# Ricci curvature, entropy and optimal transport

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'Optimal Transportation: Theory and Applications'

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These notes are the planned contents of my lectures. Some parts could be only briefly explained or skipped due to the lack of time or possible overlap with other lectures. The aim of these lectures is to review the recent development on the relation between optimal transport theory and Riemannian geometry. Ricci curvature is the key ingredient. Optimal transport theory provides a good characterization of lower Ricci curvature bounds without using differentiable structure. Then it can be considered as the 'definition' of lower Ricci curvature bounds of metric measure spaces.

In §1, we recall the definition of the Ricci curvature of a Riemannian manifold and the classical Bishop-Gromov volume comparison theorem. In §2, we start with Brunn-Minkowski inequalities in (weighted) Euclidean spaces, and show that a lower weighted Ricci curvature bound for a weighted Riemannian manifold is equivalent to some convexity inequality of entropy, called the curvature-dimension condition. In §3, we give the precise definition of the curvature-dimension condition for metric measure spaces, and see that it is stable under the measured Gromov-Hausdorff convergence. §4 is devoted to some geometric applications of the curvature-dimension condition. The final lecture will be concerned with some of related topics summarized in §5. Although we concentrate on rather geometric aspects, these lectures will be far from exhaustive. Interested readers can find more references in Further Reading at the end of each section (except §5).

## 0 Notations

First of all, we collect some notations we use for convenience. Throughout these lectures, $(M, g)$ is an $n$-dimensional complete Riemannian manifold without boundary with $n \geq 2$, $\mathrm{vol}_g$ stands for the Riemannian volume measure of $g$.

A metric space is called a *geodesic space* if any two points $x, y \in X$ can be connected by a rectifiable curve $\gamma : [0, 1] \longrightarrow X$ of length $d(x, y)$ with $\gamma(0) = x$ and $\gamma(1) = y$. Such minimizing curves parametrized proportionally to arc length are called *minimal geodesics*. Open and closed balls of center $x$ and radius $r$ will be denoted by $B(x, r)$ and $\overline{B}(x, r)$. A

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*metric measure space* will be a triple $(X, d, m)$ consisting of a geodesic space $(X, d)$ and a Borel measure $m$ on it such that $0 < m(B(x, r)) < \infty$ for all $x \in X$ and $0 < r < \infty$.

For a metric space $(X, d)$, we denote by $\mathcal{P}(X)$ the set of Borel probability measures on $X$, by $\mathcal{P}_2(X) \subset \mathcal{P}(X)$ the subset consisting of measures of finite second moment, and by $\mathcal{P}_c(X) \subset \mathcal{P}_2(X)$ the set of compactly supported measures. Then $d_2^W$ stands for the $L^2$-Wasserstein distance on $\mathcal{P}_2(X)$.

As usual in comparison geometry, the following functions will frequently appear in our discussions. For $K \in \mathbb{R}$, $N \in (1, \infty)$ and $0 < r$ ($< \pi\sqrt{(N-1)/K}$ if $K > 0$), we define

$$\mathbf{s}_{K,N}(r) := \begin{cases} \sqrt{(N-1)/K}\sin(r\sqrt{K/(N-1)}) & \text{if } K > 0, \\ r & \text{if } K = 0, \\ \sqrt{-(N-1)/K}\sinh(r\sqrt{-K/(N-1)}) & \text{if } K < 0. \end{cases}$$

In addition, for $t \in (0, 1)$, we set

$$\beta_{K,N}^t(r) := \left(\frac{\mathbf{s}_{K,N}(tr)}{t\mathbf{s}_{K,N}(r)}\right)^{N-1}, \qquad \beta_{K,\infty}^t(r) := e^{K(1-t^2)r^2/6}.$$

# 1 Ricci curvature

Take a vector field $J$ along a geodesic $\gamma : [0, 1] \longrightarrow M$. If $J$ is the variational vector field of some family of geodesics, then $J$ is called a *Jacobi field*. Jacobi fields satisfy the *Jacobi equation*

$$D_{\dot\gamma}^2 J + R(J, \dot\gamma)\dot\gamma = 0, \tag{1.1}$$

where $R : TM \otimes TM \otimes TM \longrightarrow TM$ is the curvature tensor determined by the Riemannian metric $g$. For linearly independent tangent vectors $v, w \in T_xM$,

$$\mathcal{K}(v, w) := \frac{\langle R(w, v)v, w\rangle}{|v|^2|w|^2 - \langle v, w\rangle^2}$$

is the *sectional curvature* of the 2-plane spanned by $v$ and $w$. For a unit vector $v \in T_xM$, the *Ricci curvature* of $v$ is defined as the trace of $\mathcal{K}(v, \cdot)$:

$$\mathrm{Ric}(v) := \sum_{i=1}^{n-1} \mathcal{K}(v, e_i),$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of $T_xM$ with $e_n = v$.

