ASYMPTOTIC BEHAVIOUR OF RATIONAL CURVES

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Abstract. These are a preliminary version of notes for a course delivered during the summer school on rational curves held in 2010 at Institut Fourier, Grenoble. Any comments are welcomed.

1. Introduction

1.1. **The problem.** The problem we will be concerned with, which is also considered in Peyre's lecture, may be loosely stated as follows: given an algebraic variety X (defined over a field k) possessing a lot of rational curves (by this we mean that the union of rational curves on X is not contained in a proper Zariski closed subset ; for example, this holds for rational varieties) is it possible to give a quantitative estimate of the number of rational curves on it? We expect of course an answer slighly less vague than: the number is infinite.

To give a more precise meaning to the above question, fix a projective embedding $\iota : X \subset \mathbf{P}^n$ (or, if you prefer and which amounts almost to the same, an ample line bundle $\mathcal{L}$ on X). Then given a morphism $x : \mathbf{P}^1 \to X$ we define its degree (with respect to ι)

$$\deg_\iota(x) \stackrel{\text{def}}{=} \deg((x \circ \iota)^* \mathscr{O}_{\mathbf{P}^n}(1) \qquad (1.1.1)$$

(or $\deg_{\mathcal{L}}(x) \stackrel{\text{def}}{=} \deg(x^*\mathcal{L})$). This is a nonnegative integer. Moreover, as explained in the lectures of the first week, we know from the work of Grothendieck that for any nonnegative integer there exists a quasi-projective variety $\mathbf{Hom}_{\iota,d}(\mathbf{P}^1, X)$ (or $\mathbf{Hom}_{\mathcal{L},d}(\mathbf{P}^1, X)$) parametrizing the set of morphisms $\mathbf{P}^1 \to X$ of ι-degree d. Recall in particular that for every k-extension L there is a natural 1-to-1 correspondence between the set of L-points of $\mathbf{Hom}_{\iota,d}(\mathbf{P}^1, X)$ and the set of morphisms $\mathbf{P}^1_L \to X \times_k L$ of ι-degree d.

Thus we obtain a sequence of quasi-projective varieties $\{\mathbf{Hom}_{\iota,d}(\mathbf{P}^1, X)\}_{d\in\mathbf{N}}$ and we can raise the (still rather vague) question: what can be said about the behaviour of this sequence? Note that one way to understand this question is to "specialize" the latter sequence to a numeric one, and consider the behaviour of the specialization. There are several natural examples of such numeric specializations. For instance we can consider the sequence $\{\dim(\mathbf{Hom}_{\iota,d}(\mathbf{P}^1, X))\}$ obtained by taking the dimension, or, if k is a subfield of the field of complex numbers $\mathbf{C}$, the sequence $\{\chi_c(\mathbf{Hom}_{\iota,d}(\mathbf{P}^1, X))\}$, where χ_c designates the Euler-Poincaré characteristic with compact support; if k is finite, one can also look at the sequence $\{\#\,\mathbf{Hom}_{\iota,d}(\mathbf{P}^1, X)(k)\}$.

As explained in details in Peyre's lecture, the study of the latter sequence is a particular facet of a problem raised by Manin and his collaborators in the late 1980's, namely the understanding of the asymptotic behaviour of the number of rational points of bounded height on varieties defined over global field. The degree of $x : \mathbf{P}^1 \to X$ may be interpreted as the logarithmic height of the point of $X(k(\mathbf{P}^1))$ determined by x. Note that instead of considering a variety defined over k, which may be interpreted as a constant family $X \times \mathbf{P}^1 \to \mathbf{P}^1$, we might as well look at nonconstant families, that is, varieties defined over the function field of

$\mathbf{P}^1$. In these notes, we will stick to the case of constant families. Another natural generalization would be of course to replace $\mathbf{P}^1$ by a curve of higher genus. Here we only stress that most of the results presented in these notes extend without much difficulty to the higher genus case. It is also possible to consider higher-dimensional generalization of the problem, see [Wan92].

1.2. **Batyrev's heuristic.** We retain all the notations introduced in the previous section. When the base field k is finite, Manin, his collaborators and subsequent authors made precise predictions about the asymptotic behaviour of the sequence $\{\#\,\mathbf{Hom}_{\mathcal{L},d}(\mathbf{P}^1, X)(k)\}$. Let us explain how Batyrev uses these predictions to give some heuristic insights on the asymptotic geometric properties of the sequence $\{\mathbf{Hom}_{\mathcal{L},d}(\mathbf{P}^1, X)\}$ (over an arbitrary field k). We will restrict ourselves to varieties X for which the following hypotheses hold (recall that the effective cone is the cone generated by the classes of effective divisors):

Hypotheses 1.1. *X is a smooth projective variety whose anticanonical bundle ω_X^{-1} is ample, in other words X is a Fano variety. The geometric Picard group of X is free of finite rank and the geometric effective cone of X is generated by a finite number of class of effective divisors*[1].

Moreover the degree of a morphism $x : \mathbf{P}^1 \to X$ will always be the anticanonical degree, namely $\deg(x) = \deg(x^*\omega_X^{-1})$.

For the sake of simplicity, we will assume in this section that the class ω_X^{-1} has index one in $\operatorname{Pic}(X)$, that is, $\operatorname{Min}\{d,\ \left[\omega_X^{-1}\right] \in d \operatorname{Pic}(X)\} = 1$.

In this setting, a naïve version of the predictions of Manin et al. is the asymptotic

$$\#\,\mathbf{Hom}_{\omega_X^{-1},d}(\mathbf{P}^1, X)(k) \underset{d\to+\infty}{\sim} c\, d^{\,\mathrm{rk}(\mathrm{Pic}(X))-1}\, (\#k)^d \tag{1.2.1}$$

where c is a positive constant.

We call it a naïve prediction since it was clear from the beginning that (1.2.1) could certainly not always hold because of the phenomenon of accumulating subvarieties. One of the simplest examples is the exceptional divisor of the projective plane blown-up at one point. One can check that with respect to the anticanonical degree "most" of the morphisms $x : \mathbf{P}^1 \to X$ factor through the exceptional divisor (*cf.* Peyre's lecture). Thus one is led to consider in fact the sequence $\{\mathbf{Hom}_{\omega_X^{-1},d,U}(\mathbf{P}^1, X)\}$ where U is a dense open subset of X and $\mathbf{Hom}_{\omega_X^{-1},d,U}(\mathbf{P}^1, X)$ designates the open subvariety of $\mathbf{Hom}_{\omega_X^{-1},d}(\mathbf{P}^1, X)$ parametrizing those morphisms $\mathbf{P}^1 \to X$ of anticanonical degree d which do not factor through $X \setminus U$. And one predicts that (1.2.1) holds for $\#\,\mathbf{Hom}_{\omega_X^{-1},d,U}(\mathbf{P}^1, X)(k)$ if U is a sufficiently small open dense subset of X[2].

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[1] When the characteristic of k is zero, it is true, though highly non trivial, taht the hypotheses on the Picard group and on the effective cone automatically holds for a Fano variety.

[2] In fact, one may (and will) also consider the case where the anticanonical bundle of X is not necessarily ample, but still lies in the interior of the effective cone; in this case $\mathbf{Hom}_{\omega_X^{-1},d}(\mathbf{P}^1, X)$ is not always a quasi-projective variety, but $\mathbf{Hom}_{\omega_X^{-1},d,U}(\mathbf{P}^1, X)$ is for a sufficiently small dense open set U, thus the refined prediction still makes sense in this context.

One must also stress that even with this refinement, the prediction has already been shown to fail for certain Fano varieties (see [BT96]; the proof is over a number field but adapts immediatly to our setting). Nevertheless, the class of Fano varieties for which the refined prediction holds might be expected to be quite large; in particular one might still hope that it holds for every del Pezzo surface; especially in the arithmetic setting, the analogous refined prediction was shown to be true for a large number of instances of Fano variety; here is a (far from complete) list of related work in the arithmetic setting: [BT98], [CLT02], [dlB02], [dlBF04], [dlBBP10], [dlBBD07], [FMT89], [STBT07], [Spe09], [Sal98], [ST97], [Thu08], [Thu93], [Pey95].

In order to "explain geometrically" the prediction (1.2.1), Batyrev makes use of the following heuristic:

Heuristic 1.2. *A geometrically irreducible d-dimensional variety defined over a finite field k has approximatively $(\#k)^d$ rational points defined over k.*

Of course there is the implicit assumption that the error terms deriving from this approximation will be negligible regarding our asympotic counting problem. This heuristic may be viewed as a very crude estimate deduced from the Grothendieck-Lefschetz trace formula expressing the number of k-points of X as an alternative sum of trace of the Frobenius acting on the cohomology. It is also used by Ellenberg and Venkatesh in a somewhat differentcounting problem, see [EV05].

Now for any morphism $x : \mathbf{P}^1 \to X$, its *absolute degree* is the element of $\mathrm{Pic}(X)^\vee$ defined by $\langle \deg(x)\,,\, \mathcal{L}\rangle = \deg(x^*\mathcal{L})$. For $y \in \mathrm{Pic}(X)^\vee$ and U an open dense subset of X, let $\mathbf{Hom}_y(\mathbf{P}^1, X)$ (respectively $\mathbf{Hom}_{y,U}(\mathbf{P}^1, X)$) denote the quasi-projective variety parametrizing the morphisms $\mathbf{P}^1 \to X$ with absolute degree y (respectively which do not factor through $X \setminus U$). Let us choose a finite family of effective divisors of X whose classes in $\mathrm{Pic}(X)$ generate the effective cone of X and let U be the complement of the union of the support of these divisors. Then a morphism $\mathbf{P}^1 \to X$ which does not factor by $X \setminus U$ has an absolute degree y such that $\langle y\,,\, D\rangle \geqslant 0$ for every effective class D, in other words y belongs to the dual $C_{\mathrm{eff}}(X)^\vee$ of the effective cone.

As explained in the first week of the summer school, the "expected dimension" of $\mathbf{Hom}_{y,U}(\mathbf{P}^1, X)$ is $\dim(X) + \langle y\,,\, \omega_X^{-1}\rangle$. For any algebraic variety Y, let us denote by $\rho(Y)$ the number of its geometrically irreducible components of dimension $\dim(Y)$. Assuming that $\rho(\mathbf{Hom}_{y,U}(\mathbf{P}^1, X))$ is asymptotically constant, that the dimension of $\mathbf{Hom}_{y,U}(\mathbf{P}^1, X)$ dimension coincide with the expected dimension, and that the above heuristic applies, the number of k-points of $\mathbf{Hom}_{y,U}(\mathbf{P}^1, X)$ may be approximated by

$$\mathrm{cste}\,\#\{y \in C_{\mathrm{eff}}(X)^\vee \cap \mathrm{Pic}(X)^\vee, \langle y\,,\, \omega_X^{-1}\rangle = d\}(\#k)^{d+\dim(X)}. \tag{1.2.2}$$

But we will see below that we have the asymptotic

$$\#\{y \in C_{\mathrm{eff}}(X)^\vee \cap \mathrm{Pic}(X)^\vee, \langle y\,,\, \omega_X^{-1}\rangle = d\} \underset{d\to X}{\sim} \alpha(X)\, d^{\,\mathrm{rk}(\mathrm{Pic}(X))} \tag{1.2.3}$$

where $\alpha(X)$ is a positive rational number (depending on the effective cone of X and on the class of ω_X^{-1}). Thus we see that the above geometric assumptions on the $\mathbf{Hom}_{y,U}(\mathbf{P}^1, X)$ together with the adopted heuristic are compatible with Manin's prediction.

As pointed out by Batyrev, this might lead (perhaps optimistically) to raise the following questions about the asymptotic behaviour of $\mathbf{Hom}_{y,U}(\mathbf{P}^1, X)$ and $\mathbf{Hom}_{\omega_X^{-1},d,U}(\mathbf{P}^1, X)$.

Question 1.3.
1. *is the dimension of $\boldsymbol{Hom}_{\omega_X^{-1},d,U}(\mathbf{P}^1, X)$ asymptotically equivalent to $d{+}\dim(X)$? when $\langle y\,,\, \omega_X^{-1}\rangle \to +\infty$, is the dimension of $\boldsymbol{Hom}_{y,U}(\mathbf{P}^1, X)$ asymptotically equivalent to $\langle y\,,\, \omega_X^{-1}\rangle + \dim(X)$?*
2. *is $\rho(\boldsymbol{Hom}_{\omega_X^{-1},d,U}(\mathbf{P}^1, X))$ asymptotically equivalent to $c\, d^{\mathrm{rk}(\mathrm{Pic}(X))-1}$ where c is a positive constant? when $\langle y\,,\, \omega_X^{-1}\rangle \to +\infty$, is $\rho(\boldsymbol{Hom}_{y,U}(\mathbf{P}^1, X))$ asymptotically constant?*

1.3. **A generating series: the degree zeta function.** In the previous sections, some predictions were formulated about the asymptotic behaviour of some particular specializations of the sequence $\{\mathbf{Hom}_{\omega_X^{-1},d,U}(\mathbf{P}^1, X)\}$, namely the ones obtained by considering the dimension, the number of geometrically irreducible components of maximal dimension and, in case k is finite, the number of k-points. One may of course wonder whether there exist predictions for other specializations, for instance the one deriving from the topological Euler-Poincaré characteristic with compact support. Concerning the latter, note that it has at least a common feature with the specialization "number of points over a finite field": they are both examples of maps from the set of isomorphism class of algebraic varieties to a commutative ring, which are additive in the sense that the relation $f(X) = f(X \setminus F) + f(F)$ holds whenever X is a variety and F is a closed subvariety of X, and satisfying moreover the relation $f(X \times Y) = f(X)\,f(Y)$. We call such maps *generalized Euler-Poincaré characteristic*, abbreviated in GEPC in the following . We are naturally led to consider the universal target ring for GEPC: as a group it is generated by symbols $[X]$ where X is a variety modulo the relations $[X] = [Y]$ whenever $X \xrightarrow{\sim} Y$ and $[X] = [F] + [X \setminus F]$ whenever F is a closed subvariety of X (the latter are often called *scissors relations*). We endow it with a ring structure by setting $[X]\,[Y] = [X \times Y]$. The resulting ring is called the *Grothendieck ring of varieties*[3] and denoted by $K_0(\mathrm{Var}_k)$. Thus the datum of a GEPC with value in a commutative ring A is equivalent to the datum of a ring morphism $K_0(\mathrm{Var}_k) \to A$.

For an algebraic variety V we denote by $[V]$ its class in the Grothendieck ring. Now a way to handle "all-in-one" every possible specialization of the sequence $\{\mathbf{Hom}_{\omega_X^{-1},d,U}\}$ deriving from a GEPC is to look at the sequence $\{\left[\mathbf{Hom}_{\omega_X^{-1},d,U}\right]\}$ which is thus a sequence with value in the ring $K_0(\mathrm{Var}_k)$. Note that although this is not really obvious at first sight, the knowledge of the class $[Y]$ of an algebraic variety Y allows also to recover $\dim(Y)$ and $\rho(Y)$ (though dim and ρ are certainly not GEPC), see below.

A classical and useful tool when dealing with a sequence of complex numbers $\{a_n\}$ is the associated generating series $\sum a_n\,t^n$. Indeed, it is often possible to get informations about the analytic behaviour of the meromorphic function defined by the series, which in turn yields by Tauberian theorems informations about the asymptotical behaviour of the sequence itself.

We can try a similar approach in our context by forming the generating series

$$Z(X,U,t) \overset{\text{def}}{=} \sum_{d \geqslant 0} \left[\mathbf{Hom}_{d,U}(\mathbf{P}^1, X)\right]\, t^d \in K_0(\mathrm{Var}_k)[[t]] \tag{1.3.1}$$

(from now on we will systematically drop the subscript ω_X^{-1}, reminding that the degree will always be considered with respect to the anticanonical line bundle) whose coefficients lie in the Grothendieck ring of varieties. We call it the *geometric degree zeta function.* Applying a GEPC $\chi\,:\,K_0(\mathrm{Var}_k) \to A$ to its coefficients yields a specialized degree zeta function with coefficients in A, denoted by $Z_\chi(X,U,t)$. If k is finite and the GEPC is $\#_k$ (that is, the morphism "number of k-points"), we recover the generating series associated to the counting of points of bounded anticanonical degree/height, which we will name the *classical degree zeta function.*

1.4. **Some more examples of GEPC.** So far we have given only two examples of GEPC, the topological Euler Poincaré characteristic with support compact and the number of k-rational points when k is a finite field. Both of them have of course a cohomological flavour. It turns out that cohomology theories are a natural reservoir of GEPC. Let us content ourselves to describe one particular example: fix a prime ℓ distinct from the characteristic of k, and a separable closure $\overline{k}$ of k. To every variety X defined over k are attached its ℓ-adic cohomology groups, which

[3]This ring, already considered by Grothendieck in the sixties (see [CS01]), has attracted a huge renewal of interest since Kontsevich used it fifteen years ago as a key ingredient of his theory of motivic integration. Its structure turns out to be quite difficult to understand. Let us just cite a celebrated open question, which has connections with the Zariski simplification problem: is the class of the affine line in the Grothendieck ring a zero divisor?

form a sequence of $\mathbf{Q}_\ell$-vector spaces $\{H^n(X_{\overline{k}},\mathbf{Q}_\ell)\}_{n\in\mathbf{N}}$ equipped with a continuous action of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. If X is proper, the $H^n(X_{\overline{k}},\mathbf{Q}_\ell)$ are finite dimensional and vanish for $n > 2\dim(X)$. When X is proper and smooth, one defines its *ℓ-adic Poincaré polynomial* by

$$\mathrm{Poinc}_\ell(X) \stackrel{\mathrm{def}}{=} \sum_{n\geqslant 0} \dim(H^n(X_{\overline{k}},\mathbf{Q}_\ell))t^n. \tag{1.4.1}$$

One can show that there is a ring morphism $\mathrm{Poinc}_\ell : K_0(\mathrm{Var}_k) \to \mathbf{Z}[t]$ extending Poinc_ℓ (in characteristic zero one may use the fact, proven by F.Bittner, that the class of smooth projective varieties, modulo the relations derived from blowing up along a smooth subvariety, form a presentation of $K_0(\mathrm{Var}_k)$; when k is finitely generated, one uses the weight filtration on the ℓ-adic cohomology groups with compact support; in the general case one reduces to the latter by a limiting process). For every algebraic variety X, we have $\deg(\mathrm{Poinc}_\ell([X]) = 2\dim(X)$ thus the knowledge of Poinc_ℓ allows to recover the dimension. In case k is a subfield of $\mathbf{C}$, comparison theorems between ℓ-adic cohomology and Betti cohomology show that the topological Euler-Poincaré characteristic factors through Poinc_ℓ.

In fact one can even define a refined ℓ-adic Poincaré polynomial $\mathrm{Poinc}_\ell^{\mathrm{ref}} : K_0(\mathrm{Var}_k) \to K_0(\mathrm{Gal}(\overline{k}/k) - \mathbf{Q}_\ell)$ which satisfies for X smooth and proper the relation

$$\mathrm{Poinc}_\ell^{\mathrm{ref}}(X) = \sum_{n\geqslant 0} \left[H^n(X_{\overline{k}},\mathbf{Q}_\ell)\right]\, t^n. \tag{1.4.2}$$

Here $K_0(\mathrm{Gal}(\overline{k}/k) - \mathbf{Q}_\ell)$ stands for the Grothendieck ring of the category of finite dimensional $\mathbf{Q}_\ell$-vector spaces equipped with a continuous action of the absolute Galois group. If k is finite, one can recover from this refined Poincaré polynomial the GEPC $\#_k$ by applying the trace of the Frobenius and evaluating at $t = -1$. In general, one can recover the number of geometrically irreducible components of maximal dimension from the refined Poincaré polynomial: indeed, for any algebraic variety X, $\rho(X)$ is the dimension of $(a_{2\dim(X)})^{\mathrm{Gal}(\overline{k}/k)}$, where $a_{2\dim(X)}$ is the leading coefficient of $\mathrm{Poinc}_\ell^{\mathrm{ref}}(X)$.

If the characteristic of k is zero, there exists by the work of Gillet, Soulé et al a universal "cohomological" GEPC χ_{mot} whose target is the Grothendieck ring of the category of pure motives. Recalling the construction and the basic properties of this category is beyond the scope of these notes (see [And04] for a nice introduction). Let us simply stress that one of the guiding lines of the theory of motives is that it should be a kind of universal cohomological theory for algebraic varieties, which would allow to recover any classical cohomological theory by specialization. Unfortunately, later in these notes, we will be obliged to work with the specialization $Z_{\chi_{mot}}(X,U,t)$ rather than with the original geometric degree function. Though this is certainly inacurrate in many senses, the reader unaware of motives may think of the Grothendieck ring of motives as if it was the Grothendieck ring of varieties (localized at the class of the affine line, see below).

1.5. **Completion of the Grothendieck ring of varieties and the expected analytic behaviour of the degree zeta function.** We will now define a topology on (a localization of) the Grothendieck ring of algebraic varieties. This is necessary if we want to talk about the "analytic behaviour" of the geometric zeta function. The topology we will consider is the one proposed by Kontsevich for his construction of motivic integration. We denote by $\mathbf{L}$ the class of the affine line $\mathbf{A}^1$ in the Grothendieck ring of varieties[4]. We denote by $\mathcal{M}_k$ the localization of $K_0(\mathrm{Var}_k)$

[4]The letter L stands for Lefschetz. This is because the image of $\left[\mathbf{A}^1\right]$ by the morphism χ_{mot} alluded to above coincides with the class of the so-called Lefschetz motive.

with respect to $\mathbf{L}$ (recall that it is not known whether the localization morphism $K_0(\mathrm{Var}_k) \to \mathcal{M}_k$ is injective).

Intuitively, the idea behind the definition given below might be understood as follows: if k is finite with cardinality q, the image of $\mathbf{L}$ by the "number of k-points" morphism is q; since the series $\sum_{n\geqslant 0} q^{-n}$ converges, we would like by analogy the series $\sum_{n\geqslant 0} \mathbf{L}^{-n}$ to be convergent too. Let us stress that this is really a loose analogy here, since the "number of k-points" morphism will not be continuous with the respect to the topology we will define, and thus will not extend to the completed Grothendieck ring with respect to this topology.

