}(\emptyset; q)_0}.$$

Conjecture. *The reduced Donaldson-Thomas partition function $\mathsf{Z}'_{DT}(\gamma; q)_\beta$ is a rational function of q. The change of variables*

$$q = -e^{iu}$$

relates it to the Gromov-Witten partition function

$$(-iu)^{-\operatorname{vdim}} \mathsf{Z}'_{GW}(\gamma; u)_\beta = (-q)^{-\operatorname{vdim}/2} \mathsf{Z}'_{DT}(\gamma; q)_\beta.$$

One nice feature of this conjecture is that, since the DT series is a rational function in q, the change of variables does not require any analytic continuation. Regarding the degree 0 normalization, we have a precise evaluation for the degree 0 partition function:

$$\mathsf{Z}_{DT}(\emptyset; q)_0 = M(-q)^{\int_X (c_3 - c_1 c_2)}$$

where

$$M(q) = \frac{1}{\prod (1 - q^n)^n}$$

is the Macmahon function. Note that this is very far from a rational function in q; it is only the normalization that is well-behaved.

Remark 1.6. The conjecture is known in very few cases: toric threefolds and rank 2 bundles over curves. Recently, there has been great progress in proving the rationality statement for Calabi-Yau threefolds.

Example 1.7. Let C be a smooth rigid rational curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ inside a compact Calabi-Yau threefold. Take the class $\beta = [C]$. Let us assume for convenience that C is the only effective curve in this class (or just take connected components of all the moduli spaces associated to C). So a genus g stable map to this curve class consists of a rational curve mapping isomorphically to C and contracted higher genus tails on this spine, that can move around. While I won't explain how, there is a nice formula for the Gromov-Witten series here:

$$\sum \langle \rangle^{GW}_{g,[C]} u^{2g-2} = \left(\frac{1}{2\sin(u/2)} \right)^2 = u^{-2} + 1/12 + \dots$$

If we run this through the conjecture, we get the prediction of

$$\mathsf{Z}'_{DT}(q)_{[C]} = \frac{q}{(1-q)^2} = q + 2q^2 + \dots.$$

For the normalized DT series, we will see how to do this kind of calculation later on, via localization.

1.6. **Additional comments.**

(1) Although I won't say too much about this, there is (conjecturally) a natural geometric interpretation of the normalized DT series, known as Pandharipande-Thomas theory [11], using the moduli space of stable pairs:

$$\{(\mathcal{E},\sigma)|\mathcal{E} \text{ pure torsion sheaf with one-dim. support;}$$
$$\sigma : \mathcal{O}_X \to \mathcal{E} \text{ with zero-dim. cokernel}\}.$$

It is a fun exercise to see how this moduli space handles the limits we discussed before. See [11] for more details.

(2) As discussed in Y.P. Lee's lectures, it is useful to consider not just primary invariants in Gromov-Witten theory but also descendent invariants essentially arising from the structure map to $\overline{M}_{g,n}$. While there is no direct analog in DT theory, we can take higher degree Chern classes of the tautological sheaf. Rather remarkably, due to recent work of Oblomkov, Okounkov, and Pandharipande, it appears the conjecture can be extended precisely to this setting, although it is much more complicated than the simple change of variables in the primary case. One simple statement, however, is that if we take $\deg \gamma_i > 0$, then we expect the rationality statement to hold with descendents.

(3) One feature of the change of variables is that there is no natural way of isolating the contribution of a single genus g on the Donaldson-Thomas side. However, when X is Calabi-Yau, there is a nice statement (due to Sheldon Katz [2]) using a different moduli space of sheaves. Fix a polarization L on X, and let $T(X,\beta,L)$ be the moduli space of 1-dimensional torsion sheaves on X, with $c_1 = 0, c_2 = \beta, c_3 = 1$, stable with respect to L. One can show that $T(X,\beta)$ has a perfect obstruction theoey with expected dimension 0. If we set

$$n_{0,\beta}(L) = \deg[T(X,\beta)]^{\mathrm{vir}},$$

then conjecturally we have the relation

$$\deg[\overline{M}_{0,0}(X,\beta)]^{\mathrm{vir}} = \sum_{d|\beta} \frac{1}{d^3} n_{0,\beta/d}(L).$$

It has recently been proven by Toda [13] that this conjecture follows from the general GW/DT correspondence.

2. Lecture 2

In this lecture, I want to explain some of the basic tools in the subject; unfortunately, for a general threefold, there aren't many, since we don't have access to relations from $\overline{M}_g$ like we do in Gromov-Witten theory. Still, localization and degeneration techniques in Gromov-Witten theory still apply in the DT setting.

2.1. **Virtual localization in DT theory.**

For basics on equivariant cohomology and localization, see Y.P. Lee's notes.

Let X be a smooth projective toric threefold (it is okay just to think about $\mathbb{P}^3$); by definition, X is an equivariant partial compactification of a rank three torus $T \cong (\mathbb{C}^*)^3$. Since the T-action on X induces an action on $\mathrm{Hilb}(X)$, all constructions (virtual class, DT invariants, etc.) make sense in equivariant cohomology/homology, and allow us to define equivariant DT invariants taking values in $H_T^*(\mathrm{pt}) = \mathbb{Q}[t_1, t_2, t_3]$. Because equivariant cohomology of a point does not lie only in degree zero, there is no longer a dimension constraint on equivariant insertions; in the compact case, however, we do need the inequality

$$\sum(\frac{1}{2}\mathrm{deg}\gamma_i - 1) \geq -K_X \cdot \beta,$$

in order to have nonzero invariants in the projective case.