The sectional curvature $\mathcal{K}$ naturally controls the behavior of (especially, the second order derivative of) the distance along geodesics. For instance, the lower bound $\mathcal{K} \geq k$ for $k \in \mathbb{R}$ is equivalent to that every geodesic triangle in $M$ is 'thicker' than the triangle with the same side lengths in the two-dimesional space form of constant sectional curvature $k$. This triangle comparison condition makes sense also in metric spaces. Such spaces are called *Alexandrov spaces*, and deeply investigated from the geometric and analytic viewpoints.

Since it is taking the trace, the Ricci curvature has less information and controls only the behavior of the measure $m = \mathrm{vol}_g$. Along a unit speed geodesic $\gamma : [0, l] \longrightarrow M$, consider Jacobi fields $\{J_i\}_{i=1}^{n-1}$ along $\gamma$ given by

$$J_i(t) := D(\exp_{\gamma(0)})_{t\dot{\gamma}(0)}(te_i) \in T_{\gamma(t)}M$$

using an orthonormal basis $\{e_i\}_{i=1}^n$ with $e_n = \dot{\gamma}(0)$. Define $(n-1)\times(n-1)$ matrix-valued functions

$$\mathcal{A} := (\langle J_i, J_j \rangle), \quad \mathcal{U} := \frac{1}{2}\mathcal{A}'\mathcal{A}^{-1}, \quad \mathcal{R} := (\langle R(J_i, \dot{\gamma})\dot{\gamma}, J_j \rangle).$$

Then $\mathcal{U}$ is symmetric and satisfies the (matrix) *Riccati equation*

$$\mathcal{U}' + \mathcal{U}^2 + \mathcal{R}\mathcal{A}^{-1} = 0. \tag{1.2}$$

Taking the trace yields

$$(\mathrm{tr}\,\mathcal{U})' + \mathrm{tr}(\mathcal{U}^2) + \mathrm{Ric}(\dot{\gamma}) = 0$$

which with $\mathrm{tr}(\mathcal{U}^2) \ge (\mathrm{tr}\,\mathcal{U})^2/(n-1)$ shows

$$(\mathrm{tr}\,\mathcal{U})' + \frac{(\mathrm{tr}\,\mathcal{U})^2}{n-1} + \mathrm{Ric}(\dot{\gamma}) \le 0. \tag{1.3}$$

This estimate implies (a version of) the *Bishop comparison theorem*

$$\frac{d^2}{dt^2}\Big[(\det \mathcal{A})^{1/2(n-1)}\Big] \le -\frac{\mathrm{Ric}(\dot{\gamma})}{n-1}(\det \mathcal{A})^{1/2(n-1)}. \tag{1.4}$$

Now we assume $\mathrm{Ric} \ge K$ and by integrating (1.4) find the *Bishop-Gromov volume comparison theorem*

$$\frac{m(B(x,R))}{m(B(x,r))} \le \frac{\int_0^R \mathbf{s}_{K,n}(t)^{n-1}\,dt}{\int_0^r \mathbf{s}_{K,n}(t)^{n-1}\,dt} \tag{1.5}$$

for any $x \in M$ and $0 < r < R$ ($\le \pi\sqrt{(n-1)/K}$ if $K > 0$).

The Bishop and Bishop-Gromov comparison theorems give us a nice intuition how spaces with lower Ricci curvature bounds look like. Although bounding Ricci curvature from below is essential in many analytic applications, how to characterize such spaces without using differentiable structure had been a long standing important problem. An answer to this question is the topic of §2.

**Further reading** See [Ch] for comparison theorems in Riemannian geometry. Basic references of Alexandrov spaces are [BGP], [OtS] and [BBI]. A property corresponding to the Bishop comparison theorem (1.4) was proposed as lower Ricci curvature bounds for metric measure spaces by Cheeger and Colding [CC] (as well as Gromov [Gr]), and used to study the limit spaces of Riemannian manifolds with uniform lower Ricci curvature bounds. However, its systematic investigation has not been done until [Oh1] and [St4] (see also [KS1] and [St1] for related antecedents).