We filter the elements in $\mathcal{M}_k$ by their "virtual dimension": for $n \in \mathbf{Z}$, let $\mathcal{F}^n\mathcal{M}_k$ be the subgroup of $\mathcal{M}_k$ generated by those elements which may be written as $\mathbf{L}^{-i}[X]$, where $i \in \mathbf{Z}$ and X is a k-variety satisfying $i - \dim(X) \geqslant n$ (elements whose virtual dimension is less than or equal to $-n$). Thus $\mathcal{F}^\bullet$ is a decreasing filtration, and $\underset{n\in\mathbf{Z}}{\cup} \mathcal{F}^n = \mathcal{M}_k$.

Let $\widehat{\mathcal{M}_k}$ be the completion of $\mathcal{M}_k$ with respect to the topology defined by the dimension filtration (that is, the topology for which $\{\mathcal{F}^n\mathcal{M}_k\}$ is a fondamental system of neighboroods of the origin). In other words we have

$$\widehat{\mathcal{M}_k} = \varprojlim \mathcal{M}_k/\mathcal{F}^n\mathcal{M}_k. \tag{1.5.1}$$

Thus an element of $\widehat{\mathcal{M}_k}$ may be represented as an element $(x_n) \in \prod_{n\in\mathbf{Z}} \mathcal{M}_k/\mathcal{F}^n\mathcal{M}_k$ such that for every integers n and m satisfying $m \geqslant n$ we have $\pi^n_m(x_m) = x_n$, where π^n_m is the natural projection $\mathcal{M}_k/\mathcal{F}^m\mathcal{M}_k \to \mathcal{M}_k/\mathcal{F}^n\mathcal{M}_k$. We have the natural completion morphism $\mathcal{M}_k \to \widehat{\mathcal{M}_k}$ and a natural filtration on $\widehat{\mathcal{M}_k}$ coming from the filtation $\mathcal{F}^\bullet$.

A priori $\widehat{\mathcal{M}_k}$ inherits only the group structure of the ring $\mathcal{M}_k$. Now we define a product. Let $x = (x_n)$ and $y = (y_n)$ be two elements in $\widehat{\mathcal{M}_k}$ and M be an integer such that $x, y \in \mathcal{F}^M\mathcal{M}_k$ (that is, we have $x_n = y_n = 0$ for $n \leqslant M$). Let n be an integer and $\widetilde{x_{n-M}}$, $\widetilde{y_{n-M}}$ be liftings of x_{n-M} and y_{n-M} to $\mathcal{M}_k$ respectively. Define $(x.y)_n$ as the class of $\widetilde{x_{n-M}}.\widetilde{y_{n-M}}$ modulo $\mathcal{F}^n\mathcal{M}_k$. The inclusions $\mathcal{F}^n\mathcal{M}_k.\mathcal{F}^m\mathcal{M}_k \subset \mathcal{F}^{n+m}\mathcal{M}_k$ show that this does not depend on the choices made and that this endows $\widehat{\mathcal{M}_k}$ with a ring structure compatible with the completion morphism.

For an element $x \in \mathcal{M}_k$ (respectively $x \in \widehat{\mathcal{M}_k}$), define

$$\dim(x) = -\frac{1}{2}\,\mathrm{Sup}\{n, \quad x \in \mathcal{F}^n\mathcal{M}_k\}. \tag{1.5.2}$$

Using the ℓ-adic Poincaré polynomial, one may check that if X is a k-variety then we have indeed $\dim([X]) = \dim(X)$. Note that for every integer $n \in \mathbf{Z}$ one has $\dim(\mathbf{L}^n) = n$. One may wonder whether there are nonzero elements in $\mathcal{M}_k$ with dimension $-\infty$, in other words whether the completion morphism is injective: this is an open question.

Note that a series $\sum_{n\geqslant N} x_n$ whose terms belong to $\widehat{\mathcal{M}_k}$ converges in $\widehat{\mathcal{M}_k}$ if and only if $\dim(x_n)$ goes to $-\infty$. For example $\sum_{n\geqslant 0} \mathbf{L}^n$ converges, and one checks that its limit is the inverse of $1 - \mathbf{L}$ in $\widehat{\mathcal{M}_k}$.

Note also that if k is finite with cardinality q the morphism $\#_k : \mathcal{M}_k \to \mathbf{Z}[q^{-1}] \subset \mathbf{R}$ is not continuous when we endow $\mathbf{R}$ with the usual topology; for example, for *any* sequence of integers $\{c_n\}$, the sequence $c_n\,\mathbf{L}^{-n}$ converges to zero with respect to our topology. Thus there is no hope to extend $\#_k$ to a morphism $\widehat{\mathcal{M}_k} \to \mathbf{R}$.

By contrast, the morphism $\mathrm{Poinc}_\ell : \mathcal{M}_k \to \mathbf{Z}[t, t^{-1}]$ is continuous when $\mathbf{Z}[t, t^{-1}]$ is endowed with the topology associated to the filtration by the degree, and thus extends to a morphism $\widehat{\mathcal{M}_k} \to \mathbf{Z}[[t^{-1}]]_{(t)}$.

1.6. **Some questions about the analytic behaviour of the degree zeta function.** We need a preliminary result about the characteristic function of a cone. Let N be a $\mathbf{Z}$-module of finite rank and $\mathscr{C}$ be a rational polyedral cone of N, that is, $\mathscr{C}$ is a cone in $N\otimes\mathbf{R}$ generated by a finite number of elements of N. We moreover assume that $\mathscr{C}$ is strictly convex, *i.e.* $\mathscr{C}\cap-\mathscr{C}=\{0\}$. We set

$$Z(N,\mathscr{C},t)\stackrel{\text{def}}{=}\sum_{y\in N\cap\mathscr{C}}t^y\in\mathbf{Z}[\mathscr{C}\cap N]. \tag{1.6.1}$$

When $\mathscr{C}$ is regular, that is, generated by a subset $\{y_1,\dots,y_d\}$ of a basis of N, a straightforward computation shows that

$$Z(N,\mathscr{C},t)=\prod_{i=1}^{d}\frac{1}{1-t^{y_i}}. \tag{1.6.2}$$

In general, it is known that $\mathscr{C}$ may be written as an "almost disjoint" union of regular cones (more precisely as the support of a regular fan, see below) and $Z(N,\mathscr{C},t)$ will be a finite sum of expression of the type (1.6.2). For any element $x\in N^\vee$ lying in the relative interior of $\mathscr{C}^\vee$, the level sets $\{y\in\mathscr{C}\cap N,\ \langle y\,,\,x\rangle=d\}_{d\in\mathbf{N}}$ are finite and we may define

$$Z(N,\mathscr{C},x,t)=\sum_{y\in N\cap\mathscr{C}}t^{\langle y\,,\,x\rangle}\in\mathbf{Z}[[t]] \tag{1.6.3}$$

From the above decomposition, we deduce that $Z(N,\mathscr{C},x,t)$ is a rational function of t, with a pole of order $\dim(\mathscr{C})$ at $t=1$, and whose other poles are roots of unity. For x in $N^\vee$, define the index of x in $N^\vee$ by

$$\operatorname{ind}_{N^\vee}(x)\stackrel{\text{def}}{=}\operatorname{Max}\{d\in\mathbf{N},\ x\in d\,N^\vee\}. \tag{1.6.4}$$

If $\operatorname{ind}_{N^\vee}(x)=1$, the order of any pole of $Z(N,\mathscr{C},x,t)$ distinct from 1 is less than $\dim(\mathscr{C})$. In general, a similar statement holds for the series $Z(N,\mathscr{C},x,t^{\frac{1}{\operatorname{ind}_{N^\vee}(x)}})$

Let $\alpha(N,\mathscr{C},x)$ be the leading term of $Z(N,\mathscr{C},x,t)$ at the critical point $t=1$. Thus by Cauchy estimates we obtain

$$\#\{y\in N\cap\mathscr{C},\quad\langle y\,,\,x\rangle=\operatorname{ind}_{N^\vee}(x)\,d\}\underset{d\to+\infty}{\sim}\alpha(N,\mathscr{C},x)\,(\operatorname{ind}_{N^\vee}(x)\,d)^{\dim(\mathscr{C})-1}. \tag{1.6.5}$$

Definition 1.4. Let $Z(t)\in\mathbf{C}[[t]]$, ρ a positive real number and d a nonnegative integer. We say that $Z(t)$ is *strongly* (ρ,d) *controlled* if $Z(t)$ converges absolutely in the open disc $|t|<\rho$ and the associated holomorphic function extends to a meromorphic funtion on the open disc $|t|<\rho+\varepsilon$ for a certain $\varepsilon>0$, whose all poles on the circle $|t|=\rho$ have order bounded by d. We say that $Z(t)$ is (ρ,d)-*controlled* if it is bounded by a strongly (ρ,d)-controlled series (we say that $\sum a_nt^n$ is bounded by $\sum b_nt^n$ if $|a_n|\leqslant|b_n|$ for all n).

Note that by Cauchy estimates, if $d\geqslant1$ then $\sum a_nt^n$ is (ρ,d)-controlled if and only if the sequence $\frac{a_n}{n^{d-1}\rho^{-n}}$ is bounded.

We are now in position to state a question about the analytic behaviour of the classical degree zeta function. It may be seen as a version of a refinement by Peyre of a question raised by Manin.

Question 1.5. *Let k be a finite field of cardinality q. Let X be a k-variety satisfying hypotheses 1.1. Does there exists a positive real number c and a dense open subset U such that the series*

$$Z(X,U,t)-c.Z(\operatorname{Pic}(X)^\vee,C_{\text{eff}}(X)^\vee,\left[\omega_X^{-1}\right],q\,t) \tag{1.6.6}$$

is $(q^{-1},\operatorname{rk}(\operatorname{Pic}(X))-1)$-controlled (respectively strongly $(q^{-1},\operatorname{rk}(\operatorname{Pic}(X))-1)$-controlled)?

Of course the question may be refined by asking whether the result holds for every sufficiently small dense open subset.

Note that an affirmative answer yields the estimate

$$\begin{aligned}&\#\,\mathbf{Hom}_{\mathrm{ind}_{\mathrm{Pic}(X)}(\omega_X^{-1})d,U}(\mathbf{P}^1,X)(k)\\ &\underset{d\to+\infty}{\sim} c\,\alpha(\mathrm{Pic}(X)^\vee,C_{\mathrm{eff}}(X)^\vee,\left[\omega_X^{-1}\right])\,(\mathrm{ind}_{\mathrm{Pic}(X)}(\omega_X^{-1})\,d)^{\mathrm{rk}(\mathrm{Pic}(X))-1}\,q^{\mathrm{ind}_{\mathrm{Pic}(X)}(\omega_X^{-1})\,d}\end{aligned} \tag{1.6.7}$$

Of course, in case (1.6.6) is strongly $(q^{-1},\mathrm{rk}(\mathrm{Pic}(X)-1)$-controlled, we get a more precise asymptotic expansion.

Let us add that there exists a precise description of the expected value of the constant c (see at the end of section 2.6).

Now we turn to the search for a motivic analog of the previous question. We adopt the following definition.

Definition 1.6. Let $Z(t)\in\widehat{\mathcal{M}_k}[[t]]$, $k\in\mathbf{Z}$ and d a nonnegative integer. We say that $Z(t)$ is $(\mathbf{L}^{-k},d)$ *controlled* if it may be written as a finite sum $\sum_{i\in I}Z_i(t)$ such that for every $i\in I$, there exist $d_i\leqslant d$ and d_i positive integers $a_{i,1},\dots,a_{i,d_i}$ such that the series

$$\prod_{1\leqslant e\leqslant d_i}(1-\mathbf{L}^{k\,a_{i,e}}\,t^{a_{i,e}})\,Z_i(t) \tag{1.6.8}$$

converges at $t=\mathbf{L}^{-k}$.

This definition is to be thought as a loose analog of definition 1.4.

Question 1.7. *Let k be a field and X be a k-variety satisfying hypotheses 1.1. Does there exists a nonzero element $c\in\widehat{\mathcal{M}_k}$ and a dense open subset U such that the series*

$$Z(X,U,t)-c.Z(\mathrm{Pic}(X)^\vee,C_{\mathrm{eff}}(X)^\vee,\left[\omega_X^{-1}\right],\mathbf{L}\,t) \tag{1.6.9}$$

is $(\mathbf{L}^{-1},\mathrm{rk}(\mathrm{Pic}(X))-1)$-controlled?

Does the constant c have an interpretation analogous to the one in the classical case?

Regarding tauberian statements, it is worth noting that unfortunately the situation is not as comfortable as in the case of a finite field. One would like for example to deduce from an affirmative answer to the latter question informations about the asymptotic behaviour of the dimension and the number of irreducible components of maximal dimension of $\mathbf{Hom}_{d,U}(\mathbf{P}^1,X)$, but one may check that the only statement one is able to derive is the inequality

$$\overline{\lim}\,\frac{\dim(\mathbf{Hom}_{d,U}(\mathbf{P}^1,X))}{d}\leqslant 1 \tag{1.6.10}$$

which is less precise that Batyrev's expectations. In fact, when studying the case of a toric variety X, we will be able to show that Batyrev's expectations hold before we are able to give an affirmative answer to question 1.7.

2. The case of toric varieties

In this section we explain how one can deal with the previously introduced problem in the case of toric varieties.

2.1. **Toric geometry.** Here we recall some basic facts about toric geometry. Proofs will be omitted or very roughly sketched, and are easily accessible in the classical references on the topic ([Ful93, Oda88, Ewa96]).

A (split) algebraic torus is a group variety isomorphic to a product of copies of the multiplicativ groupe $\mathbf{G}_m$. A toric variety is a normal equivariant (partial) compactification of an algebraic torus. In other words, it is a normal algebraic variety endowed with an algebraic action of an algebraic torus T and possessing an open dense subset U isomorphic to T in such a way that the action of T on U identifies with the action of T on itself by translations.

Examples 2.1. $\mathbf{A}^n$ on which $\mathbf{G}_m^n$ acts diagonally, $\mathbf{P}^n$ on which $\mathbf{G}_m^n$ acts by

$$(\lambda_1, \dots, \lambda_n)(x_0 : \dots : x_n) = (x_0 : \lambda_1 x_1 : \dots : \lambda_n x_n). \tag{2.1.1}$$

Remark 2.2. A non necessarily split algebraic torus is a group variety which becomes isomorphic to a split torus over an algebraic closure of the base field. Though the case of unsplit toric varieties, that is, compactifications of non necessarily split tori, certainly deserves consideration in the context of our problem, we will stick in these notes to the case of split toric varieties.

Let $T \xrightarrow{\sim} \mathbf{G}_m^d$ be a split torus of dimension d. The group $\mathcal{X}(T)$ of algebraic characters of T, that is, of algebraic group morphism $T \to \mathbf{G}_m$, is a free module of finite rank d.

Let X be a smooth projective equivariant compactification of T, and U its open orbit. Then $X \setminus U$ is the union of a finite number $\{D_i\}_{i \in I}$ of irreducible divisors, which are T-invariant since T is irreducible. We call the D_i's the *boundary divisors*.

Since $k[T] \xrightarrow{\sim} k[t_1, t_1^{-1}, \dots, t_d, t_d^{-1}]$ is a UFD, the Picard group of $U \xrightarrow{\sim} T$ is trivial, therefore the map which associates to D_i its class in the Picard group of X induces a short exact sequence

$$0 \to k[T]^\times / k^\times \to \bigoplus_{i \in I} \mathbf{Z} D_i \to \operatorname{Pic}(X) \to 0 \tag{2.1.2}$$

Moreover the natural morphism $\mathcal{X}(T) \to k[T]^\times$ induces an isomorphism $\mathcal{X}(T) \xrightarrow{\sim} k[T]^\times / k^\times$.

Remark 2.3. One can show that the image of $\sum_{i \in I} D_i$ in $\operatorname{Pic}(X)$ coincides with the class of the anticanonical line bundle $\left[\omega_X^{-1}\right]$.

By dualizing the exact sequence (2.1.2), we obtain

$$0 \to \operatorname{Pic}(X)^\vee \to \bigoplus_{i \in I} \mathbf{Z} D_i^\vee \to \mathcal{X}(T)^\vee \to 0. \tag{2.1.3}$$

Let ρ_i denote the image of $D_i^\vee$ in $\mathcal{X}(T)^\vee$. Let Σ_X be the set of cones generated by $\{\rho_i\}_{i \in J}$ for those $J \subset I$ such that $\cap_{i \in J} D_i \neq \varnothing$. Then Σ_X is a fan of $\mathcal{X}(T)^\vee$, in the following sense:

Definition 2.4. A *fan* Σ of $\mathcal{X}(T)^\vee$ is a finite family $\{\sigma\}_{\sigma \in Sigma}$ of polyedral rational cones of $\mathcal{X}(T)^\vee \otimes \mathbf{R}$ such that:

(1) whenever σ and σ' belong to Σ, $\sigma \cap \sigma'$ is a face of σ and σ'
(2) whenever σ belongs to Σ, every face of σ belongs to Σ

One of the most striking feature of the theory of toric varieties is that the fan Σ_X defined above allows to recover X (thus the geometry of X may be described in terms of combinatorial objects coming from convex geometry). In fact, starting from any fan in $\mathcal{X}(T)^\vee$ one may construct a normal (partial) compactification of T by glueing together the affine T-varieties $V_\sigma \stackrel{\text{def}}{=} \operatorname{Spec}(k[\sigma^\vee \cap \mathcal{X}(T)])$ along the $V_{\sigma \cap \sigma'}$, and one can show that every normal compactification of T is obtained in this way.

In our case, since we assumed X to be projective, the fan Σ_X is complete (that is, the union of its cone is the whole space) and since it was assumed to be smooth, the cones of Σ_X are regular, that is, each one is generated by a part of a $\mathbf{Z}$-basis of $\mathscr{X}(T)^{\vee}$. Note that this implies that X is covered by affine varieties isomorphic to affine spaces.

2.2. **Homogeneous coordinates on toric varieties.** When dealing with (say, projective) varieties, it may be useful to have coordinates on it, for instance to do some computations. One basic way to do this is to embed X into a projective space $\mathbf{P}^n$: the homogeneous coordinates on $\mathbf{P}^n$ yields coordinates on X. One drawback of this approach is that there are a lot of available embeddings $X \hookrightarrow \mathbf{P}^n$, and thus no canonical choice for such coordinates.

A different approach was proposed by Cox for toric varieties (and, as we will see below, subsequently generalized by other authors to larger classes of varieties). The basic idea is to observe that the homogeneous coordinates on $\mathbf{P}^n$ correspond to the quotient of the affine space $\mathbf{A}^{n+1}$ minus the origin by the diagonal action of $\mathbf{G}_m$. Let us denote by π the quotient map $\mathbf{A}^{n+1} \setminus \{0\} \to \mathbf{P}^n$. If we view $\mathbf{P}^n$ as a toric variety in the usual way, the pull back by π of a boundary divisor is the trace of a coordinates hyperplane on $\mathbf{A}^{n+1} \setminus \{0\}$.

This construction can be generalized to any smooth projective toric variety X as follows: let $\{D_i\}_{i\in I}$ be the finite set of boundary divisors. Let $T_{\mathrm{NS}}(X)$ be the torus whose character group is $\operatorname{Pic} X$, that is, the torus $\operatorname{Hom}(\operatorname{Pic}(X), \mathbf{G}_m)$ (the notation stands for *Néron-Severi torus*; in our setting the Picard group and the Néron-Severi group coincide).

The morphism $\mathbf{Z}^I \to \operatorname{Pic}(X)$ extracted from the exact sequence (2.1.2) yields by duality an algebraic group morphism $T_{\mathrm{NS}}(X) \to \mathbf{G}_m^I$. Composing with the coordinatewise action of $\mathbf{G}_m^I$ on $\mathbf{A}^I$, we get an action of $T_{\mathrm{NS}}(X)$ on $\mathbf{A}^I$. If $X = \mathbf{P}^n$, one has $\operatorname{Pic}(X) \xrightarrow{\sim} \mathbf{Z}$ and the action of $T_{\mathrm{NS}}(X) \xrightarrow{\sim} \mathbf{G}_m$ on $\mathbf{A}^{n+1}$ is the diagonal one.

We set

$$\mathscr{T}_X \overset{\text{def}}{=} \mathbf{A}^I \setminus \bigcup_{\substack{J\subset I\\ \underset{i\in J}{\cap} D_i=\varnothing}} \underset{i\in J}{\cap} \{x_i = 0\}. \tag{2.2.1}$$

Note that the condition $\cap_{i\in J} D_i = \varnothing$ may be expressed in terms of the fan Σ_X by saying that the $\{\rho_i\}_{i\in J}$ are not the rays of a cone of the fan. For $X = \mathbf{P}^n$ the only subset of $\{0, \dots, n\}$ satisfying the condition is $\{0, \dots, n\}$ itself.

One checks immediatly that the action of $T_{\mathrm{NS}}(X)$ on $\mathbf{A}^I$ leaves $\mathscr{T}_X$ invariant. Now we define a morphism $\pi : \mathscr{T}_X \to X$. First we notice that the open subsets of $\mathscr{T}_X$

$$\mathscr{T}_{X,\sigma} = \{x \in \mathbf{A}^I,\ \forall i \in I, \quad \rho_i \notin \sigma \Rightarrow x_i \neq 0\} \tag{2.2.2}$$

are T_{NS}-invariant and form a covering of $\mathscr{T}_X$ when σ varies along the maximal cones of Σ_X. Now let σ be such a cone and $\sigma(1) = \{i \in I,\ \rho_i \in \sigma\}$. Then $\{\rho_i\}_{i\in\sigma(1)}$ is a $\mathbf{Z}$-base of $\mathscr{X}(T)^{\vee}$ (recall that X is smooth, so that the fan Σ_X is regular), thus the classes of the divisors $\{D_i\}_{i\notin\sigma(1)}$ in $\operatorname{Pic}(X)$ form a $\mathbf{Z}$-basis of it, and therefore determine isomorphisms $\operatorname{Pic}(X) \xrightarrow{\sim} \mathbf{Z}^{I\setminus\sigma(1)}$ and $T_{\mathrm{NS}}(X) \xrightarrow{\sim} \mathbf{G}_m^{I\setminus\sigma(1)}$. If t is an element of $T_{\mathrm{NS}}(X)$, which we write $(\lambda_i)_{i\in I\setminus\sigma(1)}$ and $x = (x_i) \in \mathbf{A}^I$, then for all $i \notin \sigma(1)$ we have $(\lambda.x)_i = \lambda_i\, x_i$ So if x is in $\mathscr{T}_{X,\sigma}$, there is a unique $x' \in T_{\mathrm{NS}}(X).x$ such that for all $i \notin \sigma(1)$ we have $x'_i = 1$. We set

$$\pi_\sigma(x) \overset{\text{def}}{=} (x'_i)_{i\in\sigma(1)} \in \mathbf{A}^{\sigma(1)} \xrightarrow{\sim} X_\sigma = \operatorname{Spec}(k[\sigma^{\vee} \cap \mathscr{X}(T)]) \tag{2.2.3}$$

We leave to the reader the task of verifying that $\pi_\sigma : \mathscr{T}_{X,\sigma} \to X$ is indeed a morphism of algebraic variety and that the morphisms π_σ glue to a morphism $\pi : \mathscr{T}_X \to X$ which is a $T_{\mathrm{NS}}(X)$-torsor over X (here, since $T_{\mathrm{NS}}(X)$ is a split torus, it simply means that there is an open covering (X_α) of X and isomorphisms $\varphi_\alpha : U_\alpha \times T_{\mathrm{NS}}(X) \xrightarrow{\sim} \pi^{-1}(U_\alpha)$ such that $\pi \circ \varphi_\alpha = \mathrm{pr}_{U_\alpha}$ and the action of $T_{\mathrm{NS}}(X)$ on the $U_\alpha \times T_{\mathrm{NS}}(X)$ induced by φ_α^{-1} is by translations on the second factor).