The key advantage of thinking equivariantly is that the virtual fundamental class behaves nicely with respect to localization [1]:

Key Formula:.

$$[M]^{\mathrm{vir}} = \sum_k \frac{[M_k^T]^{\mathrm{vir}}}{e(N^{\mathrm{vir}})} \in H_*^T(M) \otimes \mathrm{Frac}(H_T^*(\mathrm{pt})).$$

In this equation, the sum is over connected components of the fixed locus, which inherits a virtual fundamental class from the T-fixed part of the obstruction theory of M; N^{vir} denotes the T-varying part of the obstruction theory.

Remark 2.1. One nice feature of the localization formula is that it allows us to define virtual invariants even when X is not compact (so proper pushforward is typically ill-defined). As long as M^T is proper, it makes sense to pushforward the right-hand side of the localization formula to a point. This leads to a definition of equivariant residue invariants for noncompact moduli spaces (e.g. $\mathrm{Hilb}(\mathbb{C}^3)$), which are forced to take values in $\mathrm{Frac}(H_T^*(\mathrm{pt}))$. The primary GW/DT correspondence makes sense and is expected to hold in the equivariant context. In this setting, since rational functions can have negative degree, we no longer require a degree inequality on insertions. For instance, the integral of 1 will typically be nonzero as a residue invariant.

On the space of stable maps, this localization relates Gromov-Witten invariants to Hodge integrals on the moduli space of curves; in Donaldson-Thomas theory, this localization instead has a very combinatorial flavor. Let us first see what happens for the Hilbert scheme of points on $\mathbb{C}^3$.

Lemma 2.2. *Torus-invariant subschemes of $\mathbb{C}^3$ correspond to monomial ideals of $\mathbb{C}[x, y, z]$ with finite colength.*

Given a monomial ideal $I \subset \mathbb{C}[x, y, z]$ corresponding to a zero-dimensional subscheme, we can consider the associated set of lattice points

$$\pi_I = \{(a, b, c) \in \mathbb{Z}^3_{\geq 0} | x^a y^b z^c \notin I\}.$$

This defines a (finite) three-dimensional partition.

It follows from a slightly more refined analysis that the T-fixed obstruction theory at these points is trivial, so all the data in the virtual class of $\mathrm{Hilb}(X)$ comes from the virtual normal bundle, obtained by analysing the traceless Ext-groups as T-representations.

$$N^{\mathrm{vir}} = \mathrm{Ext}^1_0(I, I) - \mathrm{Ext}^2_0(I, I)$$

The inverse of the Euler class of this virtual representation is the rational function w_π of t_1, t_2, and t_3 obtained from multiplying the linear characters that occur, weighted with the appropriate multiplicity.

Example 2.3. The simplest case is $I = (x, y, z)$; in this case, the virtual representation of traceless Ext groups is

$$e^{t_1} + e^{t_2} + e^{t_3} - e^{t_2+t_3} - e^{t_1+t_3} - e^{t_2+t_3},$$

so the localization weight is

$$\mathsf{w}_\pi = \frac{(t_1 + t_2)(t_1 + t_3)(t_2 + t_3)}{t_1 t_2 t_3}.$$

Unfortunately, while it seems like we have reduced the question to a combinatorial problem, this alone is not enough to calculate the equivariant DT series for $\mathbb{C}^3$. It turns out that the answer is extremely nice:

Proposition 2.4.

$$\sum_{\chi} \langle 1 \rangle^{DT}_{\chi} q^{\chi} = \sum_{\pi} \mathsf{w}_{\pi} q^{|\pi|} = M(-q)^{-\frac{(t_1+t_2)(t_1+t_3)(t_2+t_3)}{t_1 t_2 t_3}}.$$

The appearance of $M(q)$ is natural; it is the generating function for $3-d$ partitions of size m.

Remark 2.5. While the proof of the above proposition requires some geometric analysis, things are simpler in the Calabi-Yau specialization $t_1 + t_2 + t_3 = 0$. The tangent and obstruction spaces are equivariantly dual to each other, so

$$\mathsf{w}_{\pi} = (-1)^{\dim \mathrm{Ext}^1_0(I,I)} = (-1)^m.$$

A similar simplification happens in the global setting, making it somewhat easier to understand toric Calabi-Yau threefolds.

In the global setting, the above analysis extends, although becomes more combinatorially unwieldy. Invariant one-dimensional subschemes of $\mathbb{C}^3$ now correspond to 3-d partitions with possibly infinite legs along the axes.

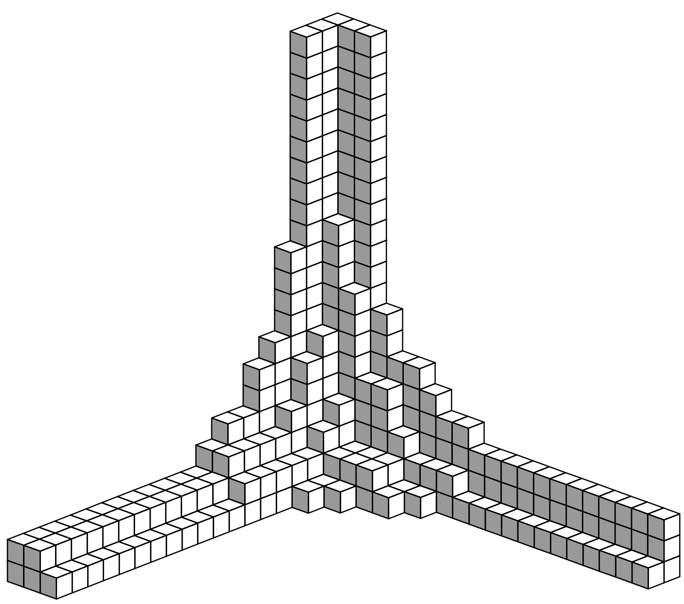

FIGURE 1. A monomial ideal in $\mathrm{Hilb}(\mathbb{C}^3)$ (picture stolen from A. Okounkov)

For a quasiprojective toric threefold X, let Δ denote the polytope associated to X. The vertices of Δ are in bijection with T-fixed points

X^T. The edges e correspond to T-invariant curves

$$C_e \subset X.$$

The fixed loci of Hilb(X) correspond to 3-d partitions at each vertex such that for two neighboring vertcies, the limiting profiles along their common edge agree.

