Extremal Laurent Polynomials*

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1 Introduction

The course is an elementary introduction to my experimental work in progress with Tom Coates, Sergei Galkin, Vasily Golyshev and Alexander Kasprzyk (http://coates.ma.ic.ac.uk/fanosearch/), funded by EPSRC. I owe a special intellectual debt to Vasily, who already several years ago insisted that we should pursue these themes.

I am an algebraic geometer. In the Fall 1988, my first "quarter" in graduate school at the University of Utah, I attended lectures by Mori on the classification of *Fano 3-folds*. (The Minimal Model Program classifies projective manifolds in three classes of manifolds with negative, zero and positive "curvature:" Fano manifolds are those of positive curvature.)

I learned from Mori that there are 105 (algebraic) deformation families of Fano 3-folds. I hope that what I tell you in this course is interesting from several perspectives, but one source of motivation for me is to get a picture of the classification of Fano 4-folds: how many families are there? Is it 100,000 families; is it 1,000,000 or perhaps 10,000,000 families? If we are seriously to do algebraic geometry in ≥ 4 dimensions, there is a dearth of examples: what does a "typical" Fano 4-fold look like?

*These notes reproduce almost exactly my 4 lectures at the Summer School on "Moduli of curves and Gromov–Witten theory" held at the Institut Fourier in Grenoble during 20th June–8th July 2011. I want to thank the organisers for a truly outstanding school and for the extremely lovely atmosphere. These notes were written in haste: please let me know if you found mistakes. You will find an updated version on my teaching page `http://www2.imperial.ac.uk/~acorti/teaching.html`

Topics

1 I give a short discussion of local systems on $\mathbb{P}^1 \setminus S$ (where $S \subset \mathbb{P}^1$ is a finite set) and introduce the notion (due to Vasily Golyshev) of *extremal local system*: that is, a local system that is nontrivial, irreducible, and of smallest possible ramification.

2 Given a Laurent polynomial $f\colon \mathbb{C}^{\times n} \to \mathbb{C}$, I explain how to construct the Picard–Fuchs differential operator L_f and its natural solution, the principal period. By definition, f is extremal if the local system of solutions of L_f is extremal. I explain the general theory and give some examples. In particular, we discovered an interesting class of Laurent polynomials where the Picard–Fuchs local systems has (conjecturally and experimentally) low ramification , called Minkowski polynomials.

3 I briefly summarize quantum cohomology of a Fano manifold X and quick-and-dirty methods of calculation. Much of the structure is encoded in a differential operator $\widehat{Q}_X$ and power series solution $\widehat{I}_X$. I motivate with examples the conjecture that $\widehat{Q}_X$ is of small (often minimal) ramification.

4 A Fano manifold X is mirror-dual to a Laurent polynomial f if $\widehat{Q}_X = L_f$. This is a very weak notion of mirror symmetry: to a Fano manifold X there correspond (infinitely) many f. I demostrate how to derive the classification of Fano 3-folds (Iskovskikh, Mori–Mukai) from the classification of 3-variable Minkowski polynomials. I outline a program to use these ideas in 4 dimensions.

References

This document contains no references. This is due in part to the fact that I am presenting a fresh new science, and partly to the fact that the notes are written in haste and I was lazy when it comes to history, attribution, and detail.

Let me at least here acknowledge my greater intellectual debts. The definition of extremal local systems (with the name "low-ramified local systems") and extremal Laurent polynomials (with the name "special Laurent polynomials") appeared first in [Vasily Golyshev, *Spectra and Strain*, arXiv:0801.0432 (hep-th)]. The view of mirror symmetry advocated here notes is modelled on [Viktor Przyjalkowski, *On Landau–Ginzburg models for Fano varieties*, arXiv:0707.3758 (math.AG)]. The webpage of Sergei Galkin `http://member.ipmu.jp/sergey.galkin/` contains a substantial amount of relevant material. All the data can be found on our group research blog at `http://coates.ma.ic.ac.uk/fanosearch/`.

2 Extremal Local Systems

A *local system* on a (topological) manifold B is a locally free sheaf of $\mathbb{Q}$-vector spaces; equivalently, it is a representation of the fundamental group $\rho\colon \pi_1(B,b) \to GL_r(\mathbb{Q})$. (Most of the time we actually work with local systems of free $\mathbb{Z}$-modules).

Let C be a compact Riemann surface, $S \subset C$ a finite set, and $\mathbb{V}$ be a local system on $U = C \setminus S$. Below I denote by $x \in U$ a point and by $j\colon U = C \setminus S \hookrightarrow C$ the natural (open) inclusion.

Definition 2.1. The *ramification* of $\mathbb{V}$ is:

$$\operatorname{rf} \mathbb{V} = \sum_{s \in S} \dim(\mathbb{V}_x / \mathbb{V}_x^{T_s})$$

(where $x \in C$ is a generic point and T_s is the monodromy around $s \in S$.)

(V. Golyshev) If $C = \mathbb{P}^1$, I say that $\mathbb{V}$ is *extremal* if it is irreducible, nontrivial, and $\operatorname{rf} \mathbb{V} = 2 \operatorname{rk} \mathbb{V}$.