Remark 2.5. One may check that the divisor $\pi^* D_i$ is the trace of the hyperplane coordinates $\{x_i = 0\}$ on $\mathscr{T}_X \subset \mathbf{A}^I$.

Remark 2.6. In the construction of $\pi : \mathscr{T}_X \to X$ we did not use the fact that X was projective, and indeed the construction is valid for any smooth toric variety. For generalization to other toric varieties and some applications we refer to Cox's paper [Cox95b].

Remark 2.7. There is a natural $\mathrm{Pic}(X)$-graduation on the polynomial ring $k[x_i]_{i\in I}$, which yields the $T_{\mathrm{NS}}(X)$-action on $\mathbf{A}^I$ used above: we set $\deg(x^{\boldsymbol{d}}) = [\sum d_i D_i]$. Now let $D = \sum a_i\, D_i$ be an integral combination of the D_i's. It is known that the set

$$\mathscr{X}(T)_D = \{m \in \mathscr{X}(T)\,,\, \forall i \in I,\, \langle m\,,\, \rho_i\rangle + a_i \geqslant 0\} \tag{2.2.4}$$

is a basis of

$$H^0(X, \mathscr{O}_X(D)) = \{f \in k(X),\, \mathrm{div}(f) + D \geqslant 0\} \cup \{0\}. \tag{2.2.5}$$

But the map $m \mapsto \prod x_i^{\langle m\,,\,\rho_i\rangle + a_i}$ is clearly a bijection from $\mathscr{X}(T)_D$ onto the set of monomials of degree $[D]$, thus the degree $[D]$ part of $k[x_i]_{i\in I}$ may be identified with the vector space of global sections $H^0(X, \mathscr{O}_X(D))$.

2.3. **Application to the description of the functor of points of a toric variety.** Now we explain the application of homogeneous coordinate rings to the description of the functor of points of a smooth projective toric variety X defined over k, that is, the functor which maps a k-scheme S to the set $\mathrm{Hom}_k(S, X)$. This is due to Cox ([Cox95a]).

Here again the case of $\mathbf{P}^n$ may serve as a basic guiding example. In fact what we will seek to generalize in a minute is the following well-known property: a morphism $S \to \mathbf{P}^n$ is determined by the datum of a line bundle on S and $n+1$ global sections of this line bundles which do not vanish simultaneously.

Now let X be a smooth toric variety. Recall that we have the exact sequence

$$0 \to \mathscr{X}(T) \to \bigoplus_{i\in I} \mathbf{Z} D_i \to \mathrm{Pic}(X) \to 0. \tag{2.3.1}$$

This means in particular that for every $m \in \mathscr{X}(T)$ we have

$$\mathrm{div}(m) = \sum_{i\in I} \langle m\,,\, \rho_i\rangle\, D_i. \tag{2.3.2}$$

Therefore $m \in \mathscr{X}(T)$ determines an isomorphism $c_m : \bigotimes_{i\in I} \mathscr{O}_X(D_i)^{\otimes\langle m\,,\,\rho_i\rangle} \xrightarrow{\sim} \mathscr{O}_X$. It is clear that $c_m \otimes c_{m'} = c_{m+m'}$.

Let $f : S \to X$ be a morphism from a k-scheme S to our toric variety X. Let $\mathcal{L}_i \stackrel{\text{def}}{=} f^*\mathscr{O}_X(D_i)$, $u_i \stackrel{\text{def}}{=} f^* s_{D_i}$ (where s_{D_i} denote the canonical section of D_i) and, for $m \in \mathscr{X}(T)$, let $d_m \stackrel{\text{def}}{=} f^* c_m$. Then the datum $((\mathcal{L}_i, u_i), (d_m)_{m\in\mathscr{X}(T)})$ is a *X-collection on S* in the following sense:

Definition 2.8. An X-collection on a k-scheme S is the datum of:

(1) a family $((\mathcal{L}_i), u_i)_{i\in I}$ where $\mathcal{L}_i$ is a line bundle on S and u_i a global section of $\mathcal{L}_i$ such that for every $J \subset I$ satisfying $\cap_{i\in J} D_i = \varnothing$ the sections $\{u_i\}_{i\in J}$ do not vanish simultaneously (non-degeneracy condition);
(2) a family of isomorphism $d_m : \otimes \mathcal{L}_i^{\otimes\langle m\,,\,\rho_i\rangle} \xrightarrow{\sim} \mathscr{O}_S$ such that $d_m \otimes d_{m'} = d_{m+m'}$

We have an obvious notion of isomorphism of X-collections on S and we denote by $\mathrm{Coll}_{X,S}$ the set of isomorphism classes of X-collections on S. Note that $\mathrm{Coll}_{X,S}$ is clearly fonctorial in S. We denote by C_X the X-collection on X given by $\{(\mathscr{O}_X(D_i)), s_{D_i}), \{c_m\}\}$.

In [Cox95a], Cox proves that the maps

$$\begin{array}{ccc} \mathrm{Hom}(S,X) & \longrightarrow & \mathrm{Coll}_{X,S} \\ f & \longmapsto & f^*C_X \end{array} \tag{2.3.3}$$

define an isomorphism between the functor of points of X and the functor which associates to a k-scheme S the set $\mathrm{Coll}_{X,S}$

Let us sketch the proof. First we describe a map $\mathrm{Coll}_{X,S} \to \mathrm{Hom}(S, \mathbf{P}^n)$. Let $((\mathcal{L}_i, u_i), d_m)$ be a representative of an element C of $\mathrm{Coll}_{X,S}$. First assume that the $\mathcal{L}_i$'s are trivial. Thus C has a representative of the form $((\mathscr{O}_S, u_i), (d_m)$. In particular (d_m) may be identified with a group morphism

$$\mathcal{X}(T) \to \mathrm{Aut}(\mathscr{O}_S) = H^0(S, \mathscr{O}_S)^{\times}, \tag{2.3.4}$$

that is, an element of $T(S)$, and if $t, t' \in T(S)$ the two X-collections $((\mathscr{O}_S, u_i), t)$ and $((\mathscr{O}_S, u_i'), t')$ are isomorphic if and only if there is an element $\lambda \in \mathbf{G}_m^I(S) = H^0(S, \mathscr{O}_S)^{\times}$ such that $\lambda.t = t'$ (recall the exact sequence of tori $1 \to T_{\mathrm{NS}} \to \mathbf{G}_m^I \to T \to 1$) and $\lambda_i.u_i = u_i'$. In particular we may choose a representative of C of the form $((\mathscr{O}_S, u_i), 1))$. The u_i's then define a morphism $S \to \mathbf{A}^I$, whose image lies in $\mathscr{T}_X$ thanks to the non degeneracy condition satisfied by the u_i's. By composition with $\pi : \mathscr{T}_X \to X$ we obtain a morphism $S \to X$. By the previous observation, the morphism $S \to \mathscr{T}_X$ depends on the choice of the representative $((\mathscr{O}_S, u_i), 1))$ but the induced morphism $S \to X$ does not because any other representative of this form differ by the action of an element of $\mathbf{G}_m^I$ whose image in T is trivial, that is, an element of $T_{\mathrm{NS}}(X)$.

If the $\mathcal{L}_i$'s are not trivial, cover S by open subset trivializing them, and glue the corresponding morphisms (this is possible thanks to fonctoriality). We thus obtain a morphism $f_C : S \to \mathbf{P}^1$ associated to C. To check that $f_C^* C_X$ and C are isomorphic, agin reduce to the case where the $\mathcal{L}_i$'s are trivial and use remark 2.5. It remains to check that if f^*C_X and C are isomorphic then $f = f_C$. This is easy if f factors through π and we reduce to the latter case by using the smoothness of π and reasoning locally with respect to the étale topology. We refer to [Cox95a] for more details.

Remark 2.9. One obtain an analogous description of the functor assigning to a k-scheme S the set of morphisms $\mathrm{Hom}(S, \mathbf{P}^1)$ which do not factor through the boundary $\cup D_i$: by remark 2.5 and the above construction they correspond to those X-collections $\{(\mathscr{L}_i, u_i), \{d_m\}$ for which no one of the u_i is the zero section. We call such collections *non degenerate X-collections.*

2.4. **Description of** $\mathrm{Hom}(\mathbf{P}^1, X)$ **for** X **toric.** Now we are ready to give a useful description of the scheme $\mathrm{Hom}(\mathbf{P}^1, X)$ where X is a smooth projective toric variety.

More precisely, for every $\boldsymbol{d} \in \mathbf{Z}^I$, we will describe the variety $\mathbf{Hom}_{\boldsymbol{d},U}(\mathbf{P}^1, X)$ parametrizing the set of morphisms $\mathbf{P}^1 \to X$ such that for $i \in I$ we have $\deg(f^*D_i) = d_i$, and which do not factor through the boundary $\cup D_i$ (recall that $U = X \setminus \cup D_i$ is the open orbit). Note that this variety will be empty if $\boldsymbol{d}$ does not belong to the image of $\mathrm{Pic}(X)^{\vee}$ in $\mathbf{Z}^I$ (recall the exact sequence (2.1.3)); if $\boldsymbol{d} \in \mathrm{Pic}(X)^{\vee}$, $\mathbf{Hom}_{\boldsymbol{d},U}(\mathbf{P}^1, X)$ parametrizes the set of morphisms with absolute degree $\boldsymbol{d}$. Moreover, if $f : \mathbf{P}^1 \to X$ does not factor through the boundary, then for all $i \in I$ the divisor $f^*(D_i)$ is effective. Thus $\mathbf{Hom}_{\boldsymbol{d},U}(\mathbf{P}^1, X)$ will be empty if $\boldsymbol{d}$ does not belong to $\mathbf{N}^I$. Note that since the D_i's generate the effective cone of X, the intersection $\mathbf{N}^I \cap \mathrm{Pic}(X)^{\vee}$ may be identified with $C_{\mathrm{eff}}(X)^{\vee} \cap \mathrm{Pic}(X)^{\vee}$.

Let L be a k-extension. Let $\boldsymbol{d} \in \mathbf{N}^I \cap \mathrm{Pic}(X)^\vee$. By the previous section, a element $f \in \mathbf{Hom}_{\boldsymbol{d},U}(\mathbf{P}^1, X)(L)$ is entirely determined by an isomorphism class of X-collection $\{(\mathcal{L}_i, u_i), \{d_m\}\}$ on $\mathbf{P}^1$, with $\deg(\mathcal{L}_i) = d_i$, and no one of the u_i's is the zero section. We may assume that $\mathcal{L}_i$ is $\mathscr{O}_{\mathbf{P}^1}(d_i)$, thus u_i may be identified with a nonzero homogeneous polynomial in two variables of degree d_i, denoted by P_i. As explained above, the datum of $\{d_m\}$ is equivalent to the datum of a point of $T(\mathbf{P}^1) = T(H^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1})) = T(k)$ and two collections $((\mathscr{O}_{\mathbf{P}^1}(d_i), P_i), t)$ and $((\mathscr{O}_{\mathbf{P}^1}(d_i), P_i'), t')$ are isomorphic if and only if there exists $\lambda = (\lambda_i) \in \mathbf{G}_m^I(L)$ such that $\lambda.t = t'$ and $\lambda_i.P_i = P_i'$.

For any nonnegative integer d, we denote by $\mathcal{H}_d^\bullet$ the variety $\mathbf{A}^{d+1} \setminus \{0\}$ (this is only to stress that we wiew a point of the latter as the coefficients of a nonzero homogeneous polynomial in two variables of degree d). For $\boldsymbol{d} \in \mathbf{N}^I$, set $\mathcal{H}_{\boldsymbol{d}}^\bullet \overset{\text{def}}{=} \prod_i \mathcal{H}_{d_i}^\bullet$.

Elimination theory shows that there exists a dense open subset $U_{\boldsymbol{d}}$ of $\mathcal{H}_{\boldsymbol{d}}^\bullet$ such that for every field L we have that (P_i) lies in $U_{\boldsymbol{d}}(L)$ if and only if the P_i's do not have a common nontrivial root in an algebraic closure of L. Thus there exists an open dense subset $\mathcal{H}_{\boldsymbol{d},X}^\bullet$ of $\mathcal{H}_{\boldsymbol{d}}^\bullet$ such that (P_i) lies in $\mathcal{H}_{\boldsymbol{d},X}^\bullet(L)$ if and only if the P_i's satisfy the non degeneracy condition of definition 2.8.

It follows that the map wich associates to the (non degenerate) collection $(\{\mathscr{O}_{\mathbf{P}^1}(d_i), P_i, t)\}$ the element $(P_i) \in \mathcal{H}_{\boldsymbol{d},\Sigma(X)}^\bullet(L)$ induces a bijection between the isomorphism classes of non degenerate collections and the set

$$\mathcal{H}_{\boldsymbol{d},X}^\bullet(L)/T_{\mathrm{NS}}(L) = (\mathcal{H}_{\boldsymbol{d},X}^\bullet/T_{\mathrm{NS}})(L) \tag{2.4.1}$$

The equality holds even if L is not algebraically closed because the torsor $\mathcal{H}_{\boldsymbol{d},\Sigma(X)}^\bullet \to \mathcal{H}_{\boldsymbol{d},\Sigma(X)}^\bullet/T_{\mathrm{NS}}$ is locally trivial for the Zariski topology.

The previous reasoning suggests that the variety $\mathcal{H}_{\boldsymbol{d},X}^\bullet/T_{\mathrm{NS}}$ is isomorphic to $\mathbf{Hom}_{\boldsymbol{d},U}(\mathbf{P}^1, X)$. It does not prove it, since we only looked at the level of points with value in a field, but with little extra work one can show that this is indeed the case.

Note in particular that for every $\boldsymbol{d} \in \mathrm{Pic}(X)^\vee \cap C_{\mathrm{eff}}(X)^\vee$ the variety $\mathbf{Hom}_{\boldsymbol{d},U}(\mathbf{P}^1, X)$ is geometrically irreducible of dimension

$$\sum d_i + \#I - \mathrm{rk}(\mathrm{Pic}(X)) = \sum d_i + \dim(X) = \langle \omega^{-1}, \boldsymbol{d} \rangle + \dim(X) \tag{2.4.2}$$

the last equality coming from remark 2.3. Hence the questions raised at the end of section 1.2 have an affirmative answer for toric varieties.

2.5. **Application to the degree zeta function.** Using the above description of $\mathrm{Hom}^{\boldsymbol{d}}(\mathbf{P}^1, X)$, we see that the geometric degree zeta function has an expression

$$Z(X, U, t) = \sum_{\boldsymbol{d} \in \mathrm{Pic}(X)^\vee \cap C_{\mathrm{eff}}(X)^\vee} \left[\mathcal{H}_{\boldsymbol{d},X}^\bullet/T_{\mathrm{NS}}(X)\right] t^{\langle \omega_X^{-1}, \boldsymbol{d} \rangle}. \tag{2.5.1}$$

Set $\mathbf{P}^{\boldsymbol{d}} = \prod \mathbf{P}^{d_i}$ and let $\mathbf{P}_X^{\boldsymbol{d}}$ be the image of $\mathcal{H}_{\boldsymbol{d},X}^\bullet$ in $\mathbf{P}^{\boldsymbol{d}}$. Since $\mathbf{P}^{\boldsymbol{d}} = \mathcal{H}_{\boldsymbol{d},X}^\bullet/\mathbf{G}_m^I$, $\mathcal{H}_{\boldsymbol{d},X}^\bullet/T_{\mathrm{NS}}(X) \to \mathbf{P}_X^{\boldsymbol{d}}$ is a $\mathbf{G}_m^I/T_{\mathrm{NS}}(X) = T$-torsor and we have

$$\left[\mathcal{H}_{\boldsymbol{d},X}^\bullet/T_{\mathrm{NS}}(X)\right] = [T] \left[\mathbf{P}_X^{\boldsymbol{d}}\right] = (\mathbf{L} - 1)^{\dim(X)} \left[\mathbf{P}_X^{\boldsymbol{d}}\right]. \tag{2.5.2}$$

The class of $\mathbf{P}^{\boldsymbol{d}}$ in the Grothendieck ring is readily computed as $\prod_{i \in I} \frac{\mathbf{L}^{d_i+1}-1}{\mathbf{L}-1}$. To evaluate the class of the open subvariety $\mathbf{P}_X^{\boldsymbol{d}}$ we will use a classical tool to "get rid" of coprimality conditions, that is, we will perform a kind of Möbius inversion. We claim that there is a unique fonction $\mu_X^{\mathrm{mot}} : \mathbf{N}^I \to K_0(\mathrm{Var}_k)$ satisfying:

$$\forall \boldsymbol{d} \in \mathbf{N}^I, \quad \left[\mathbf{P}_X^{\boldsymbol{d}}\right] = \sum_{0 \leqslant \boldsymbol{d}' \leqslant \boldsymbol{d}} \mu_X^{\mathrm{mot}}(\boldsymbol{d}') \left[\mathbf{P}^{\boldsymbol{d}-\boldsymbol{d}'}\right]. \tag{2.5.3}$$

The claim follows immediately from an induction on the lenght $\sum d_i$ of $\boldsymbol{d}$. Now we can write:

$$\sum_{\boldsymbol{d}\in \operatorname{Pic}(X)^\vee\cap C_{\text{eff}}(X)^\vee} \left[\mathbf{Hom}_{\boldsymbol{d},U}(\mathbf{P}^1,X)\right]\, t^{\left\langle \boldsymbol{d}\,,\,\omega_X^{-1}\right\rangle} \tag{2.5.4}$$

$$= \sum_{\boldsymbol{d}\in \operatorname{Pic}(X)^\vee\cap C_{\text{eff}}(X)^\vee}\ \sum_{0\leqslant \boldsymbol{d}'\leqslant \boldsymbol{d}} \mu_X^{\text{mot}}(\boldsymbol{d}') \left[\mathbf{P}^{\boldsymbol{d}-\boldsymbol{d}'}\right] t^{\left\langle \boldsymbol{d}\,,\,\omega_X^{-1}\right\rangle} \tag{2.5.5}$$

$$= \sum_{\boldsymbol{d}'\in \mathbf{N}^I} \mu_X^{\text{mot}}(\boldsymbol{d}') \sum_{\substack{\boldsymbol{d}\in \operatorname{Pic}(X)^\vee\cap C_{\text{eff}}(X)^\vee\\ \boldsymbol{d}\geqslant \boldsymbol{d}'}} \prod_{i\in I} \frac{\mathbf{L}^{d_i-d'_i+1}-1}{\mathbf{L}-1}\, t^{\left\langle \boldsymbol{d}\,,\,\omega_X^{-1}\right\rangle} \tag{2.5.6}$$

Then the computation, very roughly, goes on as follows: we approximate the summation over the truncated cone

$$\{\boldsymbol{d}\in \operatorname{Pic}(X)^\vee\cap C_{\text{eff}}(X)^\vee, \quad \boldsymbol{d}\geqslant \boldsymbol{d}'\} \tag{2.5.7}$$

by a summation over the whole cone $\operatorname{Pic}(X)^\vee\cap C_{\text{eff}}(X)^\vee$ and we approximate the quantity $\prod_{i\in I}\frac{\mathbf{L}^{d_i+1}-1}{\mathbf{L}-1}$ by $\frac{\mathbf{L}^{|\boldsymbol{d}|+\#I}}{(\mathbf{L}-1)^{\#I}}$. Of course we will have to show that the error terms resulting of these approximations do not contribute to the main term of the degree zeta function, that is, are suitably controlled in the sense of definition 1.6. Neglecting these error terms for the moment, and reminding that $\left\langle \boldsymbol{d}\,,\,\omega_X^{-1}\right\rangle = \sum d_i$ by remark 2.3, we obtain the following expression for the "leading" term of the motivic degree zeta function

$$\frac{\mathbf{L}^{\#I}}{(\mathbf{L}-1)^{\operatorname{rk}\operatorname{Pic}(X)}} \left(\sum_{\boldsymbol{d}\in\mathbf{N}^I} \mu_X^{\text{mot}}(\boldsymbol{d})\,\mathbf{L}^{-|\boldsymbol{d}|}\right) \left(\sum_{\boldsymbol{d}\in \operatorname{Pic}(X)^\vee\cap C_{\text{eff}}(X)^\vee} (\mathbf{L}\,t)^{\left\langle \boldsymbol{d}\,,\,\omega_X^{-1}\right\rangle}\right) \tag{2.5.8}$$

In view of question 1.7, the second factor is the expected one, but for the moment we are not even sure that the first factor is well defined, that is, that the series converges in the completed Grothendieck ring $\widehat{\mathcal{M}_k}$.

We will first explain what happens over a finite field for the classical degree zeta function. In this case the multiplicativity property of the Möbius function allows to easily settle the convergence of (the analogous of) the first factor of 2.5.8, by decomposing it as an Euler product. Then we will explain how this approach may be "mimicked" for the geometric degree zeta function.

