Conical Fronts and More General Curved Fronts for Homogeneous Equations in $\mathbb{R}^N$

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These notes are concerned with conical-shaped travelling fronts for homogeneous reaction-diffusion equations

$$u_t = \Delta u + f(u) \ \text{ in } \mathbb{R}^N.$$

Planar fronts are solutions of the type $u(t,x) = \phi(x\cdot e - ct)$, where the unit vector e is the direction of propagation, and c is the speed. The aim here is to show the existence of other fronts, with curved shapes, even in this homogeneous framework.

By considering the interaction of several planar fronts with different directions of propagation, we will see here how these planar fronts can give rise to more complex fronts with curved shapes. We will first be interested in conical-shaped fronts in combustion models or in Allen-Cahn equations. Then, for Fisher-KPP equations, we will point out the unexpected richness of the set of fronts with curved shapes.

In Section 1, we present a combustion model involving conical-shaped fronts. In Section 2, we prove some useful comparison principles and monotonicity results in unbounded domains. In Sections 3 to 6, we study the uniqueness, the qualitative properties, the existence, the stability of conical-shaped fronts for reaction-diffusion equations with combustion-type or bistable nonlinearities. Section 7 is concerned with more general curved fronts for KPP-type equations.

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1 An example of a conical-shaped front

A typical example of a conical front is the premixed Bunsen flame. A Bunsen flame can be divided into two parts : a diffusion flame and a premixed flame (*see* Figure 1, and [16], [17], [34], [38], [39], [50], [51], [55]). The premixed flame is itself divided into two zones : a fresh mixture (fuel and oxidant) and, above, a hot zone made of the burnt gases. For the sake of simplicity, we assume that a single global chemical reaction *fuel* + *oxidant* $\rightarrow$ *products* takes place in the mixture. The level sets of the temperature have a conical shape with a curved tip and, far away from its axis of symmetry, the flame is asymptotically almost planar. Let us assume that the flame is stabilized and stationary in an upward flow with a uniform intensity c. This uniformity assumption is reasonable at least far from the burner rim.

Because of the invariance of the shape of the flame with respect to the size of the Bunsen burner, the problem will be set in the whole space

$$\mathbb{R}^N = \{z = (x,y) \in \mathbb{R}^{N-1} \times \mathbb{R}\}.$$

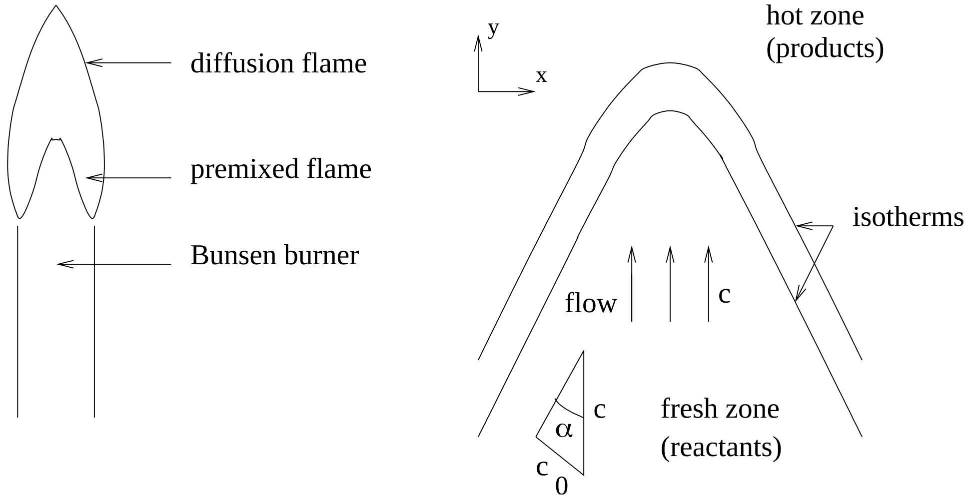

Figure 1: *Bunsen flames,* *premixed flame*

In the classical framework of the thermodiffusive model with unit Lewis number, the temperature field $u(x, y)$ satisfies the following reaction-diffusion equation :

$$\Delta u - c\frac{\partial u}{\partial y} + f(u) = 0 \tag{1.1}$$

and u is assumed to be normalized so that $0 \leq u \leq 1$ in $\mathbb{R}^N$.