Lemma 2.2 (Euler's formula)**.** *Let $\mathbb{V}$ as above be a local system on $U = C \setminus S$, where C is a Riemann surface of genus g. Then*

$$\operatorname{rf} \mathbb{V} + (2g-2) \operatorname{rk} \mathbb{V} = -\chi(C, j_\star \mathbb{V}).$$

Proof. Choose a cellular decomposition of C such that $S \subset V$ and $\mathbb{V}_{|D_f}$ is trivial for every cell D_f, $f \in F$. We get a resolution of $\mathbb{V}$:

$$0 \to \mathbb{V} \to \bigoplus_F \mathbb{Q}^r_f \to \bigoplus_E \mathbb{Q}^r_E \to \bigoplus_{V \setminus S} \mathbb{Q}^r_v \to 0$$

which implies

$$\chi(C, Rj_\star \mathbb{V}) = \chi(U, \mathbb{V}) = \big(V - E + F - |S|\big) r = \big(2 - 2g - |S|\big) \operatorname{rk} \mathbb{V}$$

On the other hand the short exact sequence of complexes:

$$0 \to j_\star \mathbb{V} \to Rj_\star \mathbb{V} \to \bigoplus_{s\in S} H^1\big(\Delta_\varepsilon(s)^\times, \mathbb{V}\big)[-1] \to 0$$

gives $\chi(C, Rj_\star\mathbb{V}) = \chi(C, j_\star\mathbb{V}) - \sum_{s\in S} \dim \mathbb{V}_x^{T_s}$, so, combining:

$$\chi(C, j_\star\mathbb{V}) = (2g-2)\operatorname{rk}\mathbb{V} + \sum_{s\in S} \dim\big(\mathbb{V}_x/\mathbb{V}_x^{T_s}\big).$$

□

Remark 2.3.

- If $\mathbb{V}$ is nontrivial irreducible, then

$$H^0(C, j_\star\mathbb{V}) = \mathbb{V}_x^{\pi_1(C\smallsetminus S, x)} = (0)$$

  and, similarly, $H^2(C, j_\star\mathbb{V}) = H^0\big(C, j_!(\mathbb{V}^*)\big)^* = (0)$. Thus, if $C = \mathbb{P}^1$ and $\mathbb{V}$ is nontrivial irreducible, then $-\chi(\mathbb{P}^1; j_\star\mathbb{V}) = h^1(\mathbb{P}^1; j_\star\mathbb{V}) \geq 0$. Thus, extremal means: smallest ramification. From the point of view of just topology, this is a very natural class of objects to consider. (In general it is also useful to look at local systems of *small* ramification.)

- The central theme of this lectures is the different ways that extremal local systems arise naturally in geometry. I hope to convince you that extremal local systems are interesting in themselves.

- My local systems support variations of (polarized, pure) Hodge structures; in particular, they are always polarised (O_r, Sp_{2r}).

We expect extremal polarised pure motivic sheaves to be rigid objects; in particular, we expect them always to be defined over number fields. Here is a very natural question that nobody, it seems, has considered before.

Problem 2.4. *Classify extremal local systems topologically. Classify extremal polarised pure motivic sheaves.*

3 Extremal Laurent Polynomials

By definition, a Laurent polynomial is a regular algebraic map (morphism) $f\colon \mathbb{C}^{\times n} \to \mathbb{C}$, that is, an element of the polynomial ring $\mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ (where $x_1, \dots, x_n$ are the standard co-ordinates on $\mathbb{C}^{\times n}$).

Definition 3.1. Let $f\colon \mathbb{C}^{\times n} \to \mathbb{C}$ be a Laurent polynomial. The *principal period* of f is:

$$\pi(t) = \Big(\frac{1}{2\pi i}\Big)^n \int_{|x_1|=\cdots=|x_n|=1} \frac{1}{1-tf(x_1,\dots,x_n)} \frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}$$

Theorem 3.2. *There principal period satisfies a ordinary differential equation $L\cdot\pi(t)\equiv 0$ where $L\in\mathbb{C}\langle t,D\rangle$ ($D=t\frac{d}{dt}$) is a polynomial differential operator*[1]*.* □

Definition 3.3. A Picard–Fuchs operator $L_f\in\mathbb{C}\langle t,D\rangle$ of f is a generator (uniquely defined up to multiplication by a constant) of the annihilator ideal of the principal period $\pi(t)$.

Definition 3.4 (V. Golyshev)**.** f is an *Extremal Laurent Polynomial (ELP)* if the local system $\operatorname{Sol} L_f$ of *solutions* of the ODE $L_f\cdot()\equiv 0$ is extremal.

Remark 3.5.

- The proof of Theorem 3.2 makes it clear that $\operatorname{Sol} L_f$ is a summand of $\operatorname{gr}^W_{n-1}R^{n-1}f_!\,\mathbb{Z}_{\mathbb{C}^{\times n}}$.
- Consider a semistable rational elliptic surface $f\colon X\to\mathbb{C}$. In general f has 12 singular fibres; Beauville classified surfaces with the smallest number, 4, of singular fibres. These surfaces of Beauville can all be realised as extremal Laurent polynomials.
- Intuitively, a Laurent polynomial is extremal if it is *maximally degenerate* in the sense that as many critical values co-incide as possible. For this reason, given a polytope P, we expect that there are (at most) finitely many ELPs f with $\operatorname{Newt}(f)=P$.

How to compute the Picard–Fuchs operator and the ramification in practice

1 Computing with the residue theorem gives $\pi(t)=\sum c_m t^m$ where $c_m = \operatorname{coeff}_{\mathbf{1}} f^m$ is the *period sequence*. Indeed, expanding $\pi(t)$ as a power series in

[1] Eventually I will add a proof of this fact relying on the combinatorics of Newton(f) also providing an estimate of the degree (in t and D of the operator.)

t and applying the residue theorem n times:

$$\pi(t) = \Big(\frac{1}{2\pi\, i}\Big)^n \int_{|x_1|=\cdots=|x_n|=1} \frac{1}{1-tf}\frac{d\,x_1}{x_1}\cdots\frac{d\,x_n}{x_n} =$$
$$= \sum_{m=0}^{\infty} t^m \Big(\frac{1}{2\pi\, i}\Big)^n \int_{|x_1|=\cdots=|x_n|=1} f^m \,\frac{d\,x_1}{x_1}\cdots\frac{d\,x_n}{x_n} = \sum_{m=0}^{\infty} c_m t^m\,. \quad (1)$$

In practice, the computation of $\operatorname{coeff}_{\mathbf{1}} f^m$, say for $1 \leq m \leq 600$, is very expensive.