In both GW and DT theory, the overall structure of T-equivariant localization formulas can be described as follows. The outermost sum in the localization formula runs over all assignments of partitions $\lambda(e)$ to the compact edges e of Δ satisfying

$$\beta = \sum_e |\lambda(e)| \cdot [C_e] \in H_2(X, \mathbb{Z}) .$$

The weight of each marking in the localization sum for the invariants equals the product of three factors:

(i) localization of the integrand,
(ii) vertex contributions associated to the 3-d partition, analogous to w_π
(iii) compact edge contributions associated to the limiting profile.

See [5, 8] for more details on the localization calculations.

2.2. **Relative DT theory.** This technology has been developed by Jun Li and Baosen Wu[3, 14], parallel to analogous constructions in Gromov-Witten theory (also due to Jun Li). In the Gromov-Witten setting, I refer again to Y.P.'s notes (which in turn refer to some older notes of Jun Li).

Given a smooth projective threefold X and a smooth divisor $D \subset X$, we are interested in studying subschemes of X which intersect D cleanly, by which we mean the natural map

$$I \otimes \mathcal{O}_D \to \mathcal{O}_D$$

is injective. Ideal sheaves with this property are called relative ideal sheaves. Geometrically this means that no one-dimensional components or embedded points of the subscheme Z lie in S. This defines an open locus of Hilb(X) which is certainly not proper.

The solution to this is to allow the threefold X to degenerate along D. More precisely, the moduli space $I_\chi(X/D, \beta)$ parametrizes stable relative ideal sheaves on degenerations $X[k]$ of X obtained by repeatedly deforming to the normal cone of D. That is, $X[k]$ is the union of X with a chain of k $\mathbb{P}^1$-bundles over D. The transversality condition above is imposed along the transform D_k of the original divisor D as well as along the singular locus of $X[k]$. Two relative ideal sheaves are isomorphic if they agree after applying an automorphism of $X[k]$ over

X. Stability means that we only consider relative ideal sheaves with finite automorphism group. One of the main theorems of [3] is

Proposition 2.6. *The moduli space $I_\chi(X/D, \beta)$ is a proper Deligne-Mumford stack equipped with a perfect obstruction theory.*

One nice feature about the space of relative ideal sheaves that we don't have in the usual setting is that there are now structural maps to other familiar moduli spaces. For instance, by construction the relative moduli space admits a natural morphism

$$\epsilon : [I] \mapsto [I \otimes \mathcal{O}_D]$$

to the Hilbert scheme of d points in D, a smooth quasiprojective variety of dimension $2d$, where $d = \beta \cdot [D]$. Cohomology classes on $\mathrm{Hilb}(D, d)$ may thus be pulled back to the relative moduli space.

We also have a more exotic structural map that involves forgetting all data of the subscheme. That is, we can send a relative ideal sheaf $(I, X[k])$ to the Artin stack $\mathcal{T}$ parametrizing target degenerations. This is a smooth zero-dimensional stack, locally of finite type, consisting of a point for each degeneration $X[k]$, with automorphism group $(\mathbb{C}^*)^k$. One can think of this stack as the open substack of $\mathfrak{M}_{0,3}$ of three-pointed semistable curves of genus 0 consisting of chains of rational curves with two marked points on one extremal component and the thrid point on the other extremal component. As before, we can pull back divisor classes and divisor relations from this stack (just as we do in Gromov-Witten theory) to obtain relations between DT invariants. We will see an example of this later.

If D is disconnected, we allow bubbling along each connected component separately, leading to a slightly different stack of target degenerations.

2.3. **Relative correspondence.** We refer the reader to [5] for more details here. In Gromov-Witten theory, relative conditions are represented by a partition μ of the number

$$d = \beta \cdot [D],$$

each part μ_i of which is marked by a cohomology class $\gamma_i \in H^*(D, \mathbb{Z})$. The numbers μ_i record the multiplicities of intersection with D while the cohomology labels γ_i record where the tangency occurs. More precisely, we integrate the pull-backs of γ_i via the evaluation maps

$$\overline{M}^\bullet_{g,r}(X/D, \beta) \to D$$

at the points of tangency.

In Donaldson-Thomas theory, the analog of these relative conditions comes from the evaluation map to $\mathrm{Hilb}(D,d)$ just discussed. A natural geometric basis to work in here is the *Nakajima basis* of $H^*(\mathrm{Hilb}(D,d),\mathbb{Q})$ indexed by a partition μ of d labeled by cohomology classes of D. For example, the class

$$\big|\mu\big\rangle \in H^*(\mathrm{Hilb}(D,d),\mathbb{Q})\,,$$

with all cohomology labels equal to the identity, is $\prod \mu_i^{-1}$ times the Poincaré dual of the closure of the subvariety formed by unions of schemes of length

$$\mu_1,\ldots,\mu_{\ell(\mu)}$$

supported at $\ell(\mu)$ distinct points of D. It turns out these cohomology classes correspond precisely to the relative tangency conditions for curves.