## 2 The curvature-dimension condition

The classical *Brunn-Minkowski inequality* in the Euclidean space $\mathbb{R}^n$ asserts the concavity of the $n$-th root of the Lebesgue measure:

$$m_L\big((1-t)A+tB\big)^{1/n} \ge (1-t)m_L(A)^{1/n} + tm_L(B)^{1/n} \tag{2.1}$$

for $t \in [0,1]$ and measurable sets $A, B \subset \mathbb{R}^n$, where

$$(1-t)A+tB := \{(1-t)x+ty \,|\, x \in A,\ y \in B\}.$$

We can prove (2.1) using optimal transport between uniform distributions on $A$ and $B$, the key ingredient is the inequality of arithmetic and geometric means

$$\Big[\det\big((1-t)I_n + t\mathcal{A}\big)\Big]^{1/n} \ge (1-t) + t(\det\mathcal{A})^{1/n}$$

for an $n\times n$ symmetric matrix $\mathcal{A}$. More careful argument shows that a weighted Euclidean space $(\mathbb{R}^n, m = e^{-\psi}m_L)$ for some $\psi \in C^\infty(\mathbb{R}^n)$ satisfies a generalization of the Brunn-Minkowski inequality

$$m\big((1-t)A+tB\big)^{1/N} \ge (1-t)m(A)^{1/N} + tm(B)^{1/N} \tag{2.2}$$

for $N \in (n,\infty)$ if (and only if)

$$\operatorname{Hess}\psi(v,v) - \frac{\langle \operatorname{grad}\psi, v\rangle^2}{N-n} \ge 0 \tag{2.3}$$

holds for all (unit) vectors $v \in T\mathbb{R}^n$.

A quantity corresponding to (2.3) is called the *weighted Ricci curvature* in the theory of weighted Riemannian manifolds $(M, g, m = e^{-\psi}\operatorname{vol}_g)$ with $\psi \in C^\infty(M)$:

$$\operatorname{Ric}_N(v) := \operatorname{Ric}(v) + \operatorname{Hess}\psi(v,v) - \frac{\langle \operatorname{grad}\psi, v\rangle^2}{N-n} \tag{2.4}$$

for unit tangent vectors $v \in TM$. The infinite dimensional case ($N=\infty$) amounts to the *Bakry-Émery tensor*

$$\operatorname{Ric}_\infty(v) := \operatorname{Ric}(v) + \operatorname{Hess}\psi(v,v). \tag{2.5}$$

We also define $\operatorname{Ric}_n(v) := \operatorname{Ric}(v)$ if $\langle \operatorname{grad}\psi, v\rangle = 0$, and $\operatorname{Ric}_n(v) := -\infty$ otherwise. Recall that the Bishop-Gromov volume comparison (1.5) with $\operatorname{Ric} \ge 0$ yields

$$\frac{m(B(x,R))}{m(B(x,r))} \le \left(\frac{R}{r}\right)^n$$

which can be regarded as the Brunn-Minkowski inequality between $\{x\}$ and $B(x,R)$ with $t = r/R$. Therefore it is natural to expect that lower Ricci curvature bounds relate to some interpolation inequalities like the Brunn-Minkowski inequality. The curvature-dimension condition $\mathsf{CD}(K,N)$ is actually a generalization of the Brunn-Minkowski inequality to pairs of (not necessarily uniformly distributed) probability measures. The precise definition of $\mathsf{CD}(K,N)$ will be given in §3, here we see that the core inequality of $\mathsf{CD}(K,N)$ is equivalent to $\mathrm{Ric}_N \ge K$ for weighted Riemannian manifolds.

Given $N \in [n,\infty)$ and absolutely continuous probability measure $\mu = \rho m \in \mathcal{P}(M)$, we define the *Rényi entropy* as

$$S_N(\mu) := -\int_M \rho^{1-1/N}\, dm. \tag{2.6}$$

We also define the *relative entropy* by

$$\mathrm{Ent}(\mu) := \int_M \rho \log \rho \, dm. \tag{2.7}$$

**Theorem 2.1** *A weighted Riemannian manifold $(M, g, m = e^{-\psi}\,\mathrm{vol}_g)$ satisfies $\mathrm{Ric}_N \ge K$ for some $K \in \mathbb{R}$ and $N \in [n,\infty)$ if and only if any pair of absolutely continuous probability measures $\mu_0 = \rho_0 m$, $\mu_1 = \rho_1 m \in \mathcal{P}_c(M)$ satisfies*

$$\begin{aligned} S_N(\mu_t) \le & -(1-t)\int_{M\times M} \beta_{K,N}^{1-t}\big(d(x,y)\big)^{1/N} \rho_0(x)^{-1/N}\, d\pi(x,y) \\ & - t\int_{M\times M} \beta_{K,N}^{t}\big(d(x,y)\big)^{1/N} \rho_1(y)^{-1/N}\, d\pi(x,y), \end{aligned} \tag{2.8}$$