2.6. **The leading term of the classical degree zeta function of a toric variety.** In this section we assume that k is a finite field, and we will study the analog of (2.5.8) for the classical degree zeta function, that is

$$\frac{q^{\#I}}{(q-1)^{\operatorname{rk}\operatorname{Pic}(X)}} \left(\sum_{\boldsymbol{d}\in\mathbf{N}^I} \#_k\mu_X^{\text{mot}}(\boldsymbol{d})\,q^{-|\boldsymbol{d}|}\right) \left(\sum_{\boldsymbol{d}\in \operatorname{Pic}(X)^\vee\cap C_{\text{eff}}(X)^\vee} (q\,t)^{\left\langle \boldsymbol{d}\,,\,\omega_X^{-1}\right\rangle}\right) \tag{2.6.1}$$

Let us stress that (2.6.1) is really to be understood as a formal analog of (2.5.8), and can not be derived from (2.5.8) by applying $\#_k$, even if one proves that the first factor of (2.5.8) converges in the completed Grothendieck ring, since $\#_k$ does not extend to the latter.

Recall that the second factor of (2.6.1) is nothing else that $Z(\operatorname{Pic}(X)^\vee, C_{\text{eff}}(X)^\vee, \left[\omega_X^{-1}\right], q\,t)$ It will be shown later that (2.6.1) is indeed the leading term of the classical degree zeta function, in other words that the difference $Z(X,U,t) - (2.6.1)$ is (strongly) $(q^{-1}, \operatorname{rk}(\operatorname{Pic}(X))-1)$-controlled. Thus the left factor of (2.6.1) will give the constant c appearing in question 1.5. But our first task is to show its convergence.

A key fact in the classical setting is that the specialized function $\#_k\mu_X^{\mathrm{mot}} : \mathbf{N}^I \to \mathbf{Z}$ can be refined as a function $\mu_X : \bigsqcup_{\boldsymbol{d}\in\mathbf{N}^I} \mathbf{P}^{\boldsymbol{d}}(k) \to \mathbf{Z}$, in the sense that for all $\boldsymbol{d}$ we will have $\#_k\mu_X^{\mathrm{mot}}(\boldsymbol{d}) = \sum\limits_{\mathcal{D}\in\mathbf{P}^{\boldsymbol{d}}(k)} \mu_X(\mathcal{D})$. Indeed, define μ_X by the relation

$$\forall \boldsymbol{d}\in\mathbf{N}^I,\quad \forall \mathcal{D}\in\mathbf{P}^{\boldsymbol{d}}(k),\qquad \sum_{(\mathcal{D}'_i)\leqslant(\mathcal{D}_i)} \mu_X((\mathcal{D}'_i)) = \mathbf{1}_{\mathbf{P}^{\boldsymbol{d}}_X}(\mathcal{D}). \tag{2.6.2}$$

Here we identify $\mathbf{P}^{\boldsymbol{d}}(k)$ with the set of I-uples of effective k-divisors $(\mathcal{D}_i)$ on $\mathbf{P}^1$ of degree (d_i): this gives a sense to the expression $(\mathcal{D}'_i)\leqslant(\mathcal{D}_i)$.

The basic properties of μ_X are listed in the following proposition The reader may check them as an easy exercise.

Proposition 2.10. *(1) μ_X is a multiplicative function: whenever $(\mathcal{D}_i)$ and $(\mathcal{D}'_i)$ are such that $\mathcal{D}_i$ and $\mathcal{D}'_i$ are coprime (that is, have disjoint supports) for each i, we have $\mu_X((\mathcal{D}_i+\mathcal{D}'_i)) = \mu_X((\mathcal{D}_i))\,\mu_X((\mathcal{D}'_i))$.*

(2) There exists a unique map $\mu_X^0 : \mathbf{N}^I \to \mathbf{Z}$ such that for all $\boldsymbol{n}\in\mathbf{N}^I$ and every closed point $\mathcal{P}$ of $\mathbf{P}^1_k$ we have

$$\mu_X((n_i\,\mathcal{P})) = \mu_X^0(\boldsymbol{n}) \tag{2.6.3}$$

(3) We have

$$\forall \boldsymbol{n}\in\{0,1\}^I,\qquad \sum_{0\leqslant\boldsymbol{n}'\leqslant\boldsymbol{n}} \mu_X^0(\boldsymbol{n}') = \begin{cases} 1 & \text{if } \bigcap\limits_{i\in I,\, n_i=1} D_i \neq \varnothing \text{ or } \boldsymbol{n}=0 \\ 0 & \text{otherwise.}\end{cases} \tag{2.6.4}$$

(4) We have $\mu_X^0(\boldsymbol{n}) = 0$ if $\sum n_i = 1$ or if there exists i such that $n_i \geqslant 2$.

(5) Denoting by $\{0,1\}^I_X$ the set of elements $\boldsymbol{n}$ of $\{0,1\}^I$ such that $\operatorname*{Min}\limits_{\sigma\in\Sigma_X}\sum_{i\notin\sigma(1)} n_i > 0$ and by $\{0,1\}^I_{X,min}$ the set of the minimal elements of $\{0,1\}^I_X$, we have

$$\forall \boldsymbol{n}\in\{0,1\}^I,\qquad \sum_{0\leqslant\boldsymbol{n}'\leqslant\boldsymbol{n}} \mu_X^0(\boldsymbol{n}') = \begin{cases} 1 & \text{if } \boldsymbol{n}=0 \\ 0 & \text{if } \boldsymbol{n}\neq 0 \text{ and } \boldsymbol{n}\notin\{0,1\}^I_X \\ (-1)^{\#\{\boldsymbol{n}'\in\{0,1\}^I_{X,min},\ \boldsymbol{n}'\leqslant\boldsymbol{n}\}} & \text{if } \boldsymbol{n}\in\{0,1\}^I_X \end{cases} \tag{2.6.5}$$

Using the classical fact that the Euler product

$$\prod_{\mathcal{P}\in(\mathbf{P}^1_k)^{(0)}} 1 + \underset{\deg(\mathcal{P})\to+\infty}{\mathcal{O}}\left(q^{-(1+\varepsilon)\,\deg(\mathcal{P})}\right) \tag{2.6.6}$$

(where $(\mathbf{P}^1_k)^{(0)}$ denotes the set of closed points of $\mathbf{P}^1_k$) converges and thanks to the previous proposition, we obtain that the series

$$\sum_{\boldsymbol{d}\in\mathbf{N}^I} \#_k\mu_X^{\mathrm{mot}}(\boldsymbol{d})\,q^{-|\boldsymbol{d}|} = \prod_{\mathcal{P}\in(\mathbf{P}^1_k)^{(0)}}\ \sum_{\boldsymbol{n}\in\{0,1\}^I} \mu_X^0(n_i\,\mathcal{P})\,q^{-\deg(\mathcal{P})(\sum_i n_i)} \tag{2.6.7}$$

is absolutely convergent. The following proposition will yield a nice interpretation of this Euler product.

Proposition 2.11. *Let L be a finite extension of k. We have the following relation:*

$$\sum_{\boldsymbol{n}\in\{0,1\}^I} \mu_X^0(n_i)\,(\#L)^{-\sum_i n_i} = (1-\#L)^{\mathrm{rk}(\mathrm{Pic}(X))}\#X(L)/(\#L)^{-\dim(X)} \tag{2.6.8}$$

Proof. We will in fact prove the following relation in the Grothendieck ring of varieties (valid over any field):

$$\sum_{\boldsymbol{n}\in\{0,1\}^I} \mu_X^0(\boldsymbol{n})\,\mathbf{L}^{\#I-\sum_i n_i} = (\mathbf{L}-1)^{\mathrm{rk}(\mathrm{Pic}(X))}\,[X]\,. \tag{2.6.9}$$

The desired relation follows immediatly by applying the realization morphism $\#_L$ and the relation $\#I = \dim(X) + \mathrm{rk}(\mathrm{Pic}(X))$.

Since the morphism $\mathscr{T}_X \to X$ is a torsor under a split torus of dimension $\mathrm{rk}(\mathrm{Pic}(X))$, we have

$$(\mathbf{L}-1)^{\mathrm{rk}(\mathrm{Pic}(X))}\,[X] = [\mathscr{T}_X] \tag{2.6.10}$$

Now for $\boldsymbol{n} \in \{0,1\}^I$ let $\mathbf{A}^I_{\boldsymbol{n}} \overset{\mathrm{def}}{=} \underset{i,\,n_i=1}{\cap} \{x_i = 0\}$. Reminding the definition of $\mathscr{T}_X$, we have (we refer to proposition 2.10 for the definition of $\{0,1\}^I_X$)

$$\mathscr{T}_X = \mathbf{A}^I \setminus \underset{\boldsymbol{n}\in\{0,1\}^I_X}{\cup} \mathbf{A}^I_{\boldsymbol{n}} = \mathbf{A}^I \setminus \underset{\boldsymbol{n}\in\{0,1\}^I_{X,\min}}{\cup} \mathbf{A}^I_{\boldsymbol{n}} \tag{2.6.11}$$

The inclusion-exclusion principle and the scissor relations now yield

$$\left[\underset{\boldsymbol{n}\in\{0,1\}^I_{X,\min}}{\cup} \mathbf{A}^I_{\boldsymbol{n}}\right] = \sum_{\varnothing\neq A\subset\{0,1\}^I_{X,\min}} (-1)^{1+\#A} \left[\underset{\boldsymbol{n}\in A}{\cap} \mathbf{A}^I_{\boldsymbol{n}}\right] \tag{2.6.12}$$

$$= \sum_{\varnothing\neq A\subset\{0,1\}^I_{X,\min}} (-1)^{1+\#A} \left[\mathbf{A}^I_{\underset{\boldsymbol{n}\in A}{\mathrm{Max}}(\boldsymbol{n})}\right]. \tag{2.6.13}$$

Note that the map which associates to a non empty subset A of $\{0,1\}^I_{X,\min}$ the element $\underset{\boldsymbol{n}\in A}{\mathrm{Max}}(\boldsymbol{n})$ is a bijection from $\mathcal{P}(\{0,1\}^I_{X,\min}) \setminus \varnothing$ onto $\{0,1\}^I_X$, whose inverse is the map associating to $\boldsymbol{n} \in \{0,1\}^I_X$ the subset $\{\boldsymbol{n'} \in \{0,1\}^I_{X,\min},\ \boldsymbol{n'} \leqslant \boldsymbol{n}\}$. Hence the above equality may be rewritten as

$$\left[\underset{\boldsymbol{n}\in\{0,1\}^I_{X,\min}}{\cup} \mathbf{A}^I_{\boldsymbol{n}}\right] = \sum_{\boldsymbol{n}\{0,1\}^I_X} (-1)^{1+\#\{\boldsymbol{n'}\in\{0,1\}^I_{X,\min},\,\boldsymbol{n'}\leqslant\boldsymbol{n}\}}\,\mathbf{L}^{\#I-|\boldsymbol{n}|} \tag{2.6.14}$$

Thus we have by proposition 2.10

$$\left[\underset{\boldsymbol{n}\in\{0,1\}^I_{X,\min}}{\cup} \mathbf{A}^I_{\boldsymbol{n}}\right] = \mathbf{L}^{\#I} - \sum_{\boldsymbol{n}\{0,1\}^I} \mu^0_X(\boldsymbol{n})\,\mathbf{L}^{\#I-|\boldsymbol{n}|} \tag{2.6.15}$$

From this and (2.6.11), the desired relation follows immediatly. $\square$

Still assuming for the moment that (2.6.1) is indeed the leading term of the degree zeta function, this shows that the constant c appearing in question 1.5 may be written as

$$\frac{q^{\dim(X)}}{(1-q^{-1})^{\mathrm{rk}\,\mathrm{Pic}(X)}} \prod_{\mathcal{P}\in(\mathbf{P}^1_k)^{(0)}} (1-q^{-\deg(\mathcal{P})})^{\mathrm{rk}(\mathrm{Pic}(X))}\,\frac{\#X(\kappa_{\mathcal{P}})}{q^{\deg(\mathcal{P})\,\dim(X)}} \tag{2.6.16}$$

where $\kappa_{\mathcal{P}}$ is the residue field of the closed point $\mathcal{P}$.

Now remark that, disregarding convergence issues, the expression (2.6.16) makes sense for any variety X satisfying hypotheses 1.1, not only the toric ones. Under suitable extra hypotheses on X, Peyre showed that the Euler product in (2.6.16) is indeed convergent and predicted that (2.6.16) should coincide with the constant c appearing in question 1.5 (in fact Peyre's construction applies to a far more general context, including the case of nonconstant families; (2.6.16) is interpreted as the volume of an adelic space associated to X, with respect to a certain Tamagawa measure; see [Pey03] for more details). Thus we have checked that it was indeed the case when X is toric. And, still in the toric case, we are going to show that the constant c appearing in question 1.7 (which is an element of the completed Grothendieck ring) has an analogous interpretation.

2.7. **The leading term of the motivic degree zeta function.** Now we come to the study of the leading term of the motivic degree zeta function of a smooth projective toric variety, whose expression is given by

$$\frac{\mathbf{L}^{\#I}}{(\mathbf{L}-1)^{\operatorname{rk}\operatorname{Pic}(X)}}\left(\sum_{\boldsymbol{d}\in\mathbf{N}^I}\mu_X^{\mathrm{mot}}(\boldsymbol{d})\,\mathbf{L}^{-\sum_i d_i}\right) Z(\operatorname{Pic}(X)^\vee, C_{\mathrm{eff}}(X)^\vee, \left[\omega_X^{-1}\right], \mathbf{L}\,t) \tag{2.7.1}$$

Recall that the proof of the fact that the ruled out terms are indeed error terms, in other words that $Z(X,U,t)-(2.7.1)$ is $(\mathbf{L}^{-1},\operatorname{rk}(\operatorname{Pic}(X))-1)$-controlled, is postponed to section 2.9. Recall also that a priori we do not even know that (2.7.1) is well defined. Our main task will be in fact to settle the convergence of the series

$$\sum_{\boldsymbol{d}\in\mathbf{N}^I}\mu_X^{\mathrm{mot}}(\boldsymbol{d})\,\mathbf{L}^{-\sum_i d_i} \tag{2.7.2}$$

in the completed Grothendieck ring. When k is finite, the analogous problem for the classical degree zeta function was easy to handle owing to the decomposition into Euler product. When working over the Grothendieck ring of varieties or its completion, there is a priori no immediate analog of the notion of Euler product. Let us now explain how to define such a notion. Let X be a quasi-projective variety defined over k. Consider the motivic Hasse–Weil zeta function

$$Z_{\mathrm{HW,mot}}(X,t)=\sum_{n\geqslant 0}\left[\operatorname{Sym}^n(X)\right]t^n \tag{2.7.3}$$

where $\operatorname{Sym}^n(X)\stackrel{\mathrm{def}}{=}X^n/\mathfrak{S}_n$. When k is finite, $\#_k Z_{\mathrm{HW,mot}}(X,t)=Z_{\mathrm{HW}}(X,t)$ is the classical Hasse–Weil zeta function attached to X and we have the decomposition into Euler product

$$\#_k Z_{\mathrm{HW,mot}}(X,t)=\prod_{\mathcal{P}\in X^{(0)}}(1-t^{\deg(\mathcal{P})})^{-1} \tag{2.7.4}$$

where $X^{(0)}$ denotes the set of closed points of X. Now, for $n\in\mathbf{N}$, let $X_n^{(0)}$ denote the set of closed points of X of degree n. Then (2.7.4) may be rewritten as

$$Z_{\mathrm{HW}}(X,t)=\prod_{n\geqslant 1}(1-t^n)^{-\#X_n^{(0)}}. \tag{2.7.5}$$

Note that the latter equality may be seen as an immediate formal consequence of the relations

$$\sum \#X(k_n)\,t^n=t\,\frac{d\log}{dt}Z_{\mathrm{HW}}(X,t) \tag{2.7.6}$$

and

$$\forall n\geqslant 1,\quad \#X(k_n)=\sum_{d|n}d\,\#X_d^{(0)} \tag{2.7.7}$$

(here k_n is an extension of k of degree n).

Now we may wonder whether there is a natural "motivic incarnation" of the family $(\#X_n^{(0)})_{n\geqslant 1}$, that is, a naturally defined family $(Y_{X,n})$ of elements in the Grothendieck ring of varieties such that when k is finite the following relation holds:

$$\forall n\geqslant 1,\quad \#_k Y_{X,n}=\#X_n^{(0)}. \tag{2.7.8}$$

If we accept to work in the Grothendieck ring of varieties with denominators (that is, tensorized with $\mathbf{Q}$), there is certainly a cheap and straightforward way of doing this. For every quasi-projective k-variety X, mimicking the relation (2.7.6) and (2.7.7) above, define families $(\Psi_n(X))_{n\geqslant 1}$ and $(\Phi_n(X))_{n\geqslant 1}$ of elements of $K_0(\mathrm{Var}_k)$ and $K_0(\mathrm{Var}_k)\otimes\mathbf{Q}$ respectively[5] by the relations

$$\sum_{n\geqslant 1}\Psi_n(X)\,t^n = t\,\frac{d\log}{dt}Z_{\mathrm{HW,mot}}(X,t) \tag{2.7.9}$$

and

$$\forall n\geqslant 1,\quad \Psi_n(X)=\sum_{d|n} d\,\Phi_d(X). \tag{2.7.10}$$

For example, $\Psi_1(X)=\Phi_1(X)=[X]$, $\Psi_2(X)=2\left[\mathrm{Sym}^2(X)\right]-\left[X^2\right]$, and $\Phi_2(X)=\left[\mathrm{Sym}^2(X)\right]-\frac{1}{2}(\left[X^2\right]-[X])$.

Lemma 2.12. *(1) There are unique group morphisms $\Psi_n : K_0(\mathrm{Var}_k)\to K_0(\mathrm{Var}_k)$ and $\Phi_n : K_0(\mathrm{Var}_k)\to K_0(\mathrm{Var}_k)\otimes\mathbf{Q}$ such that for every quasi-projective variety X we have $\Psi_n([X])=\Psi_n(X)$ and $\Phi_n([X])=\Phi_n(X)$.*

(2) Assume that k is finite. For every quasiprojective k-variety X, every $n\geqslant 1$, and every finite extension L of k we have

$$\#_L\Psi_n(X)=X(L_n)\quad\text{and}\quad \#_L\Phi_n(X)=\#X^{(0)}_{L,n}. \tag{2.7.11}$$

(3) For every $n\geqslant 1$ and $k\geqslant 0$, we have $\Psi_n(\mathbf{L}^k)=\mathbf{L}^{k\,n}$.

(4) For every $n\geqslant 1$, we have

$$\Psi_n(X)=\sum_{k=1}^{n}(-1)^{k+1}\,\frac{n}{k}\sum_{\substack{(m_1,\dots,m_k)\in(\mathbf{N}_{>0})^k\\ m_1+\dots+m_k=n}}\prod_{i=1}^{k}\left[\mathrm{Sym}^{m_i}(X)\right]. \tag{2.7.12}$$

(5) For every $n\geqslant 1$, $\Psi_n(X)$ and $\Phi_n(X)$ are in $\mathcal{F}^{-n\,\dim(X)}\mathcal{M}_k\otimes\mathbf{Q}$.

Remark 2.13. We do not claim that Ψ_n and Φ_n are ring morphisms. In fact, by considering for example the image of $\mathbf{L}$, it is straightforward to check that for $n\geqslant 2$, Φ_n is not a ring morphism. And anyway, over a finite field, it is clear that the composition of Φ_n with $\#_k$ is not a ring morphism. On the other hand, the composition of Ψ_n with $\#_k$ is a ring morphism (this amounts to the relation $\#(X\times Y)(k_n)=\#X(k_n)\#Y(k_n)$), as well as its restriction to $\mathbf{Z}[\mathbf{L}]$ when k is arbitrary. Nevertheless, it is not true that Ψ_n is a ring morphism, but the demonstration relies on a rather subtle construction of Larsen and Lunts, who proves in fact that the motivic Hasse–Weil zeta function of X is not rational in general for $\dim(X)\geqslant 2$, contrarily to the intuition that the specialization over a finite field might support (see [LL03, LL04] and [Bou10a, Remarque 2.7]). This phenomenon may be seen as an incarnation of the fact that the Grothendieck ring of varieties is definitively too big. By contrast, the specializations of $\{\Psi_n\}$ to the Grothendieck ring of motives are ring morphisms, as we will see below (and the specialization of the motivic Hasse–Weil zeta function to the Grothendieck ring of motives is conjectured to always be rational).

Now it is easy to give a motivic counterpart of (2.7.5), since by the very definition of Φ_n, we have for every quasiprojective variety X

$$Z_{\mathrm{HW,mot}}(X,t)=\prod_{n\geqslant 1}(1-t^n)^{-\Phi_n(X)} \tag{2.7.13}$$

where for every element x of $K_0(\mathrm{Var}_k)\otimes\mathbf{Q}$, $(1-t)^x$ denotes the series

$$\exp(x\,\log(1-t)). \tag{2.7.14}$$

[5] In [Bou09b], these two families were denoted the opposite way; it was a bit unfortunate choice since, as pointed out to me by E. Gorsky, what we denote by $(\Psi_n(X))$ in this text is a formal analog of the so-called Adams operations, and the letter Ψ is commonly used to denote the latter.

Note that (2.7.13) holds in $1+(K_0(\mathrm{Var}_k)\otimes\mathbf{Q})[[t]]^+$ (for any commutative ring $1+A[[t]]^+$ denotes the set of formal series with coefficients in A and constant term 1) and that more genrally for any element $P(t)\in 1+(K_0(\mathrm{Var}_k)\otimes\mathbf{Q})[[t]]^+$, $P(t)^x=\exp(x\,\log(1-P(t)))$ makes sense, as makes sense the "motivic Euler product"

$$\prod_{n\geqslant 1} P(t^n)^{\,-\Phi_n(X)}. \tag{2.7.15}$$

Now we see that an hypothetic motivic couterpart of the formula

$$\begin{aligned}\sum_{\boldsymbol{d}\in\mathbf{N}^I}\#_k\mu_X^{\mathrm{mot}}(\boldsymbol{d})\,t^{|\boldsymbol{d}|} &= \prod_{\mathcal{P}\in(\mathbf{P}^1_k)^{(0)}}\sum_{\boldsymbol{n}\in\mathbf{N}^I}\mu^0_X(n_i\,\mathcal{P})\,t^{-\deg(\mathcal{P})(\sum_i n_i)}\\ &= \prod_{n\geqslant 1}\Big(\sum_{\boldsymbol{n}\in\mathbf{N}^I}\mu^0_X(n_i)\,t^{-n(\sum_i n_i)}\Big)^{\#X_n^{(0)}}\end{aligned} \tag{2.7.16}$$

could be the (yet to be proved !) relation

$$\sum_{\boldsymbol{d}\in\mathbf{N}^I}\mu_X^{\mathrm{mot}}(\boldsymbol{d})\,t^{|\boldsymbol{d}|} = \prod_{n\geqslant 1}\left(\sum_{\boldsymbol{n}\in\mathbf{N}^I}\mu^0_X(n_i)\,t^{-n(\sum_i n_i)}\right)^{\Phi_n(\mathbf{P}^1)}. \tag{2.7.17}$$

Remark 2.14. If the latter relation holds, it follows easily that the LHS of (2.7.17) converges in the completed Grothendieck ring at $t=\mathbf{L}^{-1}$: indeed we have $\Phi_n(\mathbf{P}^1)\in\mathcal{F}^{-n}\mathcal{M}_k$, hence thanks to point 4 of proposition 2.10 the series

$$\left(\sum_{\boldsymbol{n}\in\mathbf{N}^I}\mu^0_X(\boldsymbol{n})\,t^{-n(\sum_i n_i)}\right)^{\Phi_n(\mathbf{P}^1)} \tag{2.7.18}$$

converges in $t=\mathbf{L}^{-1}$ and its limit lies in $1+\mathcal{F}^{2\,n-n}\widehat{\mathcal{M}_k}=1+\mathcal{F}^n\widehat{\mathcal{M}_k}$.