We can fix insertions and relative constraints, and define analogous normalized generating functions. The conjectural relative GW/DT correspondence [5] equates these series

$$(1)\quad (-iu)^{\mathrm{vdim}+\ell(\mu)-|\mu|}\,\mathsf{Z}'_{GW}\left(X/D,u\,\middle|\,\gamma_1,\ldots,\gamma_r\,\middle|\mu\right)_\beta \stackrel{?}{=} (-q)^{-\mathrm{vdim}/2}\,\mathsf{Z}'_{DT}\left(X/D,q\,\middle|\,\gamma_1,\ldots,\gamma_r\,\middle|\mu\right)_\beta\,,$$

after the change of variables $e^{iu}=-q$. Here, $\mathrm{vdim}=-K_X\cdot\beta$ is the virtual dimension, and μ is a cohomology weighted partition with $\ell(\mu)$ parts. As before, (1) is conjectured to be a rational function of q.

2.4. **Degeneration.** Relative theories satisfy degeneration formulas. Let

$$\mathfrak{X}\to B$$

be a nonsingular 4-fold fibered over an irreducible and nonsingular base curve B. Let X be a nonsingular fiber and

$$X_1\cup_D X_2$$

be a reducible special fiber consisting of two nonsingular 3-folds intersecting transversally along a nonsingular surface D.

If all insertions $\gamma_1,\ldots,\gamma_r$ lie in the image of

$$H^*(X_1\cup_D X_2,\mathbb{Z})\to H^*(X,\mathbb{Z})\,,$$

the degeneration formula in Gromov-Witten theory takes the form

$$(2)\quad \mathsf{Z}_{GW}\left(X\middle|\,\gamma_1,\ldots,\gamma_r\right)_\beta = \sum \mathsf{Z}'_{GW}\left(X_1\middle|\,\ldots\,\middle|\mu\right)_{\beta_1}\,\mathfrak{z}(\mu)\,u^{2\ell(\mu)}\,\mathsf{Z}'_{GW}\left(X_2\middle|\,\ldots\,\middle|\mu^\vee\right)_{\beta_2}\,,$$

where the summation is over all curve splittings $\beta = \beta_1 + \beta_2$, all splitting of the insertions γ_i, and all relative conditions μ.

In (2), the cohomological labels of $\mu^\vee$ are Poincaré duals of the labels of μ. The gluing factor $\mathfrak{z}(\mu)$ is the order of the centralizer of in the symmetric group $S(|\mu|)$ of an element with cycle type μ.

The degeneration formula in Donaldson-Thomas theory takes a very similar form,

$$\begin{aligned} &\mathsf{Z}'_{DT}\left(X \,|\, \gamma_1, \dots, \gamma_r\right)_\beta = \\ &\quad \sum \mathsf{Z}'_{DT}\left(X_1 | \, \dots \, |\mu\right)_{\beta_1} (-1)^{|\mu|-\ell(\mu)} \mathfrak{z}(\mu)\, q^{-|\mu|}\, \mathsf{Z}'_{DT}\left(X_2| \, \dots \, |\mu^\vee\right)_{\beta_2} , \end{aligned}$$

. The sum over the relative conditions μ is interpreted as the coproduct of 1,

$$\Delta 1 = \sum_\mu (-1)^{|\mu|-\ell(\mu)} \mathfrak{z}(\mu) \;\big|\mu\big\rangle \otimes \big|\mu^\vee\big\rangle \;,$$

in the tensor square of $H^*(\mathrm{Hilb}(D, \beta\cdot[D]), \mathbb{Z})$. Conjecture (1) is easily seen to be compatible with degeneration.

2.5. **Rubber space.** Consider the threefold $X = D \times \mathbb{P}^1$ (or more generally, $\mathbb{P}(L \oplus \mathcal{O}_D)$, relative to the fibers over 0 and ∞. Rather than consider the usual space of relative ideals, there is a rubber moduli space

$$I_\chi(X/D_0 \cup D_\infty, \beta)^\sim$$

parametrizing stable relative ideal sheaves on chains of $D \times \mathbb{P}^1$, up to the $\mathbb{C}^*$ action on all components.

This space arises naturally in the boundary of the original relative space and, in particular, is a constant presence when studying fixed loci of $I_\chi(X/D, \beta)$. It again carries a virtual class (of dimension one less than the usual case), and should be viewed as analogous to two-pointed stable maps. We can relate it to the usual virtual class of $I_\chi(X/D_0 \cup D_\infty; \beta)$ by a rigidification process.

2.6. **Quantum cohomology of Hilbert schemes.** Let D be a smooth projective surface and let $X = D \times \mathbb{P}^1$; we can use the degeneration formula to define a ring closely related to the cohomology of $\mathrm{Hilb}(D)$ as follows. Consider relative ideal sheaves on X with respect to the divisor

$$S = D \times \{0, 1, \infty\}.$$

If we fix a curve class $(\beta, m) \in H_2(X, \mathbb{Z})$, and take three cohomology-weighted partitions μ, ν, ρ, the relative DT partition function of X is defined to be

$$\mathsf{Z}_{DT}(X;\mu,\nu,\rho)_{(\beta,m)} = \sum_{\chi\in\mathbb{Z}} q^{\chi} \int_{[I_\chi(X/S,(\beta,m)]^{vir}} \prod_{i=1}^{k} \epsilon^*(\mu\times\nu\times\rho)$$

We can further sum over possible values of β to yield the function

$$\mathsf{Z}_{DT}(X;\mu,\nu,\rho) = \sum_{\beta} s^{\beta}\mathsf{Z}_{DT}(X;\mu,\nu,\rho)_{(\beta,m)} \in \mathbb{Q}(t_1,t_2)((q))[[s_1,\dots,s_r]],$$

where

$$s^{\beta} = s_1^{(\beta,\omega_1)}\cdot\dots\cdot s_n^{(\beta,\omega_n)}$$

is the monomial expansion of s^{β} with respect to some basis of $H^2(D)$ that is non-negative on effective curves.