The nonlinear reaction term $f(u)$ is of the "ignition temperature" type, namely f is assumed to be Lipschitz-continuous in $[0, 1]$, differentiable at 1, and

$$\exists \theta \in (0, 1), \quad f \equiv 0 \text{ on } [0, \theta] \cup \{1\}, \; f > 0 \text{ on } (\theta, 1) \text{ and } f'(1) < 0. \tag{1.2}$$

Such a profile can be derived from the Arrhenius kinetics with a cut-off for low temperatures and from the law of mass action. The real number θ is the ignition temperature, below which no reaction happens. For mathematical convenience, f is assumed to be extended by 0 outside the interval $[0, 1]$.

The temperature is low in the fresh mixture below the main reaction zone, and it is high above. The main mathematical difficulty is to translate the conical shape of the flame into some conditions on the function u. A reasonable solution is to impose conditions depending on the angle of the flame. More precisely, if $\alpha > 0$ denotes the angle of the flame (*see* Figure 1), asymptotic conical conditions like

$$\lim_{A \to -\infty} \sup_{y \leq A - |x| \cot \alpha} u(x, y) = 0, \quad \lim_{A \to +\infty} \inf_{y \geq A - |x| \cot \alpha} u(x, y) = 1 \tag{1.3}$$

can be imposed at infinity. Thus, the region where u is close to 0 corresponds to the fresh mixture and it is far below the conical surface of aperture α in the vertical direction, while the region where u is close to 1 corresponds to the burnt gases and is located far above this conical surface. In practice, the speed c of the flow at the exit of the Bunsen burner is given and it determines the angle α of the flame. We assume here that the angle α is given and the speed c is unknown. We shall see that these two formulations are equivalent.

It will turn out that these asymptotic conditions (1.3) are somehow too strong in dimensions $N \geq 3$. Several weaker conical conditions will be introduced in the sequel.

Conical fronts also arise in other contexts. For instance, such fronts, which are also called V-shaped fronts, arise in Allen-Cahn type equations (1.1) with a bistable nonlinearity. A bistable nonlinearity f on $[0,1]$ is a Lipchitz-continuous function which is differentiable at 0 and 1, and such that

$$\begin{array}{l}\exists\theta \in (0,1),\ f<0 \text{ on } (0,\theta),\ f>0 \text{ on } (\theta,1), f(0)=f(\theta)=f(1)=0\\ \qquad \text{and } f'(0)<0,\ f'(1)<0,\ f'(\theta)>0.\end{array} \tag{1.4}$$

The function f is then assumed to be differentiable at θ as well. A typical example is the cubic nonlinearity $f(s)=s(1-s)(s-\theta)$.

We will see in the course of the notes that some common properties will be valid for both the combustion model of conical premixed flame and for the Allen-Chan model for V-shaped fronts, and even for other models with more general nonlinearities f. We will present the results in a unified way.

One points out that the solutions $u(x,y)$ of (1.1) can also be viewed as travelling fronts of the type

$$v(t,x,y)=u(x,y+ct)$$

moving downwards with speed c in a quiescent medium. The function v solves the parabolic reaction-diffusion equation