2 Consider a polynomial differential operator $L = \sum t^k P_k(D)$ where $P_k(D) \in \mathbb{C}[D]$ is a polynomial in D; then $L \cdot \pi \equiv 0$ is equivalent to the *recursion relation* $\sum P_k(m-k)c_{m-k} = 0$. In practice, to compute L_f one uses knowledge of the first few periods and linear algebra to guess the recursion relation.

3 The computation of $\operatorname{rf}(\operatorname{Sol} L_f)$ is an algorithm built on standard Fuchsian theory. (I don't have the time to explain this.)

Example 3.6. Consider $f(x,y) = x + y + \frac{1}{xy}$. It is very easy to see that:

$$\pi(t) = \sum_{m\geq 0} \frac{(3m)!}{(m!)^3} t^{3m}$$

it is best to work with $u = t^3$; now $\pi(u) = \sum c_m u^m$ where the coefficients c_m satisfy the recursion relation:

$$n^2 c_n - 3(3n-1)(3n-2)c_{n-1} = 0$$

and, by what we said, this is equivalent to:

$$\Big[D^2 - 3u(3D+1)(3D+2)\Big]\pi(u) = 0\,.$$

Studying this ODE, we see that f is extremal.

Example 3.7. Consider $f(x,y) = x + xy + y + \frac{1}{xy}$. In this case

$$L_f = 8D^2 - tD - t^2(5D+8)(11D+8) -$$
$$- 12t^3(30D^2 + 78D + 47) - 4t^4(D+1)(103D+147) -$$
$$- 99t^5(D+1)(D+2)\,.$$

Here it is possible to see that $\operatorname{rf}(\operatorname{Sol} L_f) = 5 = 2\operatorname{rk}(\operatorname{Sol} L_f) + 1$ so the polynomial f is not extremal. In fact, there are no ELPs with this Newton polytope.

Research Program, Part I (a rather idealised form)

Construct examples of ELP in 3, 4 and 5 variables systematically by computer. In some cases, classify all ELP. More precisely:

- Fix a lattice polytope P. Classify all ELP f with $\text{Newt}\,(f) = P$.
- Do this for a natural class of polytopes, for instance reflexive polytopes. (Kreuzer and Skarke show that there are $4,319$ reflexive polytopes in 3 dimensions and more than 473 million in 4 dimensions.)

Minkowski polynomials

I describe (for simplicity, in 2 and 3 variables only) a class of Laurent polynomials that we discovered and christened "Minkowski polynomials" (MP) because they have something to do with Minkowski decomposition. This class is especially nice because:

- MPs have (experimentally, conjecturally) low ramification.
- In our experience, all Minkowski polynomials mirror a Fano manifold.

By a lattice *polygon* we always mean a (possibly degenerate) polytope $P \subset \mathbb{R}^n$ of dimension ≤ 2. Then $P \cap \mathbb{Z}^n$ is an affine lattice whose underlying lattice we denote by $\text{Lattice}(P)$.

Definition 3.8.

- A lattice polygon $P \subset \mathbb{R}^2$ is *admissible* if $\text{Int}(P) \cap \mathbb{Z}^2 = \emptyset$.
- A lattice polytope $P \subset \mathbb{R}^n$ is *reflexive* if one of the two equivalent conditions hold: (a) the polar polytope

$$P^* = \{f \in \mathbb{R}^{n*} \mid \langle f, v\rangle \geq -1 \,\forall v \in P\}$$

is also a lattice polytope, or (b) $\text{Int}\, P \cap \mathbb{Z}^n = \{\mathbf{0}\}$.

Definition 3.9. Let $Q \subset \mathbb{R}^n$ be a lattice polygon. A *lattice Minkowski decomposition* (LMD) of Q is:

- a Minkowski decomposition $Q = R + S$ into lattice polygons R, S, such that:
- $\text{Lattice}(Q) = \text{Lattice}(R) + \text{Lattice}(S)$.

The Minkowski ansatz Fix a reflexive polytope $P \subset \mathbb{R}^n$ of dimension ≤ 3. We describe a recipe to write down Laurent polynomials

$$f = \sum_{\mathbf{m} \in P \cap \mathbb{Z}^n} c_{\mathbf{m}} \, x^{\mathbf{m}}$$

with Newton$(f) = P$. I just need to tell you how to choose the coefficients $c_{\mathbf{m}}$. In all cases, I always take $c_{\mathbf{0}} = 0$. (This is an over-all normalization choice that corresponds to the fact that $p_1 = 0$ in the quantum period, see below.)

If P is a (reflexive or admissible) polygon, I just need to tell you how to construct the edge terms. An edge E of P lattice Minkowski decomposes into a sum of k copies of the standard unit interval $[0, 1]$ and the corresponding term is $f_E = (1 + x)^k$.

If P is a 3-tope, then I treat the edges as above. Next I need to give a recipe for the facet terms f_F, $F \subset P$ a facet. First lattice Minkowski decompose each facet into irreducibles

$$F = F_1 + \cdots + F_r \, .$$

I say that the decomposition is admissible if all F_i are admissible. Given an admissible decomposition of every facet of P, the recipe assigns a MP. The facet term corresponding to F is

$$f_F = \prod f_{F_i}$$

where f_{F_i} is given as above using the recipe for admissible polygons.