*where $(\mu_t)_{t\in[0,1]}$ is the unique minimal geodesic from $\mu_0$ to $\mu_1$ in the $L^2$-Wasserstein space $(\mathcal{P}_2(M), d_2^W)$, and $\pi$ is the unique optimal coupling of $\mu_0$ and $\mu_1$.*

*Similarly,* $\mathrm{Ric}_\infty \ge K$ *is equivalent to*

$$\mathrm{Ent}(\mu_t) \le (1-t)\,\mathrm{Ent}(\mu_0) + t\,\mathrm{Ent}(\mu_1) - \frac{K}{2}(1-t)t d_2^W(\mu_0,\mu_1). \tag{2.9}$$

A very rough sketch of the proof is as follows. For $\mathrm{Ric}_N \ge K \Rightarrow$ (2.8), we consider optimal transport maps $(\mathcal{F}_t)_{t\in[0,1]}$ with $(\mathcal{F}_t)_\sharp \mu_0 = \mu_t$, and show that its Jacobian $\mathbf{J}_t(x) := \|(D\mathcal{F}_t)_x\|$ satisfies the inequality

$$\mathbf{J}_t(x)^{1/N} \ge (1-t)\beta_{K,N}^{1-t}\big(d(x,\mathcal{F}_1(x))\big)^{1/N} + t\beta_{K,N}^{t}\big(d(x,\mathcal{F}_1(x))\big)^{1/N}\mathbf{J}_1(x)^{1/N}. \tag{2.10}$$

Then (2.8) is obtained by integration. The key inequality (2.10) can be thought of as an infinitesimal version of the Brunn-Minkowski inequality, and is shown via calculations somewhat similar to §1. We remark that the optimal transport is performed along geodesics (composition of the gradient vector field of some twice differentiable function and the exponential map), so that its variational vector fields are Jacobi fields. The converse (2.8) $\Rightarrow \mathrm{Ric}_N \ge K$ is obtained by applying (2.8) to uniform distributions on balls (i.e., generalized Brunn-Minkowski inequalities in Theorem 4.1 below).

**Further reading** See [Le] and [Ga] for the Brunn-Minkowski inequality and related topics. The Bakry-Émery tensor $\mathrm{Ric}_\infty$ was introduced in [BE], and its generalization $\mathrm{Ric}_N$ is due to Qian [Qi]. See also [Lo] for geometric and topological applications. Cordero-Erausquin, McCann and Schmuckenschläger [CMS] first showed that $\mathrm{Ric} \ge 0$ implies (2.9) with $K = 0$. Then Theorem 2.1 is due to Lott, von Renesse, Sturm and Villani [vRS], [St2], [St3], [St4], [LV1], [LV2]. Throughout, we are indebted to Brenier [Br] and McCann's [Mc] fundamental result on the shape of optimal transport maps. See [AGS], [Vi1] and [Vi2] for the basics of optimal transport theory.

## 3 Stability

We shall recall the precise definition of the curvature-dimension condition for general metric measure spaces. For simplicity, we discuss only compact spaces. Noncompact case can be treated similarly by considering $\mathcal{P}_c(X)$ instead of $\mathcal{P}(X)$, and the pointed version of the measured Gromov-Hausdorff convergence.

For $N \in [1, \infty)$, denote by $\mathcal{DC}_N$ (*displacement convex functions*) the set of continuous convex functions $U : [0, \infty) \longrightarrow \mathbb{R}$ such that $U(0) = 0$ and the function $\varphi(s) = s^N U(s^{-N})$ is convex on $(0, \infty)$. Similarly, define $\mathcal{DC}_\infty$ as the set of continuous convex functions $U : [0, \infty) \longrightarrow \mathbb{R}$ such that $U(0) = 0$ and $\varphi(s) = e^s U(e^{-s})$ is convex on $\mathbb{R}$. In both cases, it is easy to see that $\varphi$ is nonincreasing. For $\mu \in \mathcal{P}(X)$, using its Lebesgue decomposition $\mu = \rho m + \mu^s$ into absolutely continuous and singular parts, we set

$$U_m(\mu) := \int_X U(\rho)\, dm + \lim_{r \to \infty} \frac{U(r)}{r} \cdot \mu^s(X).$$

The most important element of $\mathcal{DC}_N$ is $U(r) = Nr(1 - r^{-1/N})$ which derives the Rényi entropy (2.6)

$$U_m(\rho m) = N - N \int_X \rho^{1-1/N}\, dm = N\big(1 + S_N(\rho m)\big).$$

Letting $N$ go to infinity gives $U(r) = r \log r \in \mathcal{DC}_\infty$ and the relative entropy (2.7).