Finally, we will largely be interested in the partition function obtained by normalizing these invariants with respect to the relative $\beta = 0, m = 0$ partition function:

$$\mathsf{Z}'_{DT}(X;\mu,\nu,\rho)_{(\beta,m)} = \frac{1}{\mathsf{Z}_{DT}(X;\emptyset,\emptyset,\emptyset)_{(0,0)}}\mathsf{Z}_{DT}(X;\mu,\nu,\rho)_{(\beta,m)}.$$

Proposition 2.7. *The above partition functions define the structure constants of a ring deformation of the classical cohomology* $H^*(\mathrm{Hilb}_m(S),\mathbb{Q})$ *over* $\mathbb{Q}((q))[s_1,\dots,s_r]$.

The point is that, in this geometry, the stack of target degenerations is an open substack of three-pointed semistable curves, so the same arguments we use to show, for instance, associativity of the quantum product apply here.

An extremely natural question is how this ring deformation compares to the small quantum cohomology of $H^*(\mathrm{Hilb}_m(S))$, which is also a deformation over the same number of parameters. At least in the case of holomorphic symplectic surfaces such as $\mathbb{C}^2$ and resolutions of Kleinian singularities [10, 7], one can show these deformations agree identically. This is too much to expect for a general surface (already for $m = 1$ and $S = \mathbb{P}^2$), but it seems reasonable to hope for an identification after some kind of change of variables. With a little more work, one can also define an analog of the big quantum product (allowing more insertions); again, there has not been any work along these lines beyond holomorphic symplectic surfaces.

3. Lecture 3

In this lecture, I want to explain a few of the steps that go into proving the GW/DT correspondence for toric threefolds in [8]; it is slightly simpler to focus on the rationality of the DT generating series. The basic idea is to start with a few explicit calculations as inputs and show that the degeneration and localization package allow us to determine algorithmically everything else. While the degeneration formalism is compatible with all conjectures, localization behaves quite poorly, since localization terms are no longer rational functions.

3.1. **Black boxes.** Let ζ be a primitive $(n+1)^{th}$ root of unity, for $n \geq 0$. Let the generator of the cyclic group $\mathbb{Z}_{n+1}$ act on $\mathbb{C}^2$ by

$$(z_1, z_2) \mapsto (\zeta\, z_1, \zeta^{-1} z_2)\,.$$

Let $\mathcal{A}_n$ be the minimal resolution of the quotient

$$\mathcal{A}_n \to \mathbb{C}^2/\mathbb{Z}_{n+1}.$$

The diagonal $(\mathbb{C}^*)^2$-action on $\mathbb{C}^2$ commutes with the action of $\mathbb{Z}_n$. As a result, the surfaces $\mathcal{A}_n$ are toric.

The major inputs - obtained by direct analysis - in the algorithm, are the T-equivariant relative DT invariants for the threefolds

$$\mathbb{C}^2 \times \mathbb{P}^1, \mathcal{A}_1 \times \mathbb{P}^1, \mathcal{A}_2 \times \mathbb{P}^1$$

relative to the fibers over $0, 1, \infty$, with relative insertion over 1 given by divisor classes

$$\gamma \in H^2(\mathrm{Hilb}(\mathcal{A}_n), \mathbb{Q}).$$

These have been calculated in [9] and [6]. In fact, these geometries are very similar;once one understands how to do the first threefold, analogous calculations work for all n. In what follows, we will single out one of these partition functions for $\mathcal{A}_n$: take a divisor $\Gamma \in H^2(\mathcal{A}_n, \mathbb{Q})$ and the associated divisor $(1, \Gamma) \in H^2(\mathrm{Hilb}\,\mathcal{A}_n, \mathbb{Q})$ As we vary the other relative insertions μ, ν, this partition function defines an operator

$$\mathsf{O}_{DT}(\Gamma_F) : H^*(\mathrm{Hilb}(\mathcal{A}_n)) \to H^*(\mathrm{Hilb}(\mathcal{A}_n)),$$

or rather an operator-valued function in q and the curve-class variables. It is shown in these papers that it is given by a rational function in all variables (not just q).

As mentioned in the last lecture, these invariants are precisely equal to three-pointed genus 0 Gromov-Witten invariants for $\mathrm{Hilb}(\mathcal{A}_n)$, first computed [10, 7]. Statements like rationality, for instance, can be read from explicit formulas.

The second major input is a calculation of the relative DT theory of

$$\mathcal{O}(0) \oplus \mathcal{O}(-1) \to \mathbb{P}^1,$$

relative to the fiber over ∞ [9, 8].

3.2. **Capped localization.** Let X be a smooth toric threefold. As discussed earlier, the localization weights are combinations of edge and vertex contributions coming from compatible systems of 3-d partitions at each vertex. Unfortunately, in DT theory, these edge and vertex contributions are no longer rational functions of q, so the comparison with GW theory is difficult (and requires nontrivial analytic continuation). The solution is to replace these terms by a modified localization procedure where these constituents also satisfy the conjecture. These so-called capped edge and vertex terms will be residue relative DT invariants.