MPs in 2 variables There are 16 reflexive polygons (it is a good exercise to derive this list for yourself). All support one MP. This gives 16 MPs but only 10 period sequences. These are in 1-to-1 correspondence with del Pezzo surfaces of degree ≥ 3. The 10 period sequences are extremal with two exception: the first we already met in Example 3.7 (mirror of $\mathbb{F}_1$), the other is:

Example 3.10. $f(x, y) = x + y + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}$ (mirror of dP_7). Here

$$\begin{aligned} L_f = 7D^2 + tD(31D - 3) - t^2(85D^2 + 238D + 112) - \\ - 2t^3(358D^2 + 785D + 425) - 2t^4(D + 1)(669D + 970) - \\ - 731t^5(D + 1)(D + 2) \, . \end{aligned}$$

and $\mathrm{rf}(\mathrm{Sol}\, L_f) = 5 = 2\, \mathrm{rk}(\mathrm{Sol}\, L_f) + 1$.

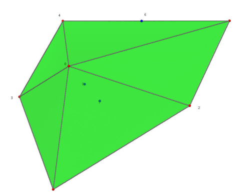
MPs in 3 variables In 3 variables, we have shown the following facts:

- There are 4,319 reflexive 3-topes;
- they have 344 distinct facets, and these have 79 lattice Minkowski irreducible pieces;
- of these, the admissible ones are A_n-triangles for $1 \leq n \leq 8$.
- There are thousands of MPs but only 165 period sequences. We are confident that they are all extremal.

Example 3.11. Consider the reflexive polytope in $\mathbb{R}^3$ with vertices:

$$\begin{pmatrix} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

This is polytope 121 in the Kreuzer–Skarke PALP list:



The pentagonal facet (as pictured, this is the base) has two Minkwoski decompositions:

and so polytope 121 supports two Minkowski polynomials:

$$f_1 = x + y + z + 3x^{-1} + x^{-1}y^{-1}z + x^{-2}z^{-1} + 2x^{-2}y^{-1} + x^{-3}y^{-1}z^{-1}$$

$$f_2 = x + y + z + 2x^{-1} + x^{-1}y^{-1}z + x^{-2}z^{-1} + 2x^{-2}y^{-1} + x^{-3}y^{-1}z^{-1}$$

The principal periods associated to f_1 and f_2 are:

$$\pi_1(t) = 1 + 6t^2 + 90t^4 + 1860t^6 + 44730t^8 + 1172556t^{10} + \cdots$$
$$\pi_2(t) = 1 + 4t^2 + 60t^4 + 1120t^6 + 24220t^8 + 567504t^{10} + \cdots$$

The corresponding Picard–Fuchs operators are:

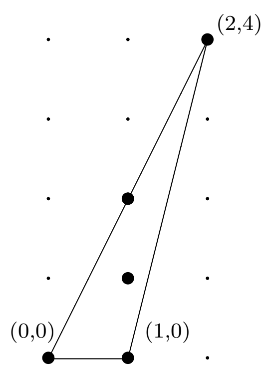
$$L_1 = 144t^4D^3 + 864t^4D^2 + 1584t^4D - 40t^2D^3 + 864t^4 - \\ - 120t^2D^2 - 128t^2D + D^3 - 48t^2$$

$$L_2 = 128t^4D^3 + 768t^4D^2 + 1408t^4D + 28t^2D^3 + 768t^4 + 84t^2D^2 + \\ + 88t^2D - D^3 + 32t^2$$

In 4 variables, there are > 473 million reflexive polytopes. We have inherited the database of Maximilian Kreuzer and we are now in the process of making a database of facets in preparation for computing their lattice Minkowski decompositions.

Not all ELP are MP

Consider the pictured polygon. This is one of the smallest facets for which



the Minkowski ansatz has nothing to say. Consider the Laurent polynomial with this Newton polygon given by:

$$f = 1 + x + 2xy^2 + x^2y^4 + axy$$

For generic a the completion of $f = 0$ is a nonsingular curve of genus 1; it becomes singular exactly when $a = \pm 4$ and in this case the geometric genus of the completion of $f = 0$ is zero. Let us take $a = 4$ and use this as a new "puzzle piece" for the Minkowski ansatz.

Consider the 3-dimensional reflexive polytope with PALP id 9. This has four faces: two smooth triangles, one A_2-triangle, and one face equal to the polygon shown above. The corresponding Laurent polynomial is:

$$F = x + y + z + x^{-4}y^{-2}z^{-1} + 2x^{-2}y^{-1} + 4x^{-1}$$

It has period sequence:

$$1, 0, 8, 0, 120, 0, 2240, 0, 47320, 0, \ldots$$

The Picard–Fuchs operator is:

$$512t^4D^3+3072t^4D^2+5632t^4D-48t^2D^3+3072t^4-144t^2D^2-160t^2D+D^3-64t^2$$

It can be seen that f is extremal, but it does not mirror any Fano 3-fold. (One can do the same thing with $a = -4$.)

4 Fano Manifolds and quantum cohomology

The quantum period

Definition 4.1. A complex projective manifold X^n of dimension n is a *Fano manifold* if the anticanonical line bundle $-K_X = \wedge^n T_X = \Omega_X^{n,\vee}$ is ample.

Remark 4.2.

- If $n = 2$ X is called a *del Pezzo* surface. It is well-known that a del Pezzo surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow up of $\mathbb{P}^2$ in ≤ 8 general points.
- It is known that there are precisely 105 deformation families of nonsingular Fano 3-folds. There are 17 families with $b_2 = 1$ (Fano, Iskovskikh) and 88 families with $b_2 \geq 2$ (Mori–Mukai).