Now we are ready to recall the precise definition of the curvature-dimension condition of the version due to Lott and Villani.

**Definition 3.1** (The curvature-dimension condition) For $K \in \mathbb{R}$ and $N \in (1, \infty]$, we say that a metric measure space $(X, d, m)$ satisfies the *curvature-dimension condition* $\mathsf{CD}(K, N)$ if, for any $\mu_0 = \rho_0 m + \mu_0^s$, $\mu_1 = \rho_1 m + \mu_1^s \in \mathcal{P}(X)$, there exists a minimal geodesic $\alpha : [0, 1] \longrightarrow \mathcal{P}(X)$ from $\mu_0$ to $\mu_1$ satisfying

$$\begin{aligned} U_m\big(\alpha(t)\big) \le (1-t) &\int_{X \times X} \beta_{K,N}^{1-t}\big(d(x,y)\big) U\left(\frac{\rho_0(x)}{\beta_{K,N}^{1-t}(d(x,y))}\right) d\pi_x(y) dm(x) \\ &+ t \int_{X \times X} \beta_{K,N}^{t}\big(d(x,y)\big) U\left(\frac{\rho_1(y)}{\beta_{K,N}^{t}(d(x,y))}\right) d\pi_y(x) dm(y) \\ &+ U'(\infty)\{(1-t)\mu_0^s(X) + t\mu_1^s(X)\} \end{aligned} \tag{3.1}$$

for any $U \in \mathcal{DC}_N$ and $t \in (0, 1)$, where $\pi$ is the optimal coupling of $\mu_0$ and $\mu_1$ induced from $\alpha$, and $\pi_x$ and $\pi_y$ denote disintegrations of $\pi$ by $\mu_0$ and $\mu_1$, i.e., $d\pi(x, y) = d\pi_x(y) d\mu_0(x) = d\pi_y(x) d\mu_1(y)$.

If both $\mu_0$ and $\mu_1$ are absolutely continuous, then (3.1) is rewritten in a more symmetric form as

$$\begin{aligned} U_m\big(\alpha(t)\big) \le (1-t) &\int_{X \times X} \frac{\beta_{K,N}^{1-t}(d(x,y))}{\rho_0(x)} U\left(\frac{\rho_0(x)}{\beta_{K,N}^{1-t}(d(x,y))}\right) d\pi(x,y) \\ &+ t \int_{X \times X} \frac{\beta_{K,N}^{t}(d(x,y))}{\rho_1(y)} U\left(\frac{\rho_1(y)}{\beta_{K,N}^{t}(d(x,y))}\right) d\pi(x,y). \end{aligned} \tag{3.2}$$

Then choosing $U(r) = Nr(1 - r^{-1/N})$ and $U(r) = r \log r$ reduces to (2.8) and (2.9), respectively. One of the most important features of $\mathsf{CD}(K,N)$ is the stability under the measured Gromov-Hausdorff convergence. This is really an important and useful property which enables us to consider singular spaces appearing as the limit of smooth spaces of a uniform lower Ricci curvature bound. This is a natural question since such Riemannian manifolds form a precompact family with respect to the measured Gromov-Hausdorff convergence (Gromov's precompactness).

A sequence of metric measure spaces $\{(X_i, d_i, m_i)\}_{i \in \mathbb{N}}$ is said to converge to a compact metric measure space $(X, d, m)$ in the sense of *measured Gromov-Hausdorff convergence* if there are sequences of positive numbers $\{\varepsilon_i\}_{i \in \mathbb{N}}$ and Borel maps $\{\varphi_i : X_i \longrightarrow X\}_{i \in \mathbb{N}}$ such that $\lim_{i \to \infty} \varepsilon_i = 0$, $\varphi_i$ is an $\varepsilon_i$-approximating map, and that $(\varphi_i)_\sharp m_i$ weakly converges to $m$. Here a map $\varphi : Y \longrightarrow X$ is said to be *$\varepsilon$-approximating* if

$$\left| d_X\big(\varphi(y), \varphi(z)\big) - d_Y(y, z) \right| \le \varepsilon$$

holds for all $y, z \in Y$ and if $\overline{B}(\varphi(Y), \varepsilon) \supset X$. If we consider only distance structures $(X_i, d_i)$, $(X, d)$ and remove the weak convergence condition of $\varphi_i$, then it is the *Gromov-Hausdorff convergence* under which the lower sectional curvature bound in the sense of Alexandrov had been known to be preserved.