Notations 2.15. Let $r\geqslant 1$ and $\boldsymbol{f}=(f_1,\dots,f_r)\in(\mathbf{N}_{>0})^r$ such that

$$f_1=f_2=\dots=f_{i_1}<f_{i_1+1}=f_{i_1+2}=\dots=f_{i_2}<f_{i_2+1}=\dots<f_{i_{k-1}+1}=\dots=f_r \tag{2.7.19}$$

Then for any sequence (x_n) with values in a $\mathbf{Q}$-algebra A we set

$$(x_{\boldsymbol{f}})\overset{\mathrm{def}}{=}\prod_{1\leqslant\ell\leqslant k}\frac{x_{f_{i_\ell}}(x_{f_{i_\ell}}-1)\dots(x_{f_{i_\ell}}-i_\ell+i_{\ell-1})}{(i_\ell-i_{\ell-1})!} \tag{2.7.20}$$

(where $i_0=0$ and $i_k=r$).

We have the following elementary combinatorial lemma.

Lemma 2.16. *Let A be a $\mathbf{Q}$-algebra, E a non empty finite set and $P=1+\sum_{\boldsymbol{n}\in\mathbf{N}^E\setminus\{0\}}a_{\boldsymbol{n}}\,\boldsymbol{t}^{\boldsymbol{n}}$ an element of $A[[(t_e)_{e\in E}]]$. Then for every sequence $(x_n)\in A^{\mathbf{N}}$ the following relation holds*

$$\begin{aligned}&\exp\left(\sum_{n\geqslant 1}a_n\,\log(P(\boldsymbol{t}^n)_{e\in E})\right)\\ &=1+\sum_{\boldsymbol{m}\in\mathbf{N}^E\setminus\{0\}}\left(\sum_{r\geqslant 1}\sum_{\substack{\boldsymbol{f}\in(\mathbf{N}_{>0})^r\\ f_1\leqslant\dots\leqslant f_r}}(x_{\boldsymbol{f}})\sum_{\substack{(\boldsymbol{n}_1,\dots,\boldsymbol{n}_r)\in(\mathbf{N}^E\setminus\{0\})^r\\ \sum\boldsymbol{n}_i\,f_i=\boldsymbol{m}}}\prod_{i=1}^r a_{\boldsymbol{n}_i}\right)\boldsymbol{t}^{\boldsymbol{m}}.\end{aligned} \tag{2.7.21}$$

For every $\boldsymbol{d} \in \mathbf{N}^I$, denote by $\widetilde{\mu_X^{\mathrm{mot}}}(\boldsymbol{d})$ the element

$$\sum_{r\geqslant 1} \sum_{\substack{\boldsymbol{f}\in \mathbf{N}_{>0}^r \\ f_1\leqslant\dots\leqslant f_r}} (\Phi_{\boldsymbol{f}}(\mathbf{P}^1)) \sum_{\substack{(\boldsymbol{n}_1,\dots,\boldsymbol{n}_r)\in(\{0,1\}^I\setminus\{0\})^r \\ \sum \boldsymbol{n}_\ell\, f_\ell=\boldsymbol{d}}} \prod_{\ell=1}^{r} \mu_X^0(\boldsymbol{n}_\ell). \tag{2.7.22}$$

Thus, by the above lemma, establishing (2.7.17) amounts to proving the following identities in $K_0(\mathrm{Var}_k)\otimes \mathbf{Q}$:

$$\forall \boldsymbol{d} \in \mathbf{N}^I, \quad \left[\mathbf{P}_X^{\boldsymbol{d}}\right] = \sum_{0\leqslant \boldsymbol{d}'\leqslant \boldsymbol{d}} \widetilde{\mu_X^{\mathrm{mot}}}(\boldsymbol{d}')\left[\mathbf{P}^{\boldsymbol{d}-\boldsymbol{d}'}\right]. \tag{2.7.23}$$

Except in some particular simple situations, including the case where X is a projective space, we do not know how to prove these relations in $K_0(\mathrm{Var}_k)\otimes\mathbf{Q}$, and we are not even sure that they indeed hold. Nevertheless, under the additional hypothesis that the characteristic of the base field is zero, we are going to explain how to prove a similar relation in the Grothendieck ring of Chow motives, using a device forged by Denef and Loeser in the context of their theory of arithmetic motivic integration.

The idea goes basically as follows: when k is finite the relation (2.7.17) certainly holds after specialization by $\#_k$ (this is because (2.7.16) is true !). We show that the involved equalities may be derived from “algebraic d-cover of formulas”, which in turns allows, thanks to Denef and Loeser's construction, to do “motivic counting” instead of “classical counting”. This motivic couting leads to a proof of (2.7.17) (in the Grothendieck ring of motive) along exactly the same way that classical counting allows to proof (2.7.17) after specialization by $\#_k$.

To illustrate the notions of d-cover and motivic couting, we begin by a very basic example, postponing the precise definitions to a little later. We refer to [Hal05] for a very nice introduction to these concepts.

Let k be a finite field of cardinality q, with q odd. The elementary fact that there are exactly $\frac{q}{2}$ nonzero squares in k may be seen as follows: let $f : \mathbf{G}_m \to \mathbf{G}_m$ the morphism $x\mapsto x^2$; then for every finite extension L of k, the morphism f induces a 2-to-1 map from $\mathbf{G}_m(L)$ onto the set of squares in $\mathbf{G}_m(L)$, which in turn may be seen as the set of elements x in $\mathbf{A}^1(L)$ satisfying the intepretation of the first order logic formula

$$\mathscr{F} : {}'(\exists y,\, x=y^2)\wedge(x\neq 0)'. \tag{2.7.24}$$

We say that f induces an algebraic 2-cover of the formula $\mathscr{F}$ by $\mathbf{G}_m$. From this derives the counting formula

$$\#\mathscr{F}(L) = \frac{1}{2}\#\mathbf{G}_m(L) \tag{2.7.25}$$

where $\mathscr{F}(L) = \{x\in L,\, (\exists y\in L,\, x=y^2)\wedge x\neq 0\}$.

Now Denef and Loeser's construction allows to deduce from the fact that $\mathbf{G}_m$ is a 2-cover of $\mathscr{F}$ not only the “classical counting” result above but far more generally a “motivic counting” result, that is,

$$[\mathscr{F}] = \frac{1}{2}[\mathbf{G}_m] \tag{2.7.26}$$

where $[.]$ denotes the class in the Grothendieck ring of motives (here the class of our formula $\mathscr{F}$ may in fact be defined by relation (2.7.26); in general, one has of course to define the class of an arbitrary formula in the Grothendieck ring of motives, which is far from trivial). In fact the more precise hypothesis under which one is able to deduce (2.7.26) is that the property that f induces a 2-to-1 map from $\mathbf{G}_m(L)$ onto $\mathscr{F}(L)$ does not hold only when L is finite but also when L is pseudo-finite. In one word, pseudo-finite fields are infinite fields satisfying any model theoretic property which holds for the finite fields.

In the next section we review briefly first order logic formula, pseudo-finite fields and the construction of Denef and Loeser.

2.8. **Pseudo-finite fields and the virtual motive of a formula.** A pseudo-finite field is a perfect infinite pseudo algebraically closed field (*i.e.* every geometrically irreducible k-variety has a k-point) which has the following property: once an algebraic closure $\overline{k}$ of k is fixed, for every $n \geqslant 1$ there is exactly one k-extension of degree n in $\overline{k}$.

One can show that every field k admits a pseudo-finite extension. Pseudo-finite fields share many properties with finite fields. For example, let k be a pseudo-finite field, $\overline{k}$ an a algebraic closure and k_n the unique extension of k of degree n in $\overline{k}$. One can show that k_n/k is cyclic and that $k_n \subset k_m$ if and only if n divides m.

A first order ring formula with coefficients in k (which from now will simply be called a k-formula) is a logical formula built from boolean combinations of polynomial equalities over k and quantifiers; for example

$$'\exists y, \forall x, x^2+y^2=z^2\,', \quad 'x^2+1=0', \quad '\forall z,\, x=y', \quad 'x^2=x^3+x+1\wedge x\neq 0'\ldots \tag{2.8.1}$$

Let φ be a k-formula with n free variables. For every k-extension L, we can define a subset $\varphi(L) \subset L^n$ (the set of "L-points of φ") consisting of all the elements in L^n satisfying the interpretation of the formula φ in L^n. Note that this defines in fact a functor ($k-$extension) $\to$ (Sets). For example if $\varphi =$ "$(\exists y,\, x=y^2)\wedge(x\neq 0)$" then $\varphi(L)$ will be the set of nonzero squares in L. Note also that if φ is quantifier free, there exists a constructible subset F of $\mathbf{A}^n$ such that for every k-extension L we have $\varphi(L) = F(L)$.

Let φ and ψ two k-formulas with free variables $x_1,\ldots,x_n$ and $y_1,\ldots,y_m$ respectively. We say that φ and ψ are equivalent if there exists a formula θ with free variables $x_1,\ldots,x_n,y_1,\ldots,y_m$ such that for every pseudo-finite k-extension K, $\theta(K)$ is the graph of a bijection between $\varphi(K)$ and $\psi(K)$. Substituting in the previous definition "d-to-1 map from $\varphi(K)$ onto $\psi(K)$" to "bijection between $\varphi(K)$ and $\psi(K)$", we obtain the definition of "φ is a d-cover of ψ". For example the formula "$y\neq 0$" is a 2-cover of the formula "$\exists y,\, (x=y^2\wedge y\neq 0)$"; here the formula θ is given by "$y=x^2$".

A very important class of formulas is given by the so-called Galois formula. Let X be a normal, affine, irreducible variety defined over k, and $\pi : Y \to X$ be an unramified Galois cover with group G. Let L be a k-extension and x be an element of $X(L)$. Recall that the decomposition subgroups of x with respect to π are the stabilizers of the action of G on the $\mathrm{Gal}(\overline{L}/L)$-orbits of the geometric fiber over x. You may then check that being given a subgroup D of G, x admits D as a decomposition subgroup if and only if x lifts to an L-point of Y/D but does not lift to an L-point of D' for every strict subgroup D' of D. Hence we see that there exists a k-formula $\varphi_{Y,X,D}$ whose L-points, for every k-extension L, are the L-points of X admitting D as a decomposition subgroup. You may check that the morphism $Y/D \to X$ makes the formula $\varphi_{Y,Y/D,D}$ a $\frac{\#N_G(D)}{\#D}$-cover of the formula $\varphi_{X,Y,D}$. Galois formulas are the key tool for eliminating quantifiers in the theory of pseudo-finite fields, see [FJ08] and [Nic07].

Let $K_0(\mathrm{PFF}_k)$ denote the Grothendieck ring of the theory of pseudo-finite fields over k: as a group, it is generated by the symbols $[\varphi]$, where φ is a k-formula, modulo the relations $[\varphi]=[\psi]$ whenever φ and ψ are equivalents and the "scissor relations" $[\varphi\vee\psi]+[\varphi\wedge\psi]=[\varphi]+[\psi]$ whenever φ and ψ have the same set of free variables. We endow it with a ring structure by defining the product of $[\varphi]$

by $[\psi]$ to be $[\varphi \vee \psi]$ if φ and ψ have disjoint sets of free variables (which of course we may always assume, by considering equivalent formulas). Now we are ready to state the result of Denef and Loeser. Their motivation for it was the construction a motivic incarnation of their theory of arithmetic motivic integration (see [DL01] and [DL02]).

Recall that when the field k has characteristic zero, there exists a unique morphism $\chi_{\mathrm{mot}} : K_0(\mathrm{Var}_k) \to K_0(\mathrm{Mot}_k)$ which maps the class of a smooth projective variety to the class of its Chow motive.

Theorem 2.17. *Let k be a characteristic zero field. There is a unique ring morphism*

$$\chi_{\mathrm{form}} : K_0(PFF_k) \longrightarrow K_0(\mathrm{Mot}_k) \otimes \mathbf{Q} \tag{2.8.2}$$

wich maps the class of a quantifier free formula to the image by χ_{mot} of the class of the associated constructible subset and which satisifies for every formulas φ, ψ such that φ is a d-cover of ψ the relation[6]

$$\chi_{\mathrm{form}}(\varphi) = d\,\chi_{\mathrm{form}}(\psi). \tag{2.8.3}$$

Recall that the reader who may not feel comfortable with motives may as well consider that the Grothendieck ring of motives is nothing else that the Grothendieck of varieties localized at the class of the affine line.

We would like to use Denef-Loeser machinery to give an other characterization of the image the family $\{\Phi_n(X)\}$ in $K_0(\mathrm{Mot}_k) \otimes \mathbf{Q}$ by the morphism χ_{mot}. By rather straightforward cut-and-paste arguments, we reduce to the case X affine, normal and irreducible.

What we have in mind is that $\Phi_n(X)$ should be the class of a formula such that for every pseudo-finite extension K of k, the K-points of this formula are in natural 1-to-1 correspondence with the closed points of degree n of X_K. Now closed points of degree n are particular instances of effective divisors of degree n, so they form a subset of the set of K-points of $\mathrm{Sym}^n(X)$ and in fact of $(\mathrm{Sym}^n(X))^0$, where $(\mathrm{Sym}^n(X))^0$ is the image of the open set $(X^n)^0$ consisting of those n-uples whose coordinates are pairwise distinct. Now the morphism $(X^n)^0 \to (\mathrm{Sym}^n(X))^0$ is plainly an unramified Galois cover with Galois group $\mathfrak{S}_n$. And we may describe the subset of $(\mathrm{Sym}^n(X))^0$ of closed points of degree n exactly as those elements of $(\mathrm{Sym}^n(X))^0(k)$ having a decomposition subgroup cyclic of order n with respect to the above Galois cover. There is therefore a Galois formula $\widetilde{\Phi}_n(X)$ whose K-points identifies naturally with the set of closed points of degree n of X_K for every pseudo-finite k-extension K. It is easy to see that its equivalence class is uniquely determined (that is, does not depend on the choice of an affine embedding of X), and we define $\Phi_{n,\mathrm{mot}}(X)$ to be the image of the class of this formula by the morphism χ_{form}.

Proposition 2.18. *Let X be a quasi-projective variety defined over k For every n, we have*

$$\chi_{\mathrm{mot}}(\Phi_n(X)) = \Phi_{n,\mathrm{mot}}(X). \tag{2.8.4}$$

In other words, we have the relation

$$\sum_{n\geqslant 1} \chi_{\mathrm{mot}}(\mathrm{Sym}^n(X))\, t^n = \prod_{n\geqslant 1} (1-t^n)^{-\Phi_{n,\mathrm{mot}}(X)} \tag{2.8.5}$$

[6] In fact Denef and Loeser proved the existence and unicity of the morphism (2.8.2) under the hypothesis that it satisfies the relation (2.8.3) only for a particular type of d-covers, those induced by Galois formulas. The fact that such a morphism satisfies (2.8.3) for every d-cover is stated without proof by Hales in [Hal05], and proved by Nicaise in [Nic07].

Proof. As before, we easily reduce to the case X affine, normal, irreducible. For every positive integer r, m and every $\boldsymbol{f}\in\mathbf{N}^r_{>0}$, denote by $\mathcal{A}_{r,\boldsymbol{f},m}$ the set

$$\left\{(n_1,\dots,n_r)\in(\mathbf{N}_{>0})^r,\quad \sum_{\ell=1}^{r} n_\ell\, f_\ell = m\right\}. \tag{2.8.6}$$

By lemma 2.16, we have to show for every positive integer m the relation

$$\chi_{\text{mot}}\left([\operatorname{Sym}^m(X)]\right)=\sum_{r\geqslant 1}\sum_{\substack{\boldsymbol{f}=(f_1,\dots,f_r)\in\mathbf{N}^r_{>0}\\ f_1\leqslant\cdots\leqslant f_r}}\left(\Phi_{\boldsymbol{f},\text{mot}}(X)\right)\#\mathcal{A}_{r,\boldsymbol{f},m}. \tag{2.8.7}$$

The latter formula may be seen as the motivic counterpart of the following relation, valid over a finite field k

$$\#\operatorname{Sym}^m(X)(k)=\sum_{r\geqslant 1}\sum_{\substack{\boldsymbol{f}=(f_1,\dots,f_r)\in\mathbf{N}^r_{>0}\\ f_1\leqslant\cdots\leqslant f_r}}\left(\#X^{(0)}_{\boldsymbol{f}}\right)\#\mathcal{A}_{r,\boldsymbol{f},m}. \tag{2.8.8}$$

Of course the latter relation follows immediatly from the decomposition of the Hasse–Weil zeta function into Euler product, but the reader may check that it can also be recovered by a direct counting argument.

Now we can apply the strategy described above: we show that this counting argument can be derived from d-covers of formulas, and apply the result of Denef and Loeser to transform the "classical counting" argument into a "motivic counting" argument.

Let $m\geqslant 1$, $r\geqslant 1$ and $\boldsymbol{f}\in(\mathbf{N}_{>0})^r$ such that $f_1\leqslant\cdots\leqslant f_r$. We use notations 2.15. There is a natural action of $\mathfrak{S}\overset{\text{def}}{=}\prod_{\ell=1}^{k}\mathfrak{S}_{i_\ell-i_{\ell-1}}$ on $\mathcal{A}_{r,\boldsymbol{f},m}$ and on $\prod_{i=1}^{r}\left(\operatorname{Sym}^{f_i}(X)\right)_0$. Let $Z_{\boldsymbol{f}}$ denote the $\mathfrak{S}_{\boldsymbol{f}}$-invariant open set of $\prod_{i=1}^{r}\operatorname{Sym}^{f_i}(X)_0$ defined by

$$Z_{\boldsymbol{f}}\overset{\text{def}}{=}\prod_{1\leqslant\ell\leqslant k}\left(\operatorname{Sym}^{f_i}(X)_0\right)_0^{i_\ell-i_{\ell-1}}$$

(recall that Y^n_0 denotes the open set of Y consisting of n-uples whose coordinates are pairwise distincts, and $\operatorname{Sym}^n(Y)_0$ the image of Y^n_0 by $Y^n\to\operatorname{Sym}^n(Y)$).