I state a well-known theorem of Mori that plays a crucial role in what follows:

Theorem 4.3. *Let X be a Fano manifold. Denote by* $\mathrm{NE}\, X \subset H_2(X;\mathbb{R})$ *the* Mori cone *of X: that, is, the covex cone generated by (classes of) algebraic curves $C \subset X$. Then* $\mathrm{NE}\, X$ *is a closed rational polyhedral cone.*

When X is Fano, denote by $\mathcal{M}_{0,k,m}$ the moduli space of stable morphisms $f\colon (C, x_1, \dots, x_k) \to X$ where C is a nodal curve of genus 0 with k marked points $x_1, \dots, x_k$, and $\deg f^\star(-K_X) = m$. This moduli space has *virtual dimension* $m - 3 + n + k$. Here we are mainly interested in $\mathcal{M}_{0,1,m}$ and the (unique) *evaluation morphism*

$$e_m\colon \mathcal{M}_{0,1,m} \to X$$

Denote by ψ the (first Chern class of) the line bundle on $M_{0,1,m}$ of *cotangent lines*: the fibre of this line bundle at $f\colon (C, x) \to X$ is the fibre of ω_C at x.

Definition 4.4. The *quantum period* of X is the power series $K(t) = \sum p_m t^m$ where $p_m = \int_{\mathcal{M}_{0,1,m}} \psi^{m-2} e_m^\star(\mathrm{pt})$. The sequence p_m is the *quantum period sequence.*

Theorem 4.5. *The quantum period satisfies a ordinary differential equation $Q \cdot K_X(t) \equiv 0$ where $Q \in \mathbb{Z}\langle t, D\rangle$ ($D = t\frac{d}{dt}$) is a polynomial differential operator.*

Proof. In short: our quantum period $K_X(t)$ is a stripped-down version of the *small J-function* of quantum cohomology. The result then follows from elementary properties of small quantum cohomology. I now explain all this in greater detail.

In what follows we denote by $\mathcal{M}_{0,k,\beta}$ the moduli space of maps of degree $\beta \in \mathrm{NE}\, X \cap H_2(X, \mathbb{Z})$. Recall that the *small quantum product* $a * b$ of (even degree) cohomology classes $a, b \in H^{\mathrm{ev}}(X; \mathbb{C})$ is defined by the following formula, which is to hold for all $c \in H^{\mathrm{ev}}(X; \mathbb{C})$:

$$(a * b, c) = (a \cup b, c) + \sum_{0 \neq \beta \in \mathrm{NE}\, X \cap H_2(X;\mathbb{Z})} q^\beta \langle a, b, c\rangle_{0,3,\beta}$$

where $(a, b) = \int_X a \cup b$ is the Poincaré inner product and

$$\langle a, b, c\rangle_{0,3,\beta} = \int_{\mathcal{M}_{0,3,\beta}} ev_1^\star(a) \cup ev_2^\star(b) \cup ev_3^\star(c)$$

is the 3-point correlator. The Frobenius manifold structure is equally well encoded in an *integrable algebraic connexion* ∇ on:

- the trivial bundle with fibre $H^{\mathrm{ev}}(X; \mathbb{C})$ on

- the torus $\mathbb{T} = \operatorname{Spec}\mathbb{C}[H_2(X,\mathbb{Z})]$.

In other words $\mathbb{T}$ is the torus with character group $\operatorname{Hom}_{\text{groups}}(\mathbb{T},\mathbb{C}^\times) = H_2(X;\mathbb{Z})$, co-character group $\operatorname{Hom}_{\text{groups}}(\mathbb{C}^\times,\mathbb{T}) = H^2(X;\mathbb{Z})$, and group of $\mathbb{C}$-valued points $\mathbb{T}(\mathbb{C}) = \mathbb{C}^\times \otimes H^2(X;\mathbb{Z})$. Note that $\operatorname{Lie}\mathbb{T} = H^2(X;\mathbb{C})$. The connexion ∇ is defined as follows: if $\psi\colon \mathbb{T} \to H^2(X;\mathbb{C})$ and $X \in \operatorname{Lie}\mathbb{T} = H^2(X;\mathbb{C})$:

$$\nabla_X \psi = X\cdot\psi - X * \psi\,.$$

The fact that this connexion is *algebraic* follows from the fact that quantum cohomology is *graded* and that $-K_X > 0$ on $\operatorname{NE} X$. The fact that the connexion is integrable (flat) is fundamental and it means that the action of $\operatorname{Lie}\mathbb{T}$ on $M = \{\psi\colon \mathbb{T} \to H^{\text{ev}}(X;\mathbb{C})\}$ extends to an action of the ring $\mathcal{D}$ of differential operators on $\mathbb{T}$: in other words, M is a $\mathcal{D}$-module, called the *quantum $\mathcal{D}$-module*. In general, out of a $\mathcal{D}$-module, we can make two local systems: the local system $\operatorname{Hom}_{\mathcal{D}}(\mathcal{O}, M)$ of *flat sections* of M, and the local system $\operatorname{Hom}_{\mathcal{D}}(M,\mathcal{O})$ of *solutions* of M. Sections of these local systems tautologically satisfy algebraic PDEs.

Recall that the (small) J-function of X is defined as follows:

$$J_X(q) = 1 + \sum_{\beta\in NEX\cap H^2(X;\mathbb{Z})} q^\beta J_\beta, \quad \text{where} \quad J_\beta = ev^\beta_\star \frac{1}{1-\psi}$$

(here $ev^\beta\colon M_{0,1,\beta}$ denotes the unique evaluation map). It is well-known that $J_X(q)$ is a solution of the quantum $\mathcal{D}$-module and therefore it tautologically satisfies an algebraic PDE. Note that $J_X(q)$ is cohomology valued but it makes sense to take its *degree-0* component $J^0_X(q) \in H^0(X,\mathbb{C})$.

Finally, the anticanonical class $-K_X \in H^2(X;\mathbb{Z})$ is a co-character of $\mathbb{T}$, that is, it "is" a group homomorphism which I denote $\kappa\colon \mathbb{C}^\times \to \mathbb{T}$, and it is clear from the definition that $K_X(t) = J^0_X \circ \kappa(t)$ (here t is the co-ordinate function on $\mathbb{C}^\times$), and the discussion above makes it clear that it satisfies an algebraic ODE. □

Definition 4.6. The *quantum differential operator* of X is the generator $Q_X \in \mathbb{Z}\langle t, D\rangle$ of the annihilator ideal of the quantum period $K(t)$.