The definition of a capped edge in Donaldson-Thomas theory is as follows. Given an edge e of the toric polytope Δ, with normal bundle of type (a, b), we consider the toric variety X_e is isomorphic to the total space of the bundle

$$\mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b) \to \mathbf{P}^1. \tag{3}$$

Given relative conditions λ, μ, the capped edge is the partition function

$$\mathsf{E}_{DT}(\lambda, \mu, t_1, t_2, t_3, t'_1, t'_2, u) = \mathsf{Z}'_{DT}(X_e/F_0 \cup F_\infty, q \mid \lambda, \mu), \tag{4}$$

defined by T-equivariant residues.

For vertices, we proceed as follows:

Let U be the T-invariant 3-fold obtained by removing the three T-invariant lines

$$L_1, L_2, L_3 \subset \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$$

passing through the point (∞, ∞, ∞),

$$U = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \ \setminus \ \cup_{i=1}^{3} L_i.$$

Let $D_i \subset U$ be the divisor with i^{th} coordinate ∞. For $i \neq j$, the divisors D_i and D_j are disjoint.

In both Gromov-Witten and Donaldson-Thomas theories, the capped vertex is the normalized partition function of U with free relative conditions imposed at the divisors D_i. While the relative geometry $U/\cup_i D_i$ is noncompact, the moduli spaces of maps $\overline{M}'_g(U/\cup_i D_i, \beta)$ and ideal sheaves $I_n(U/\cup_i D_i, \beta)$ have compact T-fixed loci. The invariants of $U/\cup_i D_i$ in both theories are well-defined by T-equivariant residues. In the localization formula for the reduced theories of $U/\cup_i D_i$, nonzero degrees can occur *only* on the edges meeting the origin $(0, 0, 0) \in Y$.

That is, the definition of the capped DT vertex is given by

$$\mathsf{C}_{DT}(\lambda, \mu, \nu, t_1, t_2, t_3, q) = \mathsf{Z}'_{DT}(U/\cup_i D_i, q \mid \lambda, \mu, \nu).$$

The partitions λ, μ, ν here represent the Nakajima basis elements of

$$H^*(\mathrm{Hilb}(D_i, [\beta] \cdot D_i), \mathbb{Q}), \quad i = 1, \ldots, 3.$$

Any primary DT invariant on a toric threefold can still be expressed in terms of these capped edge and vertex terms. Capped localization expresses the primary Gromov-Witten and Donaldson-Thomas invariants of X as a sum of capped vertex and capped edge data.

A half-edge $h = (e, v)$ is a compact edge e together with the choice of an incident vertex v. A partition assignment

$$h \mapsto \lambda(h)$$

to half-edges is *balanced* if the equality

$$|\lambda(e, v)| = |\lambda(e, v')|$$

always holds for the two halfs of e. For a balanced assignment, let

$$|e| = |\lambda(e, v)| = |\lambda(e, v')|$$

denote the *edge degree*.

The outermost sum in the capped localization formula runs over all balanced assignments of partitions $\lambda(h)$ to the half-edges h of Δ satisfying

$$\beta = \sum_e |e| \cdot [C_e] \in H_2(X, \mathbb{Z})\,. \tag{5}$$

Such a partition assignment will be called a *capped marking* of Δ. The weight of each capped marking in the localization sum for the invariants (**??**) and (**??**) equals the product of four factors:

(i) localization of the integrand,
(ii) capped vertex contributions,
(iii) capped edge contributions,
(iv) gluing terms.

The Donaldson-Thomas capped localization formula has the structure

$$\mathsf{Z}'_{DT}\Big(X, q \mid \gamma_1, \ldots, \gamma_r\Big)_\beta = \sum_{\Gamma \in \Gamma_\beta} \prod_{v \in \mathcal{V}} \prod_{e \in \mathcal{E}} \prod_{h \in \mathcal{H}} \mathsf{I}_\Gamma \; \mathsf{C}_{DT}(v, \Gamma) \; \mathsf{E}_{DT}(e, \Gamma) \; \mathsf{G}_{DT}(h, \Gamma)$$

where the evaluations $\mathsf{C}_{DT}(v, \Gamma)$ and $\mathsf{E}_{DT}(e, \Gamma)$ are defined as before. Here, I_Γ is the contribution arising from localizing primary insertions; these are pure monomials in the tangent weights supported at vertices, so are not serious. The Donaldson-Thomas gluing factors are

$$\mathsf{G}_{DT}(h_i^v, \Gamma) = (-1)^{|h_i^v| - \ell(\lambda(h_i^v))} \mathfrak{z}(\lambda(h_i^v)) \left(\frac{\prod_{j=1}^3 t_j^v}{t_i^v} \right)^{\ell(\lambda(h_i^v))} q^{-|h_i^v|},$$

which are the same factors that occur in the usual degeneration formula. There is an analogous formula for GW invariants with slightly different gluing terms.

It follows from this that, in order to understand primary invariants on all toric threefolds, it suffices to prove the following proposition, a special case of the correspondence for equivariant relative residues.

Proposition 3.1. *Capped vertex and edge DT terms are rational functions in q and satisfy the GW/DT correspondence.*

For the edge terms, this follows from the calculations for $\mathbb{C}^2 \times \mathbb{P}^1$ and the half-edge with normal bundle $(0, -1)$. Indeed, any normal bundle of type (a, b) can be obtained from these pieces by degeneration and the fact that the $(0, 1)$ edge is the inverse as a matrix of the $(0, -1)$ edge. It remains to understand the vertex terms. These are classified into three categories (one-leg, two-leg, three-leg) based on how many of the limiting partitions are empty. The one-leg vertex again follows from the existing calculations. The idea is that $\mathcal{A}_1 \times \mathbb{P}^1$ allows us to study the two-leg vertex (since only one and two-leg vertices can occur in this geometry), and then $\mathcal{A}_2 \times \mathbb{P}^1$ allow us to capture three-leg geometries.