Let $\varphi_{\boldsymbol{f}}$ be a formula whose set of K-points, for every pseudo-finite k-extension K, is $Z_{\boldsymbol{f}}(K)\cap\prod_{1\leqslant i\leqslant r}\widetilde{\Phi}_{f_i}(X)(K)$. One easily check the following relation in $K_0(\text{PFF}_k)$:

$$[\varphi_{\boldsymbol{f}}]=\prod_{1\leqslant\ell\leqslant k}\prod_{j=0}^{i_\ell-i_{\ell-1}-1}\left(\left[\widetilde{\Phi}_{f_{i_\ell}}(X)\right]-j\right)=\left[\widetilde{\Phi}_{\boldsymbol{f}}(X)\right]. \tag{2.8.9}$$

Let $\boldsymbol{n}\in\mathcal{A}_{r,\boldsymbol{f},m}$. Denote by $\mathfrak{S}_{\boldsymbol{n}}$ the stabilizator of $\boldsymbol{n}$ under the action of $\mathfrak{S}_{\boldsymbol{f}}$, and by $\pi_{\boldsymbol{f},\boldsymbol{n}}$ the k-morphism $Z_{\boldsymbol{f}}\longrightarrow\operatorname{Sym}^m(X)$ wich maps the r-uple of zero-cycles $(\mathcal{C}_1,\dots,\mathcal{C}_r)$ to $\sum_\ell n_\ell\,\mathcal{C}_\ell$. It factors through $Z_{\boldsymbol{f}}/\mathfrak{S}_{\boldsymbol{n}}$. Let $\psi_{\boldsymbol{f},\boldsymbol{n}}$ be a formula on $\operatorname{Sym}^m(X)$ whose set of K-points, for every pseudo-finite k-extension K, is $\pi_{\boldsymbol{f},\boldsymbol{n}}(\varphi_{\boldsymbol{f}}(K))$. Thus $\psi_{\boldsymbol{f},\boldsymbol{n}}(K)$ is the set of K-rationals zero-cycles which can be written $\mathcal{C}=\sum_{i=1}^{r}n_i\,\mathcal{P}_i$ where $\mathcal{P}_i$ is a closed point of degree f_i on X_K and $\mathcal{P}_i\neq\mathcal{P}_j$ whenever $f_i=f_j$. Note that $\pi^{-1}_{\boldsymbol{f},\boldsymbol{n}}(\mathcal{C})$ is then a $\mathfrak{S}_{\boldsymbol{n}}$-orbit. Therefore $\varphi_{\boldsymbol{f}}$ is a $\#\mathfrak{S}_{\boldsymbol{n}}$-covering of $\psi_{(\boldsymbol{f},\boldsymbol{n})}$ and the motivic counting formula (2.8.3) yields

$$\chi_{\text{form}}\left([\psi_{\boldsymbol{f},\boldsymbol{n}}]\right)=\frac{1}{\#\mathfrak{S}_{\boldsymbol{n}}}\chi_{\text{form}}\left([\varphi_{\boldsymbol{f}}]\right).$$

Let $\mathcal{A}^0_{r,\boldsymbol{f},m} \subset \mathcal{A}_{r,\boldsymbol{f},m}$ denote a system of representatives of $\mathcal{A}_{r,\boldsymbol{f},m}/\mathfrak{S}_{\Gamma_{\boldsymbol{f}}}$. We have

$$\sum_{\boldsymbol{n}\in\mathcal{A}^0_{r,\boldsymbol{f},m}} \chi_{\text{form}}\left([\psi_{\boldsymbol{f},\boldsymbol{n}}]\right) = \Big(\sum_{\boldsymbol{n}\in\mathcal{A}^0_{r,\boldsymbol{f},m}} \frac{1}{\#\mathfrak{S}_{\boldsymbol{n}}}\Big)\chi_{\text{form}}\left([\varphi_{\boldsymbol{f}}]\right) = \frac{\#\mathcal{A}_{r,\boldsymbol{f},m}}{\#\mathfrak{S}_{\boldsymbol{f}}}\chi_{\text{form}}\left([\varphi_{\boldsymbol{f}}]\right). \tag{2.8.10}$$

Thus from (2.8.9) we deduce the relation

$$\sum_{\boldsymbol{n}\in\mathcal{A}^0_{r,\boldsymbol{f},m}} \chi_{\text{form}}\left([\psi_{\boldsymbol{f},\boldsymbol{n}}]\right) = \left(\Phi_{\boldsymbol{f},\text{mot}}(X)\right)\#\mathcal{A}_{\boldsymbol{f},m}. \tag{2.8.11}$$

Moreover the above description of $\psi_{\boldsymbol{f},\boldsymbol{n}}(K)$ shows immediatly that every element of $\operatorname{Sym}^m(X)(K)$ is in $\psi_{\boldsymbol{f},\boldsymbol{n}}(K)$ for a unique $\boldsymbol{f}$ and a $\boldsymbol{n}\in\mathcal{A}_{r,\boldsymbol{f},m}$ unique modulo the action of $\mathfrak{S}_{\boldsymbol{f}}$. Thus the formulas

$$\left(\psi_{r,\boldsymbol{f},\boldsymbol{n}}\right)_{\substack{r\geqslant 1,\\ \boldsymbol{f}\in\mathbf{N}^r_{>0},\\ f_1\leqslant\cdots\leqslant f_r,\\ \boldsymbol{n}\in\mathcal{A}^0_{r,\boldsymbol{f},m}.}} \tag{2.8.12}$$

form a partition of $\operatorname{Sym}^m(X)$. This concludes the proof of the relation (2.8.7). □

Now we return to the case of our inital smooth projective toric variety X. In order to show the validity of the relation

$$\sum_{\boldsymbol{d}\in\mathbf{N}^I}\mu_X^{\text{mot}}(\boldsymbol{d})\,t^{|\boldsymbol{d}|} = \prod_{n\geqslant 1}\Big(\sum_{\boldsymbol{n}\in\mathbf{N}^I}\mu^0_X(n_i)\,t^{-n(\sum_i n_i)}\Big)^{\Phi_{n,\text{mot}}(\mathbf{P}^1)} \tag{2.8.13}$$

in the Grothendieck ring of motives (tensorized with $\mathbf{Q}$), we apply exactly the same strategy that in the proof of the preceding proposition. Since the proof is very similar and the only real novelty consists in dealing with more intricate notations, it will not be given in these notes and we refer to [Bou09b] for more details.

This shows that (2.7.1) makes sense in the completed Grothendieck ring of motives and (still assuming for the moment that (2.7.1) is the leading term of the degree zeta function, which will be shown below) that the constant c in (1.7) may be expressed as

$$\frac{\mathbf{L}^{\dim(X)}}{(1-\mathbf{L}^{-1})^{\operatorname{rk}\operatorname{Pic}(X)}}\prod_{n\geqslant 1}\left(\sum_{\boldsymbol{n}\in\mathbf{N}^I}\mu^0_X(n_i)\,\mathbf{L}^{-n(\sum_i n_i)}\right)^{\Phi_{n,\text{mot}}(\mathbf{P}^1)} \tag{2.8.14}$$

But a reasoning analogous to the one used to establish (2.6.9) shows that for every $n\geqslant 1$ we have

$$\sum_{\boldsymbol{n}\in\{0,1\}^I}\mu^0_X(\boldsymbol{n})\,\mathbf{L}^{n(\#I-\sum_i n_i)} = (\mathbf{L}-1)^{\operatorname{rk}(\operatorname{Pic}(X))}\,\Psi_{n,\text{mot}}(()X) \tag{2.8.15}$$

where $\Psi_{n,\text{mot}}(X)$ denote the image of $\Psi_n(X)$ by χ_{mot}. We use the fact that, contrarily to $\Psi_n(.)$, $\Psi_{n,\text{mot}}(.)$ is multiplicative, *i.e.* satisfies $\Psi_{n,\text{mot}}(Y\times Z) = \Psi_{n,\text{mot}}(Y)\Psi_{n,\text{mot}}(Z)$. One can prove this by motiving counting, see [Bou09b]. It is also an immediate consequence of the fact, proved by F.Bittner in [Hei07], that the λ-structure on $K_0(\text{Mot}_k)$ defined by the Hasse–Weil zeta function is special (see [Gor09]).

Thus the constant c may be rewritten as

$$\frac{\mathbf{L}^{\dim(X)}}{(1-\mathbf{L}^{-1})^{\operatorname{rk}\operatorname{Pic}(X)}}\prod_{n\geqslant 1}\left((1-\mathbf{L}^{-1})^{\operatorname{rk}(\operatorname{Pic}(X))}\,\frac{\Psi_{n,\text{mot}}(X)}{\mathbf{L}^{n\dim(X)}}\right)^{\Phi_{n,\text{mot}}(\mathbf{P}^1)} \tag{2.8.16}$$

and the latter may be seen as a motivic analog of (2.6.16) in the case of a toric variety X.

2.9. **The error terms.** Recall from the beginning of section 2.5 that we had written

$$Z(X,U,t) = Z(X,U,t)_{\text{lead}} + Z(X,U,t)_{\text{error}} \tag{2.9.1}$$

where

$$Z(X,U,t) = \sum_{\boldsymbol{d'}\in\mathbf{N}^I} \mu_X^{\text{mot}}(\boldsymbol{d'}) \sum_{\substack{\boldsymbol{d}\,\text{Pic}(X)^\vee\cap C_{\text{eff}}(X)^\vee \\ \boldsymbol{d}\geqslant\boldsymbol{d'}}} \prod_{i\in I} \frac{\mathbf{L}^{d_i-d'_i+1}-1}{\mathbf{L}-1}\, t^{\langle \boldsymbol{d}\,,\,\omega_X^{-1}\rangle} \tag{2.9.2}$$

and

$$Z(X,U,t)_{\text{lead}} = \frac{\mathbf{L}^{\#I}}{(\mathbf{L}-1)^{\text{rk}\,\text{Pic}(X)}} \left(\sum_{\boldsymbol{d}\in\mathbf{N}^I} \mu_X^{\text{mot}}(\boldsymbol{d})\,\mathbf{L}^{-|\boldsymbol{d}|}\right)\left(\sum_{\boldsymbol{d}\in\text{Pic}(X)^\vee\cap C_{\text{eff}}(X)^\vee} (\mathbf{L}\,t)^{\langle \boldsymbol{d}\,,\,\omega_X^{-1}\rangle}\right) \tag{2.9.3}$$

Recall that $\{D_i\}_{i\in I}$ denote the finite set of the boundary divisors of the toric variety X, and that for every element $\boldsymbol{d}\in\text{Pic}(X)^\vee\subset\mathbf{Z}^I$ and $\boldsymbol{d'}\in\mathbf{Z}^I$ the condition $\boldsymbol{d}\geqslant\boldsymbol{d'}$ may be rewritten as $\langle\boldsymbol{d}\,,\,D_i\rangle\geqslant d'_i$ for all i.

From the above expressions and the inclusion-exclusion principle we see that $Z(t)_{\text{error}}$ may be written as a finite sum and differences of the series

$$Z_{J_1,J_2}(t) = \frac{\mathbf{L}^{\#I-\#J_2}}{(\mathbf{L}-1)^{\text{rk}\,\text{Pic}(X)}} \sum_{\boldsymbol{d}\in\mathbf{N}^I} \mu_X^{\text{mot}}(\boldsymbol{d'})\mathbf{L}^{-\sum\limits_{i\notin J_2} d'_i} \sum_{\substack{\boldsymbol{d}\in\text{Pic}(X)^\vee\cap C_{\text{eff}}(X)^\vee \\ \forall i\in J_1,\quad \langle\boldsymbol{d}\,,\,D_i\rangle<d'_i \\ \forall i\in J_2,\quad \langle\boldsymbol{d}\,,\,D_i\rangle\geqslant d'_i}} \mathbf{L}^{\left\langle \boldsymbol{d}\,,\,\sum_{i\notin J_2} D_i\right\rangle} t^{\langle \boldsymbol{d}\,,\,\omega_X^{-1}\rangle} \tag{2.9.4}$$

where (J_1,J_2) runs over all the couples of subsets of I such that $J_1\cap J_2=\varnothing$ and $(J_1,J_2)\neq(\varnothing,\varnothing)$.

Now we can conclude thanks to the following elementary lemma. The key fact (already used in these notes in section 1.6) is that every polyedral rational cone may be written as the support of a regular fan (the support of a fan is the union of its cones), see [Bry80, Théorème 11]; the geometric significance of this result is the existence of equivariant resolution of singularities for toric varieties.

Note that this is not a priori clear that the above series are well defined, since their coefficients are given by infinite summations over terms of $\mathbf{Z}[\mathbf{L},\mathbf{L}^{-1}]$. The lemma will show in particular that these series are indeed convergent in the completed Grothendieck ring (of motives), thus establishing the validity of the decomposition.

Lemma 2.19. *Let N be a free $\mathbf{Z}$-module of finite rank and $\mathscr{C}$ be a polyedral rational cone in $N\otimes\mathbf{R}$ of maximal dimension. Let $x\in N$ be an element lying in the interior of $\mathscr{C}$. Let $\mathcal{S},\mathcal{U}$ be disjoiny finite sets, $\boldsymbol{x}\in(\mathscr{C}\setminus\{0\})^{\mathcal{S}\cup\mathcal{U}}$ be a finite family of* nonzero *elements in $\mathscr{C}$ and $\boldsymbol{d}\in\mathbf{N}^{\mathcal{S}\cup\mathcal{U}}$ be a finite family of nonnegative integers.*

Let Δ be a regular fan of N whose support is $\mathscr{C}^\vee$. If δ is a cone of Δ, let $\delta(1)$ denote the set of its rays, and let $\delta(1)_{\boldsymbol{x}}$ denote the subset of $\delta(1)$ consisting of those elements ρ satisfying

$$\forall s\in\mathcal{S},\ \langle y_\rho\,,\,x_s\rangle=0,\quad \textit{and}\quad \forall u\in\mathcal{U},\ \langle y_\rho\,,\,x_u\rangle=0 \tag{2.9.5}$$

(where y_ρ denotes the generator of $N\cap\rho$). For $(e_u)\in\mathbf{N}^{\mathcal{U}}$, let $\delta(\boldsymbol{x},\boldsymbol{d},\boldsymbol{e})$ denote the set of elements $y\in\sum\limits_{\rho\in\delta(1)\setminus\delta(1)_{\boldsymbol{x}}}\mathbf{N}_{>0}\,y_\rho$ satisfying

$$\forall s\in\mathcal{S},\quad \langle y\,,\,x_s\rangle\leqslant d_s\quad \textit{and}\quad \forall u\in\mathcal{U},\quad \langle y\,,\,x_u\rangle=d_u+e_u. \tag{2.9.6}$$

Let $\delta(\boldsymbol{x},\boldsymbol{d})$ *denote the set of elements* $y \in \sum\limits_{\rho\in\delta(1)\setminus\delta(1)_{\boldsymbol{x}}} \mathbf{N}_{>0}\, y_\rho$ *satisfying*

$$\forall s\in\mathcal{S},\quad \langle y\,,\,x_s\rangle \leqslant d_s. \tag{2.9.7}$$

We set

$$R_{\delta,\boldsymbol{x},\boldsymbol{d}}(\theta,t) \overset{def}{=} \sum_{\boldsymbol{e}\in\mathbf{N}^{\mathcal{U}}} \theta^{-\sum_{u\in\mathcal{U}} e_u} \sum_{y\in\delta(\boldsymbol{x},\boldsymbol{d},\boldsymbol{e})} (\theta\, t)^{\langle y\,,\,x\rangle} \quad \in \mathbf{Z}[\theta,\theta^{-1}][[t]] \tag{2.9.8}$$

if $\mathcal{U}$ *is non empty and*

$$R_{\delta,\boldsymbol{x},\boldsymbol{d}}(\theta,t) = \sum_{y\in\delta(\boldsymbol{x},\boldsymbol{d})} (\theta\, t)^{\langle y\,,\,x\rangle} \quad \in \mathbf{Z}[\theta,\theta^{-1}][[t]] \tag{2.9.9}$$

otherwise (here θ *is a variable).*

We set also $M(x,\Delta) \overset{def}{=} \operatorname*{Max}\limits_{\delta\in\Delta,\rho\in\delta(1)} \langle x\,,\,y_\rho\rangle$.

Then the following holds:

(1) *For every* $\delta\in\Delta$, $\boldsymbol{d}\in\mathbf{N}^{\mathcal{S}\cup\mathcal{U}}$ *and* $\boldsymbol{e}\in\mathbf{N}^{\mathcal{U}}$, *the cardinality of* $\delta(\boldsymbol{x},\boldsymbol{d},\boldsymbol{e})$ *is bounded by* $\sum_{s\in\mathcal{S}} d_s + \sum_{u\in\mathcal{U}} d_u + \sum_{u\in\mathcal{U}} e_u$, *and for every* $y\in\delta(\boldsymbol{x},\boldsymbol{d},\boldsymbol{e})$ *we have the inequality*

$$\langle y\,,\,x\rangle \leqslant M(x,\Delta)\left(\sum_{s\in\mathcal{S}} d_s + \sum_{u\in\mathcal{U}} d_u + \sum_{u\in\mathcal{U}} e_u\right) \tag{2.9.10}$$

(2) *Same assertion for* $\delta(\boldsymbol{x},\boldsymbol{d})$ *after dropping the* $\sum_{u\in\mathcal{U}}$

(3) *We have the decomposition*

$$\sum_{\substack{y\in\mathscr{C}^\vee\cap N^\vee\\ \forall s\in\mathcal{S},\quad \langle y\,,\,x_s\rangle\leqslant d_s\\ \forall u\in\mathcal{U},\quad \langle y\,,\,x_u\rangle\geqslant d_u}} \theta^{\left\langle y\,,\,-\sum_{u\in\mathcal{U}} x_u\right\rangle} (\theta\, t)^{\langle y\,,\,x\rangle}$$

$$= \theta^{-\sum_{u\in\mathcal{U}} d_u} \sum_{\delta\in\Delta} R_{\delta,\boldsymbol{x},\boldsymbol{d}}(\theta,t) \prod_{\rho\in\delta(1)_{\boldsymbol{x}}} \frac{\theta^{\left\langle y_\rho\,,\,-\sum_{u\in\mathcal{U}} x_u\right\rangle} (\theta\, t)^{\langle y_\rho\,,\,x\rangle}}{1-\theta^{\left\langle y_\rho\,,\,-\sum_{u\in\mathcal{U}} x_u\right\rangle} (\theta\, t)^{\langle y_\rho\,,\,x\rangle}}. \tag{2.9.11}$$

(4) *If* δ *is of maximal dimension and* $\mathcal{S}\cup\mathcal{U}$ *is non empty, the cardinality of* $\delta(1)_{\boldsymbol{x}}$ *is less than* $\dim(C)$.

Proof. We decompose

$$\sum_{\substack{y\in\mathscr{C}^\vee\cap N^\vee\\ \forall s\in\mathcal{S},\quad \langle y\,,\,x_s\rangle\leqslant d_s\\ \forall u\in\mathcal{U},\quad \langle y\,,\,x_u\rangle\geqslant d_u}} \dots = \sum_{\delta\in\Delta} \sum_{\substack{y\in\mathrm{Relint}(\delta)\cap N^\vee\\ \forall s\in\mathcal{S},\quad \langle y\,,\,x_s\rangle\leqslant d_s\\ \forall u\in\mathcal{U},\quad \langle y\,,\,x_u\rangle\geqslant d_u}} \dots \tag{2.9.12}$$

For $\delta\in\Delta$, one easily checks that every element $y\in\mathrm{Relint}(\delta)\cap N^\vee$ satisfying

$$\forall s\in\mathcal{S},\quad \langle y\,,\,x_s\rangle \leqslant d_u \quad\text{and}\quad \forall s\in\mathcal{U},\quad \langle y\,,\,x_u\rangle = d_u + e_u \tag{2.9.13}$$

may be written uniquely as y_1+y_2 where y_1 is an element of $\sum\limits_{\rho\in\delta(1)\setminus\delta(1)_{\boldsymbol{x}}} \mathbf{N}_{>0}\, y_\rho$ satisfying (2.9.13) (that is, y_1 is an element of $\delta(\boldsymbol{x},\boldsymbol{d},\boldsymbol{e})$) and y_2 is an element of $\sum\limits_{\rho\in\delta(1)_{\boldsymbol{x}}} \mathbf{N}_{>0}\, y_\rho$. The decomposition (2.9.11) follows immediatly.

Let $y \in \sum\limits_{\rho\in\delta(1)\setminus\delta(1)_{\boldsymbol{x}}} \mathbf{N}_{>0}\, y_\rho$ Thus there is a unique element $\boldsymbol{n}\in\mathbf{N}_{>0}^{\delta(1)\setminus\delta(1)_{\boldsymbol{x}}}$ such that $y = \sum\limits_{\rho\in\delta(1)\setminus\delta(1)_{\boldsymbol{x}}} n_\rho\, y_\rho$ Assume moreover that y satisfies (2.9.13). By the definition of $\delta(1)_{\boldsymbol{x}}$ there is an $s\in\mathcal{S}$ such that $\langle y_\rho\,,\,x_s\rangle\neq 0$ (hence is a positive integer) or a $u\in\mathcal{U}$ such that $\langle y_\rho\,,\,x_u\rangle\neq 0$ (same remark). In the former case, the inequality

$\langle y\,,\,x_s\rangle \leqslant d_s$ yields the inequality $n_\rho \leqslant d_s$ and similarily in the latter case we obtain $n_\rho \leqslant d_u + e_u$. Thus the first point of the proposition holds.

If δ is of maximal dimension (hence of dimension $\mathrm{rk}(N)$), $\{y_\rho\}_{\rho\in\delta(1)}$ is a $\mathbf{Z}$-basis of $N^\vee$, hence if x is a nonzero element of N we cannot have $\langle y_\rho\,,\,x\rangle = 0$ for every $\rho \in \delta(1)$. This shows the third assertion. □

Applying the lemma with $N = \mathrm{Pic}(X)$, $\mathscr{C} = C_{\mathrm{eff}}(X)$, $\mathcal{S} = J_1$ and $\mathcal{U} = J_2$ we obtain that $Z_{J_1,J_2}(t)$ may be written as a finite sum of terms of the shape

$$\left(\sum_{\boldsymbol{d}\in\mathbf{N}^I} \mu_X^{\mathrm{mot}}(\boldsymbol{d})\mathbf{L}^{-\sum_i d_i} R_{\delta,\boldsymbol{x},\boldsymbol{d}}(\mathbf{L},t)\right) \prod_{\rho\in\delta(1)_{\boldsymbol{x}}} \frac{\mathbf{L}^{\left\langle y_\rho\,,\,\sum_{i\in J_2} D_i\right\rangle} (\mathbf{L}\,t)^{\left\langle y_\rho\,,\,\omega_X^{-1}\right\rangle}}{1-\mathbf{L}^{\left\langle y_\rho\,,\,\sum_{i\in J_2} D_i\right\rangle} (\mathbf{L}\,t)^{\left\langle y_\rho\,,\,\omega_X^{-1}\right\rangle}} \tag{2.9.14}$$

You may check that

$$\sum_{\boldsymbol{d}\in\mathbf{N}^I} \mu_X^{\mathrm{mot}}(\boldsymbol{d})\mathbf{L}^{-\sum_i d_i} R_{\delta,\boldsymbol{x},\boldsymbol{d}}(\mathbf{L},t) \tag{2.9.15}$$

is indeed a well defined element of $\widehat{\mathcal{M}_k}[[t]]$, and that $R_{\delta,\boldsymbol{x},\boldsymbol{d}}(\mathbf{L},\mathbf{L}^{-1})$ converges to an element of $\mathcal{F}^0\widehat{\mathcal{M}_k}$. Assuming that the characteristic of k is zero, we obtain that $\sum_{\boldsymbol{d}\in\mathbf{N}^I} \mu_X^{\mathrm{mot}}(\boldsymbol{d})\mathbf{L}^{-\sum_i d_i} R_{\delta,\boldsymbol{x},\boldsymbol{d}}(\mathbf{L},\mathbf{L}^{-1})$ converges in the completed Grothendieck ring of motive. Thanks to (2.9.14) and the third point of the lemma, this shows that Z_{J_1,J_2} is $(\mathbf{L}^{-1}, \mathrm{rk}(\mathrm{Pic}(X))-1)$-controlled. Thus the answer to question 1.7 is positive for a smooth toric variety (after specializing to the Grothendieck ring of motives).