How to compute Q_X in practice In practice one starts by fixing a basis $\{T^a\}$ of $H^{\text{ev}}(X;\mathbb{Z})$ with $T^0 = \mathbf{1}$ the fundamental class. Let $M = M(t)$ be the matrix of quantum multiplication by $-K_X$ in this basis, written as a function on $\mathbb{C}^\times$ by composing with $\kappa\colon \mathbb{C}^\times \to \mathbb{T}$. Next consider the differential equation on $\mathbb{C}^\times$

$$\begin{cases} D\Psi(t) = \Psi M \\ \Psi(0) = I \end{cases}$$

for $\Psi\colon \mathbb{C}^\times \to \operatorname{End}\Big(H^{\mathrm{ev}}(X,\mathbb{C})\Big)$ a matrix. (Note: tautologically, the differential $\kappa_\star\colon \operatorname{Lie}\mathbb{C}^\times \to \operatorname{Lie}\mathbb{T}$ sends $D = t\,\frac{d}{dt}$ to $-K_X \in H^2(X;\mathbb{C}) = \operatorname{Lie}\mathbb{T}$.) Then the first *column* of Ψ is $J_X \circ \kappa(t)$; the first entry of the first column is our quantum period $K_X(t)$.

Remark 4.7. It is important that the matrix Ψ is on the left of M; otherwise we would be computing the flat sections of the quantum $\mathcal{D}$-module. We wish to compute the solutions instead (which is the same as the flat sections of the dual $\mathcal{D}$-module).

Example 4.8. Consider now $X = \mathbb{P}^2$ with cohomology ring $\mathbb{C}[P]/P^3$. Choose the basis $\mathbf{1}, -K = 3P, K^2 = 9\{\mathrm{pt}\}$ for the cohomology. The matrix of quantum multiplication by $-K$, in this basis, is:

$$M = \begin{pmatrix} 0 & 0 & 27t^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

where the coefficient of t^3 in the upper right corner of the matrix is calculated as a nontrivial Gromov–Witten number:

$$\langle -K * (K^2), \mathrm{pt}\rangle_{0,3,[\mathrm{line}]} = 3\langle K^2, \mathrm{pt}\rangle_{0,2,[\mathrm{line}]} = 27\langle \mathrm{pt}, \mathrm{pt}\rangle_{0,2,[\mathrm{line}]} = 27\,.$$

Next we consider the system

$$D(\psi_0, \psi_1, \psi_2) = (\psi_0, \psi_1, \psi_2)M$$

The column ψ_0 satisfies the differential operator

$$Q_X = D^3 - 27t^3 \quad \text{with solution:} \quad K(t) = \sum_{m=0}^{\infty} t^m \frac{1}{(m!)^3}\,.$$

The regularized quantum period and Mirror symmetry

The previous example suggests the following definitions:

Definition 4.9. The *regularised quantum period* is the Fourier–Laplace transform $\widehat{K}(t) = \sum (m!) p_m t^m$ of the quantum period $K(t)$.

The *regularised quantum differential operator* of X is the generator $\widehat{Q}_X \in \mathbb{Z}\langle t, D\rangle$ of the annihilator ideal of the regularised quantum period $\widehat{K}_X(t)$.

Expectation 4.10. *We hope that* $\mathbb{V} = \operatorname{Sol} \widehat{Q}_X$ *has low ramification. More precisely, we hope that* $\operatorname{rf} \mathbb{V} \leq 2 \operatorname{rk} \mathbb{V} + \dim H^{\frac{n}{2},\frac{n}{2}}_{prim}$, *the primitive cohomology of Hodge type* $n/2, n/2$. *(In particular, when* $\dim X$ *is odd, we hope that* $\mathbb{V}$ *is extremal.)*

The following is a very weak form of Mirror symmetry for Fano manifolds.

Definition 4.11. The Laurent polynomial f is mirror-dual to the Fano manifold X if $\pi(t) = \widehat{K}(t)$ (equivalently, $L_f = \widehat{Q}_X$).

Remark 4.12. It is well-known that Q_X has a pole of order 2 (an irregular singularity) at ∞ and regular singularities elsewhere. On the other hand, by a theorem of Deligne, Picard–Fuchs operators have regular singularities everywhere on $\mathbb{P}^1$: hence, regularisation is necessary before a comparison is possible. Alternatively, we could have decided to compare the quantum period directly with the oscillating integral of f. If we had done so, however, we would have missed the low ramification property.

Warning 4.13. *This is a very weak notion of mirror symmetry. To a Fano manifold* X *there correspond infinitely many mirror Laurent polynomials* f.

What we've done: Fano 3-folds

Definition 4.14. A *Minkowski period sequence* it the period sequence $\{c_m\}$ of a Minkowski polynomial f. Let $L_f = \sum_{k=0}^{D} t^k P_k(D) P_k \in \mathbb{Z}\langle t, D\rangle$ be the corresponding Picard–Fuchs operator. The period sequence is of *orbifold type* if $L_f(0) = P_0(D)$ has some nonintegral roots. Otherwise $L_f(0)$ has integral roots and we say that the period sequence is of *manifold type.*

- We made a list of all Minkowski polynomials in 3 variables supported on one of the 4,319 reflexive 3-topes. In 3 variables, there are 165 period sequences.

- In most cases, we were able to calculate the Minkowski Picard–Fuchs operators and local monodromies and we found that they are all extremal.

- Of the 165 period sequences, 67 are of orbifold type. The other 98 are in one-to-one correspondence with 98 of the 105 families of Fano 3-folds.