$$v_t=\Delta v+f(v) \ \text{ in } \mathbb{R}^N. \tag{1.5}$$

In dimension 1, problem (1.1), (1.3) reduces to the equation

$$u''-cu'+f(u)=0, \quad u(-\infty)=0,\ u(+\infty)=1. \tag{1.6}$$

We will use some basic facts about this problem. For instance, if f satisfies (1.2) or (1.4), then there is a unique solution $(c_0,u_0)=(c(f),u(f))$, which depends on f only. Furthermore, the function $u_0=u(f)$ is increasing and unique up to translation, and the speed $c_0=c(f)$ has the sign of $\displaystyle\int_0^1 f(s)ds$ ([2], [5], [9], [35]). These results can be obtained by a shooting method or a study in the phase plane. The above existence, uniqueness and monotonicity results have been generalized by Berestycki, Larrouturou, Lions [7] and Berestycki, Nirenberg [11] in the multidimensional case of a straight infinite cylinder $\Sigma=\omega\times\mathbb{R}=\{z=(x,y),\ x\in\omega,\ y\in\mathbb{R}\}$, for equations of the type

$$\left\{\begin{array}{rcl}\Delta u-(c+\beta(x))\partial_y u+f(u) & = & 0 \ \text{ in } \Sigma=\omega\times\mathbb{R}\\ \partial_\nu u & = & 0 \ \text{ on } \partial\Sigma\\ u(\cdot,-\infty) \ = \ 0, \quad u(\cdot,+\infty) & = & 1,\end{array}\right. \tag{1.7}$$

where β is a given continuous function defined on the bounded and smooth section $\overline{\omega}$ of the cylinder, and $\partial_\nu u$ denotes the partial derivative of u with respect to the outward unit normal ν on $\partial\Sigma$. Under the above conditions, there exists a unique solution (c,u) of (1.7), and the function $u=u(x,y)$ is increasing in y and unique up to translation in y. Variational formulas for the unique speed exist in the one-dimensional case [25] and in the multidimensional case [26], [33].

Recently, generalizations of the above results have been obtained for *pulsating fronts* in periodic domains and media with periodic coefficients by Berestycki and Hamel [4] and Xin [53], [54].

Let us now come back to problem (1.1) with *conical* conditions (1.3). Note that, although the underlying flow is here uniform, the solutions are nevertheless non-planar, because of the conical conditions, such as (1.3), which are imposed at infinity. Formal analyses had been done, especially using asymptotic expansions in some singular limits. The mathematical difficulties come on the one hand from the fact that the problem is set in the whole space $\mathbb{R}^N$ and on the other hand from the *non-standard* conical conditions at infinity. These conditions are rather weak and do not a priori impose anything about the behavior of the function u in the directions making an angle α with respect to the unit vector $-e_N = (0, \cdots, 0, -1)$.

We here want to establish some existence or uniqueness results for this problem by using PDE methods. In the theory of Bunsen flames, one of the tasks is to establish a relationship between the speed c of the outgoing flow and the angle α of the flame.

In Section 3, we first prove several qualititave properties for the solutions of (1.1) satisfying some conical conditions at infinity, with combustion-type or bistable-type nonlinearities. Some of the conditions will be weaker than (1.3). We especially prove the uniqueness of the speed, given the angle α, the relationship between α and c, some monotonicity properties in cones of directions and some uniqueness results under conditions (1.3). Most of these results rely on some versions of the maximum principle which are established in Section 2. We then give some existence results in the case of combustion or bistable nonlinearities and we discuss the stability of these conical fronts. Lastly, Section 7 is concerned with the case of a KPP-type nonlinearity f. The set of conical fronts turns out to be much richer than for combustion or bistable-type nonlinearities, and more general curved fronts will be constructed.

2 Maximum principles for elliptic and parabolic problems on unbounded domains

In this section, we give some generalizations of the weak maximum principle for elliptic or time-global parabolic equations in domains which are unbounded in the space variables. We then apply these comparison principles to prove monotonicity results for solutions of elliptic or parabolic equations in cylindrical domains.