Now we turn to the classical case, showing that the answer to question 1.5 is positive for a toric variety X, when U is the open orbit. We use a decomposition of the error term formally analogous to the one used for the motivic term. Note that the latter was a decomposition into a finite sum of series with coefficients in the completed Grothendieck ring of variety, and thus, strictly speaking, we can not apply the morphism $\#_k$ to it in order to obtain a decomposition of the classical degree zeta function. Nevertheless, it happens that doing so formally give the right answer, that is, the error term of the classical degree zeta function may be decomposed as a finite sum of terms of the shape

$$\left(\sum_{\boldsymbol{d}\in\mathbf{N}^I} \mu_X^{\mathrm{mot}}(\boldsymbol{d})\mathbf{L}^{-\sum_i d_i} R_{\delta,\boldsymbol{x},\boldsymbol{d}}(q,t)\right) \prod_{\rho\in\delta(1)_{\boldsymbol{x}}} \frac{q^{\left\langle y_\rho\,,\,\sum_{i\in J_2} D_i\right\rangle} (\mathbf{L}\,t)^{\left\langle y_\rho\,,\,\omega_X^{-1}\right\rangle}}{1-q^{\left\langle y_\rho\,,\,\sum_{i\in J_2} D_i\right\rangle} (\mathbf{L}\,t)^{\left\langle y_\rho\,,\,\omega_X^{-1}\right\rangle}} \tag{2.9.16}$$

Here we have to show that

$$\sum_{\boldsymbol{d}\in\mathbf{N}^I} \mu_X^{\mathrm{mot}}(\boldsymbol{d})q^{-\sum_i d_i} R_{\delta,\boldsymbol{x},\boldsymbol{d}}(q,t) \tag{2.9.17}$$

is indeed a well defined element of $\mathbf{R}[[t]]$ (this is a bit subtler than in the motivic case, since in the latter we had dealt with a non-archimedean norm). Moreover, still using the above lemma, we see easily that there exists an $\varepsilon > 0$ such that for every $\eta \leqslant \varepsilon$ $R_{\delta,\boldsymbol{x},\boldsymbol{d}}(q, q^{-1+\eta})$ is absolutely convergent and its sum is bounded by

$$\sum_{(e_i)\in\mathbf{N}^{J_2}} \left(\sum_{i\in J_1\cup J_2} d_i + \sum_{i\in J_2} e_i\right) q^{-\sum_{i\in J_2} e_i - \varepsilon\, M(-\mathscr{K}_X,\Delta)\,(\sum_{i\in J_1\cup J_2} d_i + \sum_{i\in J_2} e_i)} \tag{2.9.18}$$

$$\leqslant (1+\sum_{i\in I} d_i) q^{\varepsilon M(\omega_X^{-1},\Delta)\sum_{i\in I} d_i} \big(\sum_{\boldsymbol{e}\in\mathbf{N}^{J_2}} (1+\sum e_i) q^{-(1-\varepsilon\, M(\omega_X^{-1},\Delta))\sum e_i}\big) \tag{2.9.19}$$

By the properties of μ_X^{mot} (proposition 2.10), the series

$$\sum_{\boldsymbol{d}\in\mathbf{N}^I} \#_k \mu_X^{\mathrm{mot}}(\boldsymbol{d})(1+\sum d_i) q^{\varepsilon M(\omega_X^{-1},\Delta)\sum d_i} q^{-\sum_i d_i} \tag{2.9.20}$$

converges for every sufficiently small $\varepsilon > 0$. Hence we obtain that the error term of the classical degree zeta function is strongly $(q^{-1}, \mathrm{rk}(\mathrm{Pic}(X)) - 1)$-controlled.

3. The general case

In this section, we want to explain how the use of homogeneous coordinates for the study of the degree zeta function of a smooth projective toric varieties might be generalized to other varieties. The first natural question is of course: how does the notion of homogeneous coordinates generalize? The general notion of homogeneous coordinate rings emerged from the work of Cox in the toric case, and began to be intensively studied in the last ten years. The terms *Cox rings* or *total coordinate rings* are often found in the literature to designate homogeneous coordinate rings[7]. The notion is tightly connected with that of universal torsor, introduced by Colliot-Thélène and Sansuc in the 1970's in order to study weak approximation and Hasse principle on rational varieties (see *e.g.* [CTS80, Sko01]). One owes to Salberger the idea of using universal torsors in the context of Manin's conjecture on rational points of bounded height. He showed in [Sal98] that this approach was indeed fructuous for toric varieties (defined over $\mathbf{Q}$) and the first non toric example of a succesful application of the method is due to de la Breteche ([dlB02]). Since then, the use of universal torsors/homogeneous coordinate rings has allowed to settle the arithmetic version of Manin's conjectures for a certain number of non toric varieties (especially in dimension 2), see *e.g.* [Bro07].

In the arithmetic setting, the use of homogeneous coordinate rings reduces the counting of rational points of bounded height to the counting of integral points of an affine space satisfying certain algebraic relations, coprimality conditions and norm inequalities. In the geometric setting, we will explain below how it similarily reduces the counting of morphism $\mathscr{C} \to X$ of bounded degree to the counting of global sections of line bundles of $\mathscr{C}$ satisfying certain algebraic relations, non degeneracy conditions, and degree conditions. This will generalize the case of a toric variety X, for which there are indeed *no* algebraic relations. For the sake of simplicity we will limit ourselves to the case $\mathscr{C} = \mathbf{P}^1$.

For more about homogeneous coordinate rings and examples of computations, see *e.g.* [BH03, BH07, Bri07, Has04, HT04].

3.1. **A bief survey of the theory homogeneous coordinate rings.** Let k be a perfect field and X be a smooth projective variety. We hereby assume that the Picard group of X coincides with its geometric Picard group and that it is free of finite rank (the theory of homogeneous coordinate rings can be developed in a more general context, see *e.g.* [EKW04, BH03]).

Very roughly, the idea behind the theory of homogeneous coordinate rings is that instead of working with a particular choice of coordinates coming from a morphism from X to a projective space, which in turn corresponds to a subspace of the space of global sections of a particular invertible sheaf on X, we could as well work simultaneously with the space of global sections of all the invertible sheaves on X.

[7]Though "Cox ring" is probably the most commonly used, I will stick to the terminology "homogeneous coordinate ring" which I find more appealing, even though there might be confusion with the homogeneous coordinate ring associated to one particular projective embedding. Note that what is called an homogeneous coordinate ring in [BH03] is in fact the ring we discuss here plus an extra structure

Let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be a basis of $\mathrm{Pic}(X)$. We define the homogeneous coordinate ring of X by

$$\mathrm{HCR}(X) \stackrel{\text{def}}{=} \bigoplus_{\boldsymbol{d} \in \mathbf{Z}^n} H^0(X, \mathcal{L}_1^{\otimes d_1} \otimes \cdots \otimes \mathcal{L}_n^{\otimes d_n}). \tag{3.1.1}$$

This is a k-algebra naturally graded by $Pic(X)$: just impose that $H^0(X, \mathcal{L}_1^{\otimes d_1} \otimes \cdots \otimes \mathcal{L}_n^{\otimes d_n})$ is homogeneous of degree the class of $\mathcal{L}_1^{\otimes d_1} \otimes \cdots \otimes \mathcal{L}_n^{\otimes d_n}$. The degree of the nonzero graded pieces are precisely the effective classes in $\mathrm{Pic}(X)$. The definition depends of course on a particular choice of a basis of $\mathrm{Pic}(X)$. Nevertheless, one can easily shows that two different choices give rise to isomorphic $\mathrm{Pic}(X)$-graded k-algebras.

Example 3.1. The first example of homogeneous coordinate ring is due to Cox who worked oud the toric case in [Cox95b]. Let X be a smooth toric variety and let $\{D_i\}_{i \in I}$ be the irreducible divisors of the boundary. For $i \in I$ let s_i be the canonical section of $\mathscr{O}_X(D_i)$. Then the s_i's generate $\mathrm{HCR}(X)$, and there are no nontrivial relation between them, thus $\mathrm{HCR}(X)$ is a polynomial ring in $\#I$ variables in this case (this is essentially the content of remark 2.7).

Example 3.2. Let X be the projective plane blown up at three collinear points, D_0 be the strict transform of the line L joining the points, D_1, D_2 and D_3 the exceptional divisors and D_4, D_5, and D_6 the strict transform of the lines joining a point not lying on L to the blown up points. Let s_i be the canonical section of $\mathscr{O}_X(D_i)$. Then one can show that the s_i generate $\mathrm{HCR}(X)$ and that (after a suitable normalization of the s_i's) one has $\mathrm{HCR}(X) \xrightarrow{\sim} k[s_0, \ldots, s_6]/s_1\, s_4 + s_2\, s_5 + s_3\, s_6$. (see [Has04] and [Der06])

In the previous examples, the homogeneous coordinate ring happens to be finitely generated. The relevance of the property of finite generatedness of the homogeneous coordinate ring was stressed by Hu and Keel in the context of Mori theory. In [HK00], they call varieties with finitely generated homogeneous coordinate rings *Mori dream spaces*, showing in particuler that they behave very well with respect to the minimal model program. The question of deciding whether the homogeneous coordinate ring of a variety is finitely generated is a difficult one. The finiteness of the canonical ring, a very deep result proved recently by Birkar, Cascini, Hacon and McKernan, implies that the homogeneous coordinate ring of a Fano variety is finitely generated. Another difficult issue is to compute explicitely generators and relations for the homogeneous coordinate ring.

In the following, we will denote by X a smooth projective variety defined over a perfect field k such that the Picard group is free of finite rank, coincide with the geometric Picard group, and such that $\mathrm{HCR}(X)$ is finitely generated by sections invariant under the action of the absolute Galois-group. Hu an Keel showed the following generalization of the fact that a toric variety may be recovered as a quotient of an open set of an affine space by the action of a torus. Recall that $T_{\mathrm{NS}}(X) = \mathrm{Hom}(\mathrm{Pic}(X), \mathbf{G}_m) \xrightarrow{\sim} \mathbf{G}_m^{\mathrm{rk}(\mathrm{Pic}(X))}$.

Proposition 3.3. *Let D be an ample class in* $\mathrm{Pic}(X)$*. It corresponds to a character of $T_{NS}(X)$, hence to a $T_{NS}(X)$-linearization of the trivial bundle on* $\mathrm{Spec}(\mathrm{HCR}(X))$*. The GIT quotient of the open set* $\mathrm{Spec}(\mathrm{HCR}(X))^{ss}$ *of semi-stable points by the action of $T_{NS}(X)$ is a geometric quotient isomorphic to X and the quotient map is a $T_{NS}(X)$-torsor.*

We refer to [HK00, Proposition 2.9] for a proof of this proposition. We will not review here the tools of Geometric Invariant Theory necessary to understand the statement and its proof (see [MFK94, Dol03]). Let us just explain how the open set of semi-stable points appearing in the above statement may be explicitely discribed.

In our situation, the semi-stable points are by definitions those points x admitting a $T_{\mathrm{NS}}(X)$ invariant section s of some power of the D-linearized trivial line bundle such that $s(x) \neq 0$. The $T_{\mathrm{NS}}(X)$-invariant sections $s \in \mathrm{HCR}(X)$ are those sections s which satisify $t.s = t([D]).s$: these are exactly the elements of the D-graded part of $\mathrm{HCR}(X)$. Hence the semi-stable points are those points x for which there exists $s \in \underset{m \geqslant 1}{\cup} \mathrm{HCR}(X)_{m\,D}$ such that $s(x) \neq 0$; in other words the open set of semi-stable points is the complement of the closed subscheme with ideal

$$\mathscr{I}_D \overset{\text{def}}{=} \langle \underset{m \geqslant 1}{\cup} \mathrm{HCR}(X)_{m\,D} \rangle. \tag{3.1.2}$$

Note that one easily checks that the radical $\sqrt{\mathscr{I}_D}$ (and thus the open set of semi-stable points) does not depend on the choice of the ample class D. We denote by $\mathscr{T}_X$ the open set of semi-stable points.

Now we choose a presentation of $\mathrm{HCR}(X)$. Let $\{s_i\}_{i \in I}$ denote a finite family of global (non constant) sections generating $\mathrm{HCR}(X)$. They induce an isomorphism $\mathrm{HCR}(X) \xrightarrow{\sim} k[x_i]_{i \in I}/\mathscr{I}_X$, where $\mathscr{I}_X$ is a $\mathrm{Pic}(X)$-homogeneous ideal, and an embedding $\mathrm{Spec}(\mathrm{HCR}(X)) \hookrightarrow \mathbf{A}^I$.

For $i \in I$, let D_i denote the divisor of s_i. Let U denote the complement of the union of the D_i. Since the s_i's generate $\mathrm{HCR}(X)$, the class of the D_i's generate $\mathrm{Pic}(X)$ as a group and $C_{\mathrm{eff}}(X)$ as a cone, and $\mathrm{Pic}(U)$ is trivial. It is moreover known that $\mathrm{HCR}(X)$ is an UFD ([EKW04, BH03]), thus we may assume that the s_i are irreducible elements of $\mathrm{HCR}(X)$, and that no two of them are associate.

Therefore we obtain an exact sequence of free modules of finite rank:

$$0 \to k[U]^{\times}/k^{\times} \to \underset{i \in I}{\oplus} \mathbf{Z}\, D_i \to \mathrm{Pic}(X) \to 0 \tag{3.1.3}$$

which is a generalization of (2.1.2) valid in the toric case.

Now comes an even more explicit description of $\mathscr{T}_X$, viewed as a locally closed subvariety of $\mathbf{A}^I$. For an ample class D denote by $\mathcal{I}_D$ the class of subset J of I such that there exists $\lambda_i \in \mathbf{N}^I_{>0}$ and $m \in \mathbf{N}_0$ satisfying $[\sum \lambda_i\, D_i] = [m\, D]$. Then the ideals $\langle \prod_{i \in J} s_i \rangle_{J \in I_D}$ and $\mathscr{I}_D$ have the same radical, and therefore $\mathcal{T}_X$ may be described as the open subset of the variety $\mathrm{Spec}(\mathrm{HCR}(X))$ given by the union over $J \in \mathcal{I}_D$ of the trace of the open subset $\prod_{i \in J} x_i \neq 0$. Setting

$$\widetilde{\mathcal{I}_D} = \{J \subset I, \quad \forall K \in \mathcal{I}_D, \quad J \cap K \neq \varnothing\}, \tag{3.1.4}$$

we therefore have

$$\mathscr{T}_X = \mathrm{Spec}(\mathrm{HCR}(X)) \setminus \underset{J \in \widetilde{\mathcal{I}_D}}{\cup} \ \underset{i \in J}{\cap} \{x_i = 0\}. \tag{3.1.5}$$

In fact one may check that denoting by π the quotient morphism $\mathscr{T}_X \to X$ the divisor $\pi^* D_i$ is the trace of the hyperplane $\{x_i = 0\}$ on $\mathscr{T}_X$. Hencewe have

$$\mathscr{T}_X = \mathrm{Spec}(\mathrm{HCR}(X)) \setminus \underset{\substack{J \subset I \\ \underset{i \in J}{\cap} D_i = \varnothing}}{\cup} \ \underset{i \in J}{\cap} \{x_i = 0\}. \tag{3.1.6}$$

For a toric variety X, we thus recover the previous definition (2.2.1) of $\mathscr{T}_X$. For the plane blown up at three collinear points, we have

$$\begin{gathered}\mathscr{T}_X = \mathrm{Spec}(k[x_0, \dots, x_6]/(x_1 x_4 + x_2 x_5 + x_3 x_6)) \setminus \\ \underset{4 \leqslant i \neq j \leqslant 6}{\cup} \{x_i = 0\} \cap \{x_0 = 0\} \cup \underset{1 \leqslant i \neq j \leqslant 3}{\cup} \{x_i = 0\} \cap \{x_j = 0\} \cup \underset{\substack{1 \leqslant i \leqslant 3, \\ 4 \leqslant j \leqslant 6, \\ j \neq i+3}}{\cup} \{x_i = 0\} \cap \{x_j = 0\}.\end{gathered} \tag{3.1.7}$$

3.2. **Description of the functor of points of a variety whose homogeneous coordinate ring is finitely generated.** Retain all the notations of the previous section. We now want to describe the functor of points of X in terms of its homogeneous coordinate ring. We follow closely the approach used in the toric case. In fact, the only novelty is the nontrivial relations satisfied by the generator

Recalling exact sequence (3.1.3), similarily to the toric case, every element m of $k[U]^\times/k^\times$ determines an isomorphism $c_m : \underset{i\in I}{\otimes}\mathscr{O}_X(D_i)^{\otimes v_{D_i}(m)} \xrightarrow{\sim} \mathscr{O}_X$ (where $v_{D_i}(m)$ is the order of annulation of the rational function m along D_i), and we have $c_m \otimes c_{m'} = c_{m+m'}$.

Let $f : S \to X$ be a morphism from a k-scheme S to X. Let $\mathcal{L}_i \overset{\text{def}}{=} f^*\mathscr{O}_X(D_i)$, $u_i \overset{\text{def}}{=} f^* s_i$ and for $m \in k[U]^\times/k^\times$, $d_m \overset{\text{def}}{=} f^* c_m$ The datum $((\mathcal{L}_i, t_i), (d_m)_{m\in k[U]^\times/k^\times}$ is then a X*-collection on* S in the following sense:

Definition 3.4. An X-collection on a k-scheme S is the datum of:

(1) a family $((\mathcal{L}_i), u_i)_{i\in I}$ where $\mathcal{L}_i$ is a line bundle on S and u_i a global section of $\mathcal{L}_i$
(2) a family of isomorphisms $\{d_m : \otimes\mathcal{L}_i^{\otimes v_{D_i}(m)} \xrightarrow{\sim} \mathscr{O}_S\}_{m\in k[U]^\times/k^\times}$

satisfying the following conditions:

(1) $d_m \otimes d_{m'} = d_{m+m'}$;
(2) for every $J \subset I$ such that $\cap_{i\in J} D_i = \varnothing$ the sections $\{u_i\}_{i\in J}$ do not vanish simultaneously;
(3) For every homogeneous element F of $\mathscr{I}_X$, the section $F(u_i)_{i\in I}$ is zero.

Note that the datum of the trivializations $\{d_m\}$ allows to give a sense to the latter condition. We have a canonical X-collection C_X on X given by $((\mathscr{O}_X(D_i), s_i), \{c_m\})$ and similarily to the toric case one shows that the maps

$$\begin{array}{ccc} \operatorname{Hom}(S,X) & \longrightarrow & \operatorname{Coll}_{X,S} \\ f & \longmapsto & f^*C_X \end{array} \tag{3.2.1}$$

define an isomorphism between the functor of points of $\mathbf{P}^n$ and the functor which associates to a k-scheme S the set $\operatorname{Coll}_{X,S}$ of isomorphism classes of X-collections on S. Moreover (3.2.1) induces a bijection between the element of $\operatorname{Hom}(S,X)$ which do not factor through the boundary $\cup D_i$ and the non-degenerate X-collections on S (those for which no one of the sections u_i is the zero section).

Now we should examine the functor $\operatorname{Hom}(\mathbf{P}^1, X)$, or more precisely the open subfonctor given by morphisms who do not factor through the boundary. Such a morphism is entirely determined by an equivalence class of non-degenerate X-collections on $\mathbf{P}^1$. Let $\boldsymbol{d} \in \mathbf{N}^I \cap \operatorname{Pic}(X)^\vee = \operatorname{Pic}(X)^\vee \cap C_{\text{eff}}(X)^\vee$ (here of course we view $\operatorname{Pic}(X)^\vee$ as a subgroup of $\mathbf{Z}^I$ through the dual of the exact sequence (3.1.3)). Denote by $\widetilde{\mathscr{Z}_X^{\boldsymbol{d}}}$ the $T_{\text{NS}}(X)$-invariant closed subscheme of $\mathcal{H}_{\boldsymbol{d}}^\bullet \xrightarrow{\sim} \prod_{i\in I}\mathbf{A}^{d_i+1}\setminus\{0\}$ defined by the equations

$$F(P_i) = 0 \tag{3.2.2}$$

where F varies along the homogeneous elements of $\mathscr{I}_X$. Denote by $\mathscr{Z}_X^{\boldsymbol{d}}$ the image of $\widetilde{\mathscr{Z}_X^{\boldsymbol{d}}}$ in $\mathbf{P}^{\boldsymbol{d}}$.

Denote by $\mathcal{H}_{\boldsymbol{d},X}^\bullet$ the open subset of $\mathcal{H}_{\boldsymbol{d}}^\bullet$ consisting of I-uple (P_i) such that for every $J \subset I$ such that $\cap_{i\in I} D_i = \varnothing$, the $\{P_i\}_{i\in J}$ are coprime.

Then one can show that the variety $\mathcal{H}_{\boldsymbol{d},X}^\bullet \cap \widetilde{\mathscr{Z}_X^{\boldsymbol{d}}}/T_{\text{NS}}(X)$ is isomorphic to $\mathbf{Hom}_{\boldsymbol{d},U}(\mathbf{P}^1, X)$. Hence, if T_X denotes the torus $\operatorname{Hom}(k[U]^\times/k^\times, \mathbf{G}_m)$, $\mathbf{Hom}_{\boldsymbol{d},U}(\mathbf{P}^1, X)$ is a T_X-torsor over $\mathbf{P}_X^{\boldsymbol{d}} \cap \mathscr{Z}_X^{\boldsymbol{d}}$.