Remark 4.15. For the remaining 7 families, we know mirror Laurent polynomials whose Newton polytopes are nonreflexive.

Research Program, Part II (a rather idealised form)

- Make a list of all Minkowski polynomials in 4 variables (or, perhaps, a larger meaningful class of Laurent polynomials) and calculate the (first few hundred terms of their) period sequences.

- Compute the Minkowski Picard–Fuchs operators and verify that they are of low ramification.

- Use the list of Minkowski period sequences as a (partial) *directory* of Fano 4-folds: the first few (say 10) terms of the period sequence are like a phone number of a residence that may be occupied by a Fano 4-fold.

- Extract information (Hilbert function, Chern numbers, Betti cohomology) about the (potential) Fano 4-folds from the Picard–Fuchs operators.

- Construct the Fano 4-folds as smoothings of the singular toric Fano 4-fold with fan polytope Newton(f). (This ought to become clearer once we make contact with the Gross–Siebert program.)

Methods of Calculation

I explain how to calculate the quantum period of a Fano complete intersection in a toric manifold using the quantum Lefschetz theorem of Givental and Coates–Givental.

Toric varieties For us, a toric variety is a GIT quotient:

$$X = \mathbb{C}^r /\!/_\chi (\mathbb{C}^\times)^b$$

where $(\mathbb{C}^\times)^b$ acts via a group homomorphism $\rho\colon (\mathbb{C}^\times)^b \to (\mathbb{C}^\times)^r$, where $(\mathbb{C}^\times)^r$ acts on $\mathbb{C}^r$ via its natural diagonal action. The group homomorphism ρ is given dually by a $b \times r$ integral matrix:

$$D = (D_1, \dots D_r)\colon \mathbb{Z}^r \to \mathbb{Z}^b$$

that we call the *weight data* of the toric variety X.

The weight data alone does not determine X: it is necessary to choose a $(\mathbb{C}^\times)^b$-linearized line bundle L on $\mathbb{C}^r$: this choice is equivalent to the choice of a character $\chi \in \mathbb{Z}^b$ of $(\mathbb{C}^\times)^b$: denoting by L_χ the corresponding line bundle, we have

$$H^0(\mathbb{C}^r; L_\chi)^{(\mathbb{C}^\times)^b} = \Big\{ f \in \mathbb{C}[x_1, \dots, x_r] \mid f(\lambda\, x) = \chi(\lambda)\, f(x)\ \forall \lambda \in (\mathbb{C}^\times)^b \Big\} .$$

Having made this choice, the set of stable points is

$$U^s(\chi) = \Big\{ \mathbf{a} \in \mathbb{C}^r \mid \exists N \gg 0,\, \exists f \in H^0(\mathbb{C}^r; L_\chi)^{(\mathbb{C}^\times)^b},\ f(\mathbf{a}) \neq 0 \Big\} .$$

The set of $\chi \in \mathbb{Z}^b$ for which $U^s(\chi) \neq \emptyset$ generates a rational polyhedral cone in $\mathbb{R}^b$ with a partition in locally closed rational polyhedral chambers defined such that $U^s(\chi)$ depends only on the chamber containing χ. We always choose χ in the interior of a chamber of maximal dimension, and then we define $X = U^s(\chi)/(\mathbb{C}^\times)^b$. We have an identification $\mathbb{Z}^b = H^2(X;\mathbb{Z}) = \operatorname{Pic}(X)$ and the chamber containing χ is then identified with the *ample cone* $\operatorname{Amp} X$. The appropriate Euler sequence shows that $-K_X = \sum_{i=1}^r D_i$.

Theorem 4.16 (Givental)**.** *Let X be a toric Fano manifold. Then*

$$K_X(t) = \sum_{\mathbf{k} \in \mathbb{Z}^b \cap \operatorname{NE} X} t^{-K_X \cdot \mathbf{k}} \frac{1}{(D_1 \cdot \mathbf{k})! \cdots (D_r \cdot \mathbf{k})!} .$$

□

Theorem 4.17 (Givental)**.** *Let F be a Fano manifolds and $L_1, \dots, L_c \in \operatorname{Nef} F$. Consider:*

$$X = (f_1 = \cdots = f_c = 0) \subset F, \quad \textit{where} \quad f_i \in H^0(F; L_i) .$$

Assume:

- *X is nonsingular of the expected codimension $c = \operatorname{codim}_X F$, and*
- *$A = -(K_X + \sum_{i=0}^{c})L_i) \in \operatorname{Amp} F$.*

Then $K_X(t) = \exp(-a_1 t) I_X(t)$ where

$$I_X(t) = \sum_{\mathbf{k} \in \mathbb{Z}^b \cap \operatorname{NE} F} t^{A\cdot\mathbf{k}} \frac{(L_1 \cdot \mathbf{k})! \cdots (L_c \cdot \mathbf{k})!}{(D_1 \cdot \mathbf{k})! \cdots (D_r \cdot \mathbf{k})!} = 1 + a_1 t + O(t^2)\,.$$

□

5 Three examples

I give three examples illustrating the program in 3 dimensions.