**Theorem 3.2** (Stability) *If a sequence of metric measure spaces $\{(X_i, d_i, m_i)\}_{i \in \mathbb{N}}$ uniformly satisfies $\mathsf{CD}(K,N)$ for some $K \in \mathbb{R}$ and $N \in (1, \infty]$, and converges to $(X, d, m)$ in the sense of measured Gromov-Hausdorff convergence, then $(X, d, m)$ also satisfies $\mathsf{CD}(K,N)$.*

The proof of stability goes as follows. First of all, we can restrict ourselves to measures with continuous density, for it implies by approximation the general case. Given continuous measures $\mu = \rho m, \nu = \sigma m \in \mathcal{P}(X)$, we consider

$$\mu_i = \frac{\rho \circ \varphi_i}{\int_{X_i} \rho \circ \varphi_i \, dm_i} \cdot m_i, \qquad \nu_i = \frac{\sigma \circ \varphi_i}{\int_{X_i} \sigma \circ \varphi_i \, dm_i} \cdot m_i$$

and take minimal geodesics $\alpha_i : [0,1] \longrightarrow \mathcal{P}(X_i)$ satisfying (3.2). Extracting a subsequence if necessary, $\alpha_i$ converges to a minimal geodesic $\alpha : [0,1] \longrightarrow \mathcal{P}(X)$ from $\mu$ to $\nu$. Then the right-hand side of (3.2) for $\mu_i, \nu_i$ converges to that for $\mu, \nu$ by virtue of the continuous densities, while the lower semi-continuity $U_m(\alpha(t)) \le \liminf_{i \to \infty} U_{m_i}(\alpha_i(t))$ holds in general. Thus we obtain (3.2) for $\alpha$.

**Further reading** Lott and Villani called the condition as in Definition 3.1 *$N$-Ricci curvature bounded from below by $K$* ([LV1], [LV2]). The term 'curvature-dimension condition' is used by Sturm ([St3], [St4]) following Bakry and Émery's celebrated work [BE]. Sturm independently introduced a similar condition that (3.1) holds for all absolutely continuous measures and $U = S_{N'}$ for all $N' \in [N, \infty]$. These conditions are equivalent in non-branching spaces (see §5). Both Lott-Villani and Sturm proved the stability in their own setting. See [Fu], [Gr] and [BBI] for the basics of (measured) Gromov-Hausdorff convergence. See also celebrated work of Cheeger and Colding [CC] for related geometric approach toward the investigation of limit spaces of Riemannian manifolds of Ricci curvature bounded below.

# 4 Geometric applications

Metric measure spaces satisfying the curvature-dimension condition $\mathsf{CD}(K,N)$ enjoy many properties common to '$N$-dimensional spaces of Ricci curvature $\ge K$'. Proofs based on optimal transport theory themselves are interesting. Here we concentrate on rather geometric applications.

Our first application is a generalization of the Brunn-Minkowski inequality (2.1), (2.2) to curved spaces. Given two sets $A, B \subset X$ and $t \in (0,1)$, we denote by $Z_t(A,B)$ the set of points $\gamma(t)$ such that $\gamma : [0,1] \longrightarrow X$ is a minimal geodesic with $\gamma(0) \in A$ and $\gamma(1) \in B$.

**Theorem 4.1** (Generalized Brunn-Minkowski inequalities) *Let a metric measure space $(X,d,m)$ satisfy $\mathsf{CD}(K,N)$ and take Borel sets $A, B \subset X$ with $0 < m(A), m(B) < \infty$.*

(i) *If $N < \infty$, then we have*

$$m\big(Z_t(A,B)\big)^{1/N} \ge (1-t) \inf_{x\in A,\, y\in B} \beta_{K,N}^{1-t}\big(d(x,y)\big)^{1/N} \cdot m(A)^{1/N} + t \inf_{x\in A,\, y\in B} \beta_{K,N}^{t}\big(d(x,y)\big)^{1/N} \cdot m(B)^{1/N}$$

*for all $t \in (0,1)$.*

(ii) *If $N = \infty$, then we have*

$$\log m\big(Z_t(A,B)\big) \ge (1-t)\log m(A) + t \log m(B) + \frac{K}{2}(1-t)t d_2^W\left(\frac{m|_A}{m(A)}, \frac{m|_B}{m(B)}\right)^2$$