3.3. **Application to the degree zeta function.** Let us know explain how this description of $\mathrm{Hom}(\mathbf{P}^1, X)$ gives rise to an expression of the degree zeta function similar to the one we obtained in the toric case. We will assume that the base field k is a finite field of cardinality q and restrict ourselves to the case of the classical degree zeta function. We have

$$\frac{\#_k Z(X,U,t)}{(q-1)^{\dim(T_X)}} \tag{3.3.1}$$

$$= \sum_{\boldsymbol{d}\in C_{\mathrm{eff}}(X)^\vee\cap \mathrm{Pic}(X)^\vee} \#\left(\mathbf{P}^{\boldsymbol{d}}_X \cap \mathscr{Z}^{\boldsymbol{d}}_X(k)\right)\, t^{\langle \boldsymbol{d}\,,\,\omega_X^{-1}\rangle} \tag{3.3.2}$$

$$= \sum_{\boldsymbol{d}\in C_{\mathrm{eff}}(X)^\vee\cap \mathrm{Pic}(X)^\vee} \sum_{\boldsymbol{\mathfrak{D}}\in \mathbf{P}^{\boldsymbol{d}}(k)} \mathbf{1}_{\mathbf{P}^{\boldsymbol{d}}_X(k)}(\boldsymbol{\mathfrak{D}})\mathbf{1}_{\mathscr{Z}^{\boldsymbol{d}}_X(k)}(\boldsymbol{\mathfrak{D}})\, t^{\langle \boldsymbol{d}\,,\,\omega_X^{-1}\rangle} \tag{3.3.3}$$

$$= \sum_{\boldsymbol{d}\in C_{\mathrm{eff}}(X)^\vee\cap \mathrm{Pic}(X)^\vee} \sum_{\boldsymbol{\mathfrak{D}}\in \mathbf{P}^{\boldsymbol{d}}(k)} \sum_{0\leqslant \boldsymbol{\mathfrak{D}}'\leqslant \boldsymbol{\mathfrak{D}}} \mu_X(\boldsymbol{\mathfrak{D}}')\mathbf{1}_{\mathscr{Z}^{\boldsymbol{d}}_X(k)}(\boldsymbol{\mathfrak{D}})\, t^{\langle \boldsymbol{d}\,,\,\omega_X^{-1}\rangle} \tag{3.3.4}$$

where μ_X is the function determined by the relation

$$\forall \boldsymbol{d}\in \mathbf{N}^I, \quad \forall \mathcal{D}\in \mathbf{P}^{\boldsymbol{d}}(k), \qquad \sum_{(\mathcal{D}'_i)\leqslant(\mathcal{D}_i)} \mu_X((\mathcal{D}'_i)) = \mathbf{1}_{\mathbf{P}^{\boldsymbol{d}}_X(k)}(\mathcal{D}), \tag{3.3.5}$$

for which proposition 2.10 remains valid. After a straightforward change of variables, the previous expression becomes

$$\sum_{\boldsymbol{\mathfrak{D}}\in \mathrm{Div}_{\mathrm{eff}}(\mathbf{P}^1)^I} \mu_X(\boldsymbol{\mathfrak{D}}) \sum_{\substack{\boldsymbol{d}\in C_{\mathrm{eff}}(X)^\vee\cap \mathrm{Pic}(X)^\vee,\\ \forall i\in I,\quad \langle \boldsymbol{d}\,,\,D_i\rangle\geqslant \deg(\mathcal{D}_i)}} \sum_{\boldsymbol{\mathfrak{D}}'\in \mathbf{P}^{\boldsymbol{d}-\deg(\boldsymbol{\mathfrak{D}})}} \mathbf{1}_{\mathscr{Z}^{\boldsymbol{d}}_X(k)}(\boldsymbol{\mathfrak{D}}+\boldsymbol{\mathfrak{D}}')\, t^{\langle \boldsymbol{d}\,,\,\omega_X^{-1}\rangle}. \tag{3.3.6}$$

For $\boldsymbol{\mathfrak{D}}\in \mathrm{Div}_{\mathrm{eff}}(\mathbf{P}^1)$ and $\boldsymbol{d}\in C_{\mathrm{eff}}(X)^\vee\cap \mathrm{Pic}(X)^\vee$ such that $\langle \boldsymbol{d}\,,\,D_i\rangle \geqslant \deg(\mathcal{D}_i)$ let us denote by $\mathscr{N}_X(\boldsymbol{\mathfrak{D}},\boldsymbol{d})$ the cardinality of the set

$$\{(P_i)\in \mathcal{H}^{\bullet}_{\boldsymbol{d}-\deg(\boldsymbol{\mathfrak{D}})}(k),\, \forall F\in \mathscr{I}_X^{\mathrm{homog}},\, F(P_i.P_{\mathcal{D}_i})=0\} \tag{3.3.7}$$

(where $P_{\mathcal{D}_i}\in \mathcal{H}^{\bullet}_{\deg(\mathcal{D}_i)}(k)$ denotes a representative of $\mathcal{D}_i\in \mathbf{P}^{\deg(\mathcal{D}_i)}(k)$). Then $\#_k Z(X,U,t)$ may be rewritten as

$$\frac{1}{(q-1)^{\mathrm{rk}(\mathrm{Pic}(X))}} \sum_{\boldsymbol{\mathfrak{D}}\in \mathrm{Div}_{\mathrm{eff}}(\mathbf{P}^1)^I} \mu_X(\boldsymbol{\mathfrak{D}}) \sum_{\substack{\boldsymbol{d}\in C_{\mathrm{eff}}(X)^\vee\cap \mathrm{Pic}(X)^\vee,\\ \forall i\in I,\quad \langle \boldsymbol{d}\,,\,D_i\rangle\geqslant \deg(\mathcal{D}_i)}} \mathscr{N}_X(\boldsymbol{\mathfrak{D}},\boldsymbol{d})\, t^{\langle \boldsymbol{d}\,,\,\omega_X^{-1}\rangle}. \tag{3.3.8}$$

Compare this expression with the one we obtained in the toric case (that is, (2.5.4) to which we apply the morphism $\#_k$), and note that this is indeed a generalization. In the toric case, the ideal $\mathscr{I}_X$ is zero and $\mathscr{N}_X(\boldsymbol{\mathfrak{D}},\boldsymbol{d})$ is nothing else than the cardinality of $\mathcal{H}^{\bullet}_{\boldsymbol{d}-\deg(\boldsymbol{\mathfrak{D}})}$. Since the behaviour of the Möbius function μ_X is easily handled whether the variety X is toric or not, the fundamental difference between the toric and non toric case in the study of the degree zeta function is that we have to deal with the non trivial relations satisfied by the generators of the homogeneous coordinate ring. Thus $\mathscr{N}_X(\boldsymbol{\mathfrak{D}},\boldsymbol{d})$ is really the hard part to undersand in the above expression; as far as I know, there is yet no general procedure to handle these kind of relations; every succesful attempt to settle Manin's conjecture using this method is highly dependent on the particular shape of the equations defining the homogeneous coordinate ring on the involved variety or family of varieties.

Remark 3.5. It is not clear (at least to me) what a sensible analog of expression (3.3.8) for the geometric degree zeta funtion could be.

3.4. **Application to the projective plane blown up at three collinear points.** I will now describe very sketchly how the expression (3.3.8) leads to the expected estimates for the degree zeta function in a very particular case, namely the case of the projective plane blown up at three collinear points. So let X be the plane blown up at three collinear points, D_0 be the strict transform of the line L joining the points, D_1, D_2 and D_3 the exceptional divisors and D_4, D_5, and D_6 the strict transform of the lines joining a point not on L to the blown up points. As already stated, one may find sections $\{s_i\}_{0\leqslant i\leqslant 7}$ with divisor D_i generating $\mathrm{HCR}(X)$ and such that $x_i\mapsto s_i$ induces an isomorphism $k[x_0,\ldots,x_6]/x_1\,x_4+x_2\,x_5+x_3\,x_6]\xrightarrow{\sim}\mathrm{HCR}(X)$. Note that (D_0,D_1,D_2,D_3,D_4) is a basis of $\mathrm{Pic}(X)$ and that we have the linear equivalence relations

$$D_4\sim D_0+D_2+D_3,\quad D_5\sim D_0+D_1+D_3,\quad D_6\sim D_0+D_1+D_2. \tag{3.4.1}$$

Moreover an anticanonical divisor is easily computed as $3\,D_0+2\,D_1+2\,D_2+2\,D_3$. Note that its class coincide with the class of the sum of the boundary divisors minus class of the degree of the relation defining $\mathrm{HCR}(X)$; this in fact a special case of a generalized adjunction formula, see [BH07, proposition 8.5].

Now let $\boldsymbol{\mathcal{D}}\in\mathrm{Div}_{\mathrm{eff}}(\mathbf{P}^1)^7$ and let $\boldsymbol{d}\in C_{\mathrm{eff}}(X)^\vee\cap\mathrm{Pic}(X)^\vee\subset\mathbf{Z}^7$ such that $\boldsymbol{d}\geqslant\deg(\boldsymbol{\mathcal{D}})$; note that the condition $\boldsymbol{d}\in C_{\mathrm{eff}}(X)^\vee\cap\mathrm{Pic}(X)^\vee$ means here that $\boldsymbol{d}$ satisifies $d_i\geqslant 0$ for $0\leqslant i\leqslant 7$ and

$$d_4=d_0+d_2+d_3,\quad d_5=d_0+d_1+d_3,\quad d_6=d_0+d_1+d_2. \tag{3.4.2}$$

Let $Q_i\in\mathcal{H}^\bullet_{\deg(\mathcal{D}_i)}$ be a representative of $\mathcal{D}_i$. We have to estimate the number of elements $P_0,\ldots,P_7\in\mathcal{H}^\bullet_{\boldsymbol{d}-\deg(\boldsymbol{\mathcal{D}})}$ satisfying $P_1\,P_4\,Q_1\,Q_4+P_2\,P_5\,Q_2\,Q_5+P_3\,P_6\,Q_3\,Q_6=0$. We make a first "approximation" by allowing P_4, P_5 and P_6 to be zero and use the following elementary lemma.

Lemma 3.6. *Let D be a nonnegative integer, e_1, e_2 and e_3 be nonnegative integers such that $e_i\leqslant D$. Moreover we assume that $e_i+e_j\leqslant D$ holds whenever $i\neq j$. Let (R_1,R_2,R_3) be an element of $\mathcal{H}^\bullet_{(e_1,e_2,e_3)}(k)$. Then the cardinality of the set*

$$\{(R'_1,R'_2,R'_3)\in\mathcal{H}^{(D-e_1,D-e_2,D-e_3)},\quad R_1\,R'_1+R_2\,R'_2+R_3\,R'_3=0\} \tag{3.4.3}$$

is

$$q^{2+2\,D-(e_1+e_2+e_3)+\deg(\gcd(P_1,P_2,P_3))} \tag{3.4.4}$$

We apply this lemma to the above situation, setting $R_i=P_i\,Q_i\,Q_{i+3}$ and $R'_i=P_{i+3}$ (hence $e_i=d_i+\deg(\mathcal{D}_{i+3})$ and $D=d_i+d_{i+3}=d_0+d_1+d_2+d_3$), and we find that under the conditions

$$\deg(\mathcal{D}_i)+\deg(\mathcal{D}_j)\leqslant d_0+d_k\quad\{i,j,k\}=\{1,2,3\} \tag{3.4.5}$$

we have

$$\begin{aligned}\mathscr{N}_X(\boldsymbol{d},\boldsymbol{\mathcal{D}})&=q^{2+2\,d_0+d_1+d_2+d_3-\deg(\mathcal{D}_4)-\deg(\mathcal{D}_5)-\deg(\mathcal{D}_6)}\\&\quad\times\sum_{\boldsymbol{\mathcal{E}}\in\mathbf{P}^{(d_i-\deg(\mathcal{D}_i))_{0\leqslant i\leqslant 3}}}q^{\deg(\gcd(\mathcal{E}_1+\mathcal{D}_1+\mathcal{D}_4,\mathcal{E}_2+\mathcal{D}_2+\mathcal{D}_5,\mathcal{E}_3+\mathcal{D}_3+\mathcal{D}_6))}.\end{aligned} \tag{3.4.6}$$

Our second "approximation" will be to assume that (3.4.6) holds regardless (3.4.5) are satisfied or not.

Now for $\boldsymbol{d}\in\mathbf{N}^4$ and $\boldsymbol{\mathcal{D}}\in\mathrm{Div}_{\mathrm{eff}}(\mathbf{P}^1)^7$ we want to estimate the quantity

$$\sum_{\boldsymbol{\mathcal{E}}\in\mathbf{P}^{\boldsymbol{d}}}q^{\deg(\gcd(\mathcal{E}_1+\mathcal{D}_1+\mathcal{D}_4,\mathcal{E}_2+\mathcal{D}_2+\mathcal{D}_5,\mathcal{E}_3+\mathcal{D}_3+\mathcal{D}_6))} \tag{3.4.7}$$

We consider the generating series

$$\sum_{\boldsymbol{d}\in\mathbf{N}^4}\sum_{\boldsymbol{\mathcal{E}}\in\mathbf{P}^{\boldsymbol{d}}} q^{\deg(\gcd(\mathcal{E}_1+\mathcal{D}_1+\mathcal{D}_4,\mathcal{E}_2+\mathcal{D}_2+\mathcal{D}_5,\mathcal{E}_3+\mathcal{D}_3+\mathcal{D}_6))}\prod_{0\leqslant i\leqslant 3} t_i^{d_i}$$
$$= \sum_{\boldsymbol{\mathfrak{D}}\in\mathrm{Div}_{\mathrm{eff}}(\mathbf{P}^1)^4} q^{\deg(\gcd(\mathcal{E}_1+\mathcal{D}_1+\mathcal{D}_4,\mathcal{E}_2+\mathcal{D}_2+\mathcal{D}_5,\mathcal{E}_3+\mathcal{D}_3+\mathcal{D}_6))}\prod_{0\leqslant i\leqslant 3} t_i^{\deg(\mathcal{E}_i)} \qquad (3.4.8)$$

wich decomposes into an Euler product

$$\prod_{\mathcal{P}\in(\mathbf{P}^1_k)^{(0)}}\sum_{\boldsymbol{n}\in\mathbf{N}^4} q^{\deg(\mathcal{P})\underset{1\leqslant i\leqslant 3}{\mathrm{Min}}(n_i+\mathrm{ord}_{\mathcal{P}}(\mathcal{D}_i)+\mathrm{ord}_{\mathcal{P}}(\mathcal{D}_{i+3}))}\prod_{0\leqslant i\leqslant 3} t_i^{\deg(\mathcal{P})\,n_i} \qquad (3.4.9)$$

Let us explain what happens in the case $\boldsymbol{\mathfrak{D}}=0$. It is rather easy to check the identity

$$\sum_{\boldsymbol{n}\in\mathbf{N}^4}\theta^{\mathrm{Min}(n_1,n_2,n_3)}\prod_{0\leqslant i\leqslant 3} t_i^{n_i} = \frac{1-t_1\,t_2\,t_3}{1-\theta\,t_1\,t_2\,t_3}\prod_{1\leqslant i\leqslant 3}\frac{1}{1-t_i}. \qquad (3.4.10)$$

Thus (3.4.9) may be rewritten as

$$\prod_{\mathcal{P}\in(\mathbf{P}^1_k)^{(0)}}\frac{1-(t_1\,t_2\,t_3)^{\deg(\mathcal{P})}}{1-(q\,t_1\,t_2\,t_3)^{\deg(\mathcal{P})}}\prod_{0\leqslant i\leqslant 3} Z_{\mathrm{HW}}(\mathbf{P}^1_k,t) \qquad (3.4.11)$$

(recall that $Z_{\mathrm{HW}}(\mathbf{P}^1_k,t)=\frac{1}{(1-t)(1-q\,t)}$ is the Hasse–Weil zeta function of $\mathbf{P}^1_k$). Note that the first factor of the above expression defines a holomorphic function F in the polydisc $\prod\{|t_i|\leqslant q^{-1+\varepsilon}\}$ for sufficiently small $\varepsilon>0$. Using Cauchy estimates, one obtains the approximation

$$\sum_{\boldsymbol{\mathfrak{D}}\in\mathrm{Div}_{\mathrm{eff}}(\mathbf{P}^1)^{\boldsymbol{d}}} q^{\deg(\gcd(\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3))}\sim F(q^{-1},\dots,q^{-1})\,q^{d_0+d_1+d_2+d_3} \qquad (3.4.12)$$

In case $\boldsymbol{\mathfrak{D}}\neq 0$, an analogous reasoning shows the approximation

$$\sum_{\boldsymbol{\mathfrak{D}}\in\mathrm{Div}_{\mathrm{eff}}(\mathbf{P}^1)^{\boldsymbol{d}}} q^{\deg(\gcd(\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3))}\sim F_{\boldsymbol{\mathfrak{D}}}(q^{-1},\dots,q^{-1})\,q^{d_0+d_1+d_2+d_3} \qquad (3.4.13)$$

where $F_{\boldsymbol{\mathfrak{D}}}(q^{-1},\dots,q^{-1})$ has an explicit expression as an Euler product $\prod_{\mathcal{P}}\widetilde{F}_{\boldsymbol{\mathfrak{D}}}(q^{-\deg(\mathcal{P})})$ where $\widetilde{F}_{\boldsymbol{\mathfrak{D}}}$ is a rational function. Note that $F_{\boldsymbol{\mathfrak{D}}}$ and $\widetilde{F_{\boldsymbol{\mathfrak{D}}}}$ depends only on the 7-uple of integers $\mathrm{ord}_P(\boldsymbol{\mathfrak{D}})$.

As a third approximation we will assume that the above estimation is in fact equality, thus obtaining

$$\mathscr{N}_X(\boldsymbol{d},\boldsymbol{\mathfrak{D}}) = F_{\boldsymbol{\mathfrak{D}}}(q^{-1},\dots,q^{-1})\,q^{2+3d_0+2\,d_1+2\,d_2+2\,d_3-\sum_{0\leqslant i\leqslant 6}\deg(\mathfrak{D}_i)} \qquad (3.4.14)$$

$$\mathscr{N}_X(\boldsymbol{d},0) = F(q^{-1},\dots,q^{-1})\,q^{2+3d_0+2\,d_1+2\,d_2+2\,d_3-\sum_{0\leqslant i\leqslant 6}\deg(\mathfrak{D}_i)} \qquad (3.4.15)$$

Our last approximation will be to drop the conditions $\langle \boldsymbol{d}\,,\,D_i\rangle\geqslant\deg(\mathfrak{D}_i)$ appearing in the summation in (3.3.8).

Modulo all the previous approximations, (3.3.8) may be now written as

$$\sum_{\boldsymbol{\mathfrak{D}}\in\mathrm{Div}_{\mathrm{eff}}(\mathbf{P}^1)^I}\mu_X(\boldsymbol{\mathfrak{D}})F_{\boldsymbol{\mathfrak{D}}}(q^{-1},\dots,q^{-1})q^{-\sum_{0\leqslant i\leqslant 6}\deg(\mathfrak{D}_i)}$$
$$\times\sum_{\boldsymbol{d}\in C_{\mathrm{eff}}(X)^\vee\cap\mathrm{Pic}(X)^\vee} q^{2+3\,d_0+2\,d_1+2\,d_2+2\,d_3}\,t^{\langle\boldsymbol{d}\,,\,\omega_X^{-1}\rangle} \qquad (3.4.16)$$

Recalling that the anticanonical class is given by $3\,D_0+D_1+D_2+D_3$, the second factor is $Z(\mathrm{Pic}(X)^\vee, C_{\mathrm{eff}}(X)^\vee, \left[\omega_X^{-1}\right], q\,t)$.

Now the main task we are left with in order to show that the answer to question 1.5 is indeed positive, is to establish that all the above approximations yield error terms which are indeed $(q^{-1}, \operatorname{rk}(\operatorname{Pic}(X))-1)$ controlled. Roughly, this can be done using a regular decomposition of the effective cone analogous to the one used in the toric case, but there is a certain amount of technical subtelties that will not be discussed here (see [Bou09a, Bou10b]).

Regarding Peyre's refinement of Manin's conjecture discussed at the end of section 2.6, another task is to show that the constant given by the first factor of (3.4.16) may be expressed as the Tamagawa number

$$\frac{q^{\dim(X)}}{(1-q^{-1})^{\operatorname{rk}\operatorname{Pic}(X)}} \prod_{\mathcal{P}\in(\mathbf{P}^1_k)^{(0)}} (1-q^{-\deg(\mathcal{P})})^{\operatorname{rk}(\operatorname{Pic}(X))} \frac{\#X(\kappa_{\mathcal{P}})}{q^{\deg(\mathcal{P})\dim(X)}}. \tag{3.4.17}$$

But using properties of μ_X, this factor may be rewritten as the Euler product

$$\prod_{\mathcal{P}\in(\mathbf{P}^1_k)^{(0)}} \sum_{\boldsymbol{n}\in\{0,1\}^7} \mu^0_X(\boldsymbol{n})\widetilde{F}_{\boldsymbol{n}}(q^{-\deg(\mathcal{P})})q^{-\deg(\mathcal{P})\sum n_i} \tag{3.4.18}$$

Hence we must check, for every $\mathcal{P}\in(\mathbf{P}^1_k)^{(0)}$, the following identity

$$(1-q^{-\deg(\mathcal{P})})^{\operatorname{rk}(\operatorname{Pic}(X))}\frac{\#X(\kappa_{\mathcal{P}})}{q^{\deg(\mathcal{P})\dim(X)}} = \sum_{\boldsymbol{n}\in\{0,1\}^7} \mu^0_X(\boldsymbol{n})\widetilde{F}_{\boldsymbol{n}}(q^{-\deg(\mathcal{P})})q^{-\deg(\mathcal{P})\sum n_i}. \tag{3.4.19}$$

Note that $\#X(\kappa_{\mathcal{P}}) = 1 + 4\,q^{\deg(\mathcal{P})} + q^{2\deg(\mathcal{P})}$, hence (3.4.19) may be seen as a formal identity between two rational functions in the variable $q^{\deg(\mathcal{P})}$, which may be checked in a finite amount of time (recall that we have in fact an explicit expression for the functions $\widetilde{F}_{\boldsymbol{n}}$; of course a symbolic computation software may be helpful...). One can also try to exploit the following relation, which holds for every finite k-extension L. This is a generalization of proposition 2.11 to the nontoric case, valid for every k-variety X having a finitely generated homogeneous coordinate ring:

$$\sum_{\boldsymbol{n}\in\{0,1\}^I} \mu^0_X(\boldsymbol{n})\frac{\#\mathscr{T}_{X,\boldsymbol{n}}(L)}{(\#L)^{\dim(\mathscr{T}_X)}} = (1-\#L)^{\operatorname{rk}(\operatorname{Pic}(X))}\frac{\#X(L)}{(\#L)^{\dim(X)}} \tag{3.4.20}$$

Here we denote by $\mathscr{T}_{X,\boldsymbol{n}}$ the intersection of $\mathscr{T}_X\subset\mathbf{A}^I$ with the subspace $\underset{i,\,n_i=1}{\cap}\{x_i=0\}$. The proof goes along the same line that the proof of proposition 2.11 and from (3.4.20) one may derive a slightly more conceptual proof of (3.4.19) (see [Bou09a]). But to our mind this still does not explain clearly why (3.4.19) holds, and it would be nice to find a genuine conceptual explaination.

One of the key ingredient in the above (sketch of) proof of the geometric Manin's conjecture for the plane blown up at three collinear points was in fact the property that the homogeneous coordinate ring has one relation and that there exists $I_0\subset I$ such that the classes of $\{D_i\}_{i\in I_0}$ form a basis of $\operatorname{Pic}(X)$ and the relation is linear with respect to the variables $\{s_i\}_{i\in I\setminus I_0}$. In a sense, in the context of our counting problem, this situation may be considered as the simplest one once the case of toric varieties (for which there are no relations) has been excluded. One might hope that the techniques employed may lead to an kind of uniform proof of Manin's conjecture for varieties satisfying the above requirements (see [Bou10b] for a beginning of justification), though even under these hypotheses the control of the error terms seems to be a very hard task in general. Note that along varieties for which the hypotheses hold one finds a lot of generalized del Pezzo surfaces whose homogeneous coordinate ring has one relation (see [Der06] for their complete classification).

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