Example 1

Proposition 5.1. *The period $\pi_1(t)$ of the MP f_1 of Example 3.11 is the regularized quantum period of the Fano 3-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. $J(t) = \sum_{k,l,m\geq 0} \frac{t^{2k+2l+2m}}{k!k!l!l!m!m!}$ hence:

$$\widehat{J}(t) = \sum_{k,l,m\geq 0} t^{2k+2l+2m} \frac{(2k+2l+2m)!}{k!k!l!l!m!m!}$$
$$= 1 + 6t^2 + 90t^4 + 1860t^6 + 44730t^8 + 1172556t^{10} + \cdots$$

□

Example 2

Proposition 5.2. *The period $\pi_2(t)$ of the MP f_2 of Example 3.11 is the regularized quantum period of the Fano 3-fold W, where W is a divisor of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.*

Proof. Quantum Lefschetz gives $J(t) = \sum_{k,l\geq 0} t^{2k+2l} \frac{(k+l)!}{k!k!k!l!l!l!}$ hence:

$$\widehat{J}(t) = \sum_{k,l\geq 0} t^{2k+2l} \frac{(k+l)!(2k+2l)!}{k!k!k!l!l!l!}$$
$$= 1 + 4t^2 + 60t^4 + 1120t^6 + 24220t^8 + 567504t^{10} + 14030016t^{12} + \cdots$$

□

Example 3

Consider now the polytope with vertices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & -1 & -1 & 0 & 1 & -2 \\ 0 & 0 & 2 & -2 & 0 & -1 & 1 & -2 \end{pmatrix}$$

This is polytope 2093 in the Kreuzer–Skarke PALP list. It supports a unique Minkowski polynomial:

$$f = yz^2 + 2yz + x + y + 2z + x^{-1}yz + xz^{-1} + 2z^{-1} + y^{-1} + 3x^{-1} + \\ + y^{-1}z^{-1} + y^{-1}z^{-2} + 3x^{-1}y^{-1}z^{-1} + x^{-1}y^{-2}z^{-2}$$

The principal period of f is:

$$\pi(t) = 1 + 22t^2 + 174t^3 + 2514t^4 + 34200t^5 + 501070t^6 + 7586880t^7 + \cdots$$

The corresponding Picard–Fuchs operator is:

$$\begin{aligned} L = {} & 1156602627t^{10}D^4 + 11566026270t^{10}D^3 + 2432454171t^9D^4 + \\ & + 40481091945t^{10}D^2 + 21710463138t^9D^3 + 2180832814t^8D^4 + \\ & + 57830131350t^{10}D + 69451424553t^9D^2 + 17076830696t^8D^3 + \\ & + 1081614794t^7D^4 + 27758463048t^{10} + 92867844258t^9D + \\ & + 49152182076t^8D^2 + 7275730258t^7D^3 + 320495624t^6D^4 + \\ & + 42694428672t^9 + 60631223566t^8D + 18475102414t^7D^2 + \\ & + 1803663274t^6D^3 + 56396924t^5D^4 + 26375039372t^8 + \cdots \\ & + 20646075298t^7D + 3923559726t^6D^2 + 257499448t^5D^3 + \\ & + 5230066t^4D^4 + 8365088348t^7 + 3859944956t^6D + \\ & + 453227034t^5D^2 + 19311296t^4D^3 + 113734t^3D^4 + \\ & + 1418470580t^6 + 369455180t^5D + 22953224t^4D^2 + \\ & + 641894t^3D^3 - 18907t^2D^4 + 115564004t^5 + 12261988t^4D \\ & - 49938t^3D^2 + 24976t^2D^3 - 1031tD^4 + 2204080t^4 - \\ & - 358692t^3D - 28323t^2D^2 + 2174tD^3 + 16D^4 - \\ & - 165208t^3 - 16128t^2D - 55tD^2 - 16D^3 - 2816t^2. \end{aligned}$$

Proposition 5.3. *$\pi(t)$ is the regularized quantum period of the Fano 3-fold X which is number 9 in Mori–Mukai's list of rank-2 Fano 3-folds. Here X is the blow up of $\mathbb{P}^3$ in a curve Γ of degree 7 and genus 5.*

Proof. Γ is cut out by the equations:

$$\operatorname{rk}\begin{pmatrix} l_0 & l_1 & l_2 \\ q_0 & q_1 & q_2 \end{pmatrix} < 2$$

where the l_i are linear forms and the q_j are quadratics. Write:

$$\begin{aligned} y_0 &= l_0 q_1 - l_1 q_0 \\ y_1 &= l_2 q_0 - l_0 q_2 \\ y_2 &= l_0 q_1 - l_1 q_0 \ . \end{aligned}$$

The relations (szyzgies) between these equations are generated by:

$$\begin{aligned} l_0 y_0 + l_1 y_1 + l_2 y_2 = 0 \ , \\ q_0 y_0 + q_1 y_1 + q_2 y_2 = 0 \ . \end{aligned}$$

Thus X is defined by these two equations in $\mathbb{P}^3 \times \mathbb{P}^2$, where the first factor has co-ordinates x_0, x_1, x_2, x_3 and the second factor has co-ordinates y_0, y_1, y_2.

Since X is a complete intersection in $\mathbb{P}^3 \times \mathbb{P}^2$ of type $(1,1) \cdot (2,1)$, we have $-K_X = (1,1)$. Quantum Lefschetz gives:

$$\begin{aligned} I_X(t) = \sum_{l,m \geq 0} t^{l+m} \frac{(l+m)!(2l+m)!}{l!l!l!l!m!m!m!} = \\ = 1 + 3t + \frac{37t^2}{2} + \frac{769t^3}{6} + \frac{7307t^4}{8} + \frac{786991t^5}{120} + \frac{33872833t^6}{720} + \\ + \frac{188039513t^7}{560} + \frac{165697813t^8}{70} + \cdots \end{aligned}$$

We recover the quantum period of X as:

$$\begin{aligned} K_X(t) = \exp(-3t)\, I_X(t) = \\ = 1 + 14t^2 + \frac{245t^3}{3} + 602t^4 + \frac{64796t^5}{15} + \frac{1114619t^6}{36} + \frac{46294021t^7}{210} + \\ \frac{6920662871t^8}{4480} + \cdots \end{aligned}$$

and the regularized quantum period of X as:

$$\widehat{K}(t) = 1 + 22t^2 + 174t^3 + 2514t^4 + 34200t^5 + 501070t^6 + 7586880t^7 + \\ + 117858370t^8 + \cdots$$

$\square$