*for all $t \in (0,1)$.*

These follow from (3.2) applied to $S_N$ or $\mathrm{Ent}_m$ between uniform distributions on $A$ and $B$. In the particular case of $K = 0$, we obtain the concavity of $m^{1/N}$ or $\log m$ as in (2.1), (2.2). Under $\mathsf{CD}(K,N)$ with $N < \infty$, applying (i) to thin annuli shows the Bishop-Gromov volume comparison (see (1.5))

$$\frac{m(B(x,R))}{m(B(x,r))} \le \frac{\int_0^R \mathbf{s}_{K,N}(t)^{N-1}\,dt}{\int_0^r \mathbf{s}_{K,N}(t)^{N-1}\,dt},$$

where $0 < r < R$ ($\le \pi\sqrt{(N-1)/K}$ if $K > 0$). In particular, we have the Bonnet-Myers diameter bound $\operatorname{diam} X \le \pi\sqrt{(N-1)/K}$ if $K > 0$.

Another interesting geometric application is the Lichnerowicz inequality. Spaces satisfying $\mathsf{CD}(K,\infty)$ with $K > 0$ are known to enjoy several functional inequalities, such as the Talagrand inequality, logarithmic Sobolev inequality and global Poincaré inequality. There are also applications in the concentration of measure phenomenon. Among these inequalities, the global Poincaré inequality

$$\int_X f^2\,dm \le \frac{1}{K}\int_X |\nabla^- f|^2\,dm$$

is improved under $\mathsf{CD}(K,N)$ with $N < \infty$.

**Theorem 4.2** (A generalized Lichnerowicz inequality) *Suppose that a metric measure space* $(X, d, m)$ *satisfies* $\mathsf{CD}(K, N)$ *for some* $K > 0$ *and* $N \in (1, \infty)$*. Then we have*

$$\int_X f^2 \, dm \leq \frac{N-1}{KN} \int_X |\nabla^- f|^2 \, dm \tag{4.1}$$

*for any Lipschitz function* $f : X \longrightarrow \mathbb{R}$ *with* $\int_X f \, dm = 0$.

Here $|\nabla^- f|$ is the generalized gradient of $f$ defined by

$$|\nabla^- f|(x) := \liminf_{y \to x} \frac{\max\{f(x) - f(y), 0\}}{d(x, y)}.$$

The proof is done via careful calculations using (3.2) for $S_N$ between $m(X)^{-1} \cdot m$ and its perturbation $(1 + \varepsilon f)m(X)^{-1} \cdot m$ for $\varepsilon \in (-1, 1)$. The inequality (4.1) means that the lowest positive eigenvalue of the Laplacian is larger than or equal to $KN/(N-1)$. The constant $(N-1)/KN$ in (4.1) is sharp. Moreover, in Riemannian geometry, it is known that this best constant (with $N = \dim M$) is achieved only for spheres. For metric measure spaces, we know only that the maximal diameter $\pi\sqrt{(N-1)/K}$ is achieved only for spherical suspensions ([Oh2]).

Among others, a challenging problem is to show (some appropriate variant of) the Lévy-Gromov isoperimetric inequality using optimal transport. Most known proofs appeal to the deep existence and regularity theory of minimal surfaces, and it can not be expected in singular spaces. For instance, let us consider the *isoperimetric profile* $I_M$ of a weighted Riemannian manifold $(M, g, m = e^{-\psi} \mathrm{vol}_g)$ with $m(M) < \infty$, i.e., $I_M(V)$ is the least perimeter of sets with volume $V$. Then the differential inequality

$$(I_M^{N/(N-1)})'' \leq -\frac{KN}{N-1} I_M^{1/(N-1)-1} \tag{4.2}$$

holds if $\mathrm{Ric}_N \geq K$. This immediately implies the corresponding *Lévy-Gromov isoperimetric inequality*

$$\frac{I_M(t \cdot m(M))}{m(M)} \geq I_{K,N}(t) \tag{4.3}$$

for $t \in [0, 1]$, where $I_{K,N}$ is the isoperimetric profile of the $N$-dimensional space form of constant sectional curvature $K/(N-1)$ equipped with the normalized measure (extended to non-integer $N$ numerically). The inequality (4.2) seems to be related to the Brunn-Minkowski inequality, however, known proof of (4.2) is based on the variational formula of minimal surfaces.

**Further reading** The generalized Brunn-Minkowski inequalities are established by von Renesse and Sturm [vRS] ($N = \infty$) and Sturm [St4] ($N < \infty$). Some more related interpolation inequalities can be found in [CMS]. They all were new even for Riemannian manifolds. The relation between $\mathsf{CD}(K, \infty)$ and the Talagrand, logarithmic Sobolev and global Poincaré inequalities are studied by Otto and Villani [OV] and Lott and Villani [LV1]. The latter duo also shows the generalized Lichnerowicz inequality ([LV2]). The differential inequality (4.2) is due to Bayle [Ba].