In order to execute this, we take the relative DT theory of $\mathcal{A}_n \times \mathbb{P}^1$, relative to the fiber over ∞ - these invariants are essentially trivial - and apply capped localization. This requires a new capped localization contribution, due to the relative divisor.

3.3. **Capped rubber.** We start with the tube,

$$\pi : \mathcal{A}_n \times \mathbf{P}^1 \to \mathbf{P}^1$$

relative to the fibers over $0, \infty \in \mathbf{P}^1$. In the space of relative ideal sheaves, we define a T-equivariant open subset

$$V^{CR}_{\chi,\beta} \subset I_\chi(\mathcal{A}_n \times \mathbf{P}^1 / F_0 \cup F_\infty, \beta)$$

is defined to be the locus consisting of subschemes with *no positive* $\mathcal{A}_n$-*degree components* in the destabilization of the fiber over $0 \in \mathbf{P}^1$. The T-equivariant residue theory of V^{CR} is well-defined since the T-fixed loci are compact.

We define partition functions with relative conditions over the divisors F_0, F_∞ lying over $0, \infty \in \mathbf{P}^1$:

$$(-q)^d \sum_{\sigma \in H_2(\mathcal{A}_n, \mathbb{Z})} s^\sigma \, \mathsf{Z}'_{DT} \left(V^{CR}, q \,\middle|\, \middle|\lambda, \mu \right)_{d[\mathbf{P}^1]+\sigma}.$$

We can think of these partition functions as matrix elements of an operator

$$\mathsf{O}_{DT}(\mathsf{CR}) : H^*(\mathrm{Hilb}(\mathcal{A}_n)) \to H^*(\mathrm{Hilb}(\mathcal{A}_n)).$$

Lemma 3.2. *The capped $\mathcal{A}_n$-rubber invariants are rational functions of q and satisfy the GW/DT correspondence.*

3.4. **Differential equations.** We will prove this statement from the $\mathcal{A}_n$ geometry by using divisor relations on the space of relative ideal sheaves.

Let t_3 be the weight of the T-action on the trivial normal bundle of

$$F_0 \subset \mathcal{A}_n \times \mathbf{P}^1.$$

By localization, every class $\gamma \in H^*_T(\mathcal{A}_n \times \mathbb{P}^1, \mathbb{Z})$ satisfies

$$t_3\, \gamma = \gamma_0 - \gamma_\infty \ \in H^*_T(\mathcal{A}_n \times \mathbb{P}^1, \mathbb{Z}) \tag{6}$$

where γ_0, γ_∞ are the restrictions of γ to F_0 and F_∞ respectively.

We can analyze what a primary insertion γ_0 does to relative DT theory as follows. Let $\mathcal{R} \to \mathcal{T}$ be the universal target over the stack $\mathcal{T}$ of target degenerations. We can think of this as the moduli space of target degenerations, along with a point on the (possibly degenerate) target that does not lie on relative divisors or the singular locus. There is a T-equivariant contraction map

$$\pi : \mathcal{R} \to \mathcal{A}_n \times \mathbb{P}^1.$$

There are two divisors on $\mathcal{R}$ related to $0 \in \mathbb{P}^1$. The first is $\pi^*([F_0])$. The second is the boundary divisor $\mathcal{R}_0 \subset \mathcal{R}$ where extra point lies on a destabilization over $0 \in \mathbb{P}^1$. The following result can be easily seen by comparing the divisors on smooth charts for $\mathcal{R}$.

Lemma 3.3. $\pi^*([F_0]) = \mathcal{R}_0$ *in* $\mathrm{Pic}(\mathcal{R})$.

If we pull this relation back to the space of relative ideal sheaves, and factor the virtual class over the $\mathcal{R}_0$, we get an identity

$$\left\langle \lambda \middle| \gamma_0 \middle| \mu \right\rangle'_\beta = \sum_{\nu,\, \sigma_1+\sigma_2=\sigma} \left\langle \lambda \middle| \gamma_0 \middle| \nu \right\rangle^{\sim}_{\beta_1}{}' \,(-1)^{|\nu|-\ell(\nu)}\, \mathfrak{z}(\nu)\, q^{-|\nu|} \left\langle \nu^\vee \right| \left| \mu \right\rangle'_{\beta_2} \tag{7}$$

where $\sim$ denotes rubber relative invariants coming from the degenerate part of the target. We apply this identity when γ is a divisor $\Gamma \in H^2(\mathcal{A}_n, \mathbb{Q})$. Inserting γ multiplies an invariant in class $d[\mathbb{P}^1] + \sigma$ by $\Gamma \cdot \sigma$. In particular this insertion may be interpreted as the action of a linear differential operator ∂_Γ on the partition function, over the variables indexing the $\mathcal{A}_n$ curve classes.

If we will pull this relation back to the open subset V^{CR} in the case where γ is a divisor Γ on $\mathcal{A}_n$ and use the operator formalism O everywhere, we get a differential equation:

$$t_3\, \partial_\Gamma \mathsf{O}_{DT}(\mathsf{CR}) = \mathsf{O}_{DT}(\Gamma_F)_0 \cdot \mathsf{O}_{DT}(\mathsf{CR}) - \mathsf{O}_{DT}(\mathsf{CR}) \cdot \mathsf{O}_{DT}(\Gamma_F)\,, \tag{8}$$

where $\mathsf{O}_{DT}(\Gamma_F)_0$ denotes the contribution of curves with degree 0 in the $\mathcal{A}_n$-direction.