# 5 Related topics

If there is time, I would discuss some of the following topics.

## 5.1 Non-branching spaces

We say that a metric space $(X, d)$ is *non-branching* if geodesics do not branch, more precisely, if each quadruple of points $z, x_0, x_1, x_2 \in X$ with $d(x_0, x_1) = d(x_0, x_2) = 2d(z, x_i)$ $(i = 0, 1, 2)$ must satisfy $x_1 = x_2$. In such a space, a.e. $x \in X$ has unique minimal geodesic from $x$ to a.e. $y \in X$. Therefore we can localize the inequality (3.1), and then (3.1) for single $U = S_N$ implies that for all $U \in \mathcal{DC}_N$. Riemannian (or Finsler) manifolds and Alexandrov spaces are clearly non-branching. However, as $n$-dimensional Banach spaces satisfy $\mathsf{CD}(0, n)$, the curvature-dimension condition does not prevent the branching phenomenon. A big open problem after Cheeger and Colding's work is whether any limit space of Riemannian manifolds with a uniform lower Ricci curvature bound is non-branching or not.

## 5.2 Alexandrov spaces

As is briefly explained in §1, Alexandrov spaces are metric spaces whose sectional curvature is bounded from below by some constant. One interesting fact is that a compact geodesic space $(X, d)$ is an Alexandrov space of nonnegative curvature if and only if so is the Wasserstein space $(\mathcal{P}(X), d_2^W)$ over it ([St3], [LV1]). We remark that this relation can not be extended to positive or negative curvature bounds. Optimal transport in Alexandrov spaces is further studied in [Be] and [Oh3] (see also [Sa]). Since the Ricci curvature is the trace of the sectional curvature, it is natural to expect that Alexandrov spaces satisfy the curvature-dimension condition. Petrunin [Pe] recently asserts that it is indeed the case.

## 5.3 Finsler manifolds

The equivalence between $\mathrm{Ric}_N \geq K$ and $\mathsf{CD}(K, N)$ (Theorem 2.1) is extended to a general Finsler manifold $(M, F, m)$ with an arbitrary positive smooth measure $m$ (Ohta [Oh4]). Here $F$ is only positively homogeneous, so that the associated distance function may not be symmetric. The point is how to generalize the weighted Ricci curvature in this setting. Together with the stability, every $n$-dimensional Banach (or even Minkowski) space $(\mathbb{R}^n, \| \cdot \|)$ satisfies $\mathsf{CD}(0, n)$ for the Lebesgue measure $m = m_L$ (this was first demonstrated by Cordero-Erausquin [Vi2, Theorem in page 908]), and $\mathsf{CD}(0, \infty)$ for a Gaussian measure $m = e^{-\|\cdot\|^2/2} m_L$. These should be compared with Cheeger and Colding's observation that Banach spaces can not appear as the limit space of Riemannian manifolds with a uniform lower Ricci curvature bound. Then the situation is very different from Alexandrov spaces, it is not known whether there is an Alexandrov space which can not be approximated by Riemannian manifolds with a uniform lower sectional curvature bound (when we allow collapsing).

### 5.4 The measure contraction property

For $k \in \mathbb{R}$ and $N \in (1, \infty)$, a metric measure space is said to satisfy the *measure contraction property* $\mathsf{MCP}(K, N)$ if the Bishop inequality (1.4) holds in an appropriate sense (see Ohta [Oh1] and Sturm [St4] for the precise definition). $\mathsf{MCP}(K, N)$ can be regarded as the curvature-dimension condition $\mathsf{CD}(K, N)$ applied only for pairs of a Dirac measure and a uniform distribution on a set, and $\mathsf{CD}$ actually implies $\mathsf{MCP}$ in non-branching spaces. It is known that Alexandrov spaces satisfy $\mathsf{MCP}$ (see [Oh1] and [KS2] as well). For $n$-dimensional Riemannian manifolds, $\mathsf{MCP}(K, n)$ is equivalent to $\mathrm{Ric} \geq K$, however, $\mathsf{MCP}(K, N)$ with $N > n$ does not imply $\mathrm{Ric} \geq K$. This is one drawback of $\mathsf{MCP}$. An advantage of $\mathsf{MCP}$ is its simpleness. There are several facts known for $\mathsf{MCP}$, but unknown for $\mathsf{CD}$, such as the propagation of $\mathsf{MCP}$ to product spaces and cones ([Oh2]).

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