While it is not obvious, this differential equation is exactly parallel to the quantum differential equation in genus 0 Gromov-Witten theory, related to the J-function in YP's notes.

This differential equation uniquely reconstructs the capped rubber operators from their horizontal parts (which we assume calculated). If one works inductively on the vertical curve class σ, the differential equation can be written as an invertible linear equation for the contribution $\mathsf{O}^{\mathsf{CR}}_{DT,\sigma}$ in terms of lower degree terms (known to be rational functions by induction). Therefore it is also given by an operator-valued rational function.

If one makes analogous arguments for Gromov-Witten theory, we have a parallel capped GW rubber determined by the same differential equation. Therefore, knowing the GW/DT correspondence for $\mathsf{O}_{GW/DT}(\Gamma_F)$ determines it for capped rubbers as well.

3.5. **Invertibility.** How does the capped $\mathcal{A}_n$ rubber help us? It shows up as part of the capped localization for the relative DT theory of $\mathcal{A}_n \times \mathbb{P}^1$. If we apply capped localization to relative invariants that we already know, we get useful identities for the capped vertex, since every other capped localization term is determined.

If we consider the cap geometry of $\mathcal{A}_1 \times \mathbb{P}^1$ relative to the fiber F_∞, then it globally satisfies rationality (in fact, it nearly vanishes identically for dimension reasons). On the other hand, if we apply capped localization, we have the diagram shown in Figure **??**

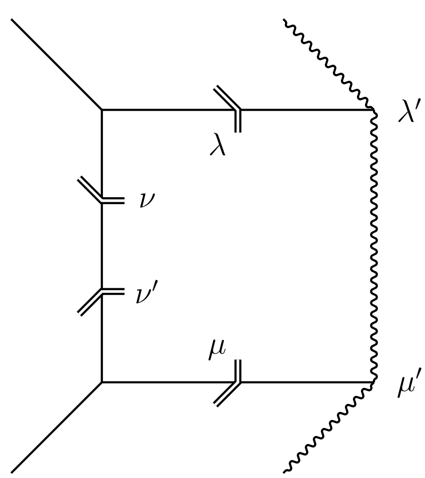

FIGURE 2. Capped localization for the $\mathcal{A}_1$-cap

The lines in Figure 2 represent the edges of the toric polyhedron; the squiggly lines belong to the relative divisor. Over 0, the usual capped vertices and edges occur; over ∞, a single capped $\mathcal{A}_1$ rubber occurs.

Since O^{CR} is an invertible operator given by rational functions, this diagram quickly leads to a quadratic constraint for the two-leg capped vertex. It is not hard to show it is invertible (by an induction on the size of the partitions at each leg). See [8] for details. Again, the point is that the same diagram and analysis applies for Gromov-Witten theory; since these constraints have a unique solution.

For the three-leg vertex, we use the $\mathcal{A}_2$ vertex, with the localization diagram shown in Figure 3: Here, since there is a unique vertex with

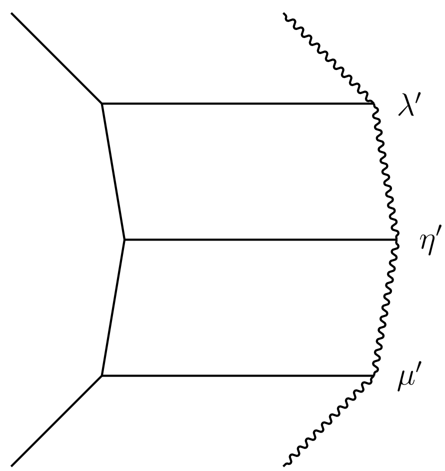

FIGURE 3. Capped localization for the $\mathcal{A}_2$-cap

three legs, so this is in fact a linear constraint on the unknown vertex, so the analysis is easier.

3.6. **Refined conjectures.** This argument while complicated does lead to some refined questions that we do not know how to answer using this procedure. The most interesting for me is the following.

Let X be a nonsingular projective 3-fold with very ample line bundle L. The Chow variety of curves parameterizes cycles of class $\beta \in H_2(X, \mathbb{Z})$ in X.

For both the moduli space of stable maps $\overline{M}'_g(X, \beta)$ and the Hilbert scheme of curves $I_\chi(X, \beta)$, the associated (seminormalized) varieties admit maps to $\mathrm{Chow}(X,\beta)$ for all g and n,

$$\begin{tikzcd} \overline{M}_g(X,\beta)_{\mathrm{sn}} \arrow[dr, "\rho_{GW}"] & & I_\chi(X,\beta)_{\mathrm{sn}} \arrow[dl, "\rho_{DT}"'] \\ & \mathrm{Chow}(X,\beta) & \end{tikzcd}.$$

Conjecture. *Let X be a nonsingular projective toric 3-fold. We have*

$$\sum_g u^{2g-2}\,\rho_{GW*}\left(\left[\overline{M}'_g(X,\beta)\right]^{\mathrm{vir}}\right) = \frac{1}{\mathsf{Z}_{DT}(X,q)_0}\sum_\chi q^\chi\,\rho_{DT*}\left([I_\chi(X,\beta)]^{\mathrm{vir}}\right) \tag{9}$$

in $H_*(\mathrm{Chow}(X,\beta),\mathbb{Q})\otimes\mathbb{Q}(q)$, after the variable change $e^{iu}=-q$.

In the toric case, this is an immediate corollary of capped localization formalism. In general, it seems even more inaccessible than the original conjecture.

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