We will use some notations and assumptions throughout this section. Let Ω be an open connected subset of $\mathbb{R}^N$. We define

$$Pu(t,x) := \partial_t u - a_{ij}(t,x)\partial_{ij}u - b_i(t,x)\partial_i u - f(t,x,u)$$

under the usual summation convention for repeated indices, where the coefficients a_{ij}, b_j are of class $L^\infty \cap C^{0,\alpha}(\mathbb{R}\times\overline{\Omega})$ (with $\alpha > 0$) and there exists $c_0 > 0$ such that

$$a_{ij}(t,x)\xi_i\xi_j \geq c_0|\xi|^2 \text{ for all } \xi \in \mathbb{R}^N \text{ and } (t,x) \in \mathbb{R}\times\overline{\Omega}. \tag{2.1}$$

We denote $\partial_t u = \frac{\partial u}{\partial t} = u_t$, $\partial_i u = \frac{\partial u}{\partial x_i} = u_{x_i}$, $\partial_{ij} u = \frac{\partial^2 u}{\partial x_i x_j} = u_{x_i x_j}$. We assume that, for each $M \geq 0$, there exists $C_M \geq 0$ such that

$$|f(t,x,s) - f(t,x,s')| \leq C_M\ |s - s'| \text{ for all } (t,x) \in \mathbb{R}\times\overline{\Omega} \text{ and } s,\ s' \in [-M, M]. \tag{2.2}$$

These assumptions are made troughout this section.

Theorem 2.1 *Let $\underline{u}(t,x)$ and $\overline{u}(t,x)$ be two bounded uniformly continuous functions defined in $\mathbb{R}\times\overline{\Omega}$, such that the partial derivatives $\partial_t\underline{u}$, $\partial_t\overline{u}$, $\partial_i\underline{u}$, $\partial_i\overline{u}$, $\partial_{ij}\underline{u}$, $\partial_{ij}\overline{u}$ exist and are of class $C^{0,\alpha}(\mathbb{R}\times\overline{\Omega})$. Assume that*

$$\begin{array}{rcll} P\underline{u} & \leq & P\overline{u} & \text{in } \mathbb{R}\times\Omega, \\ \underline{u} & \leq & \overline{u} & \text{on } \mathbb{R}\times\partial\Omega, \end{array}$$

and

$$\limsup_{t\in\mathbb{R},\ x\in\Omega,\ dist(x,\partial\Omega)\to+\infty} \big(\underline{u}(t,x)-\overline{u}(t,x)\big)\le 0.$$

Lastly, we assume that, for all $(t,x)\in\mathbb{R}\times\overline{\Omega}$, $f(t,x,s)$ is nonincreasing in s for $s\in(-\infty,\sup\underline{u}]$.

Then

$$\underline{u}\le\overline{u}\quad in\quad \mathbb{R}\times\overline{\Omega}.$$

Remark 2.2 *In the case where all coefficients a_{ij}, b_i, f and the functions $\underline{u}$, $\overline{u}$ do not depend on t, Theorem 2.1 can then be viewed as a weak elliptic maximum principle in the set $\Omega\subset\mathbb{R}^N$.*

Proof. Denote

$$\underline{u}_\varepsilon=\underline{u}-\varepsilon$$

for any $\varepsilon>0$. Since both $\underline{u}$ and $\overline{u}$ are bounded, one has $\underline{u}_\varepsilon\le\overline{u}$ in $\mathbb{R}\times\overline{\Omega}$ for $\varepsilon>0$ large enough. Let us set

$$\varepsilon^*=\inf\{\varepsilon>0,\ \underline{u}_\varepsilon\le\overline{u}\text{ in }\mathbb{R}\times\overline{\Omega}\}.$$

We have $\underline{u}_{\varepsilon^*}\le\overline{u}$ and our goal is to prove now that $\varepsilon^*=0$.

Assume by contradiction that $\varepsilon^*>0$. We can then find a sequence of positive numbers $(\varepsilon_k)_{k\in\mathbb{N}}$ such that $\varepsilon_k\nearrow\varepsilon^*$ and a sequence of points $(t_k,x_k)\in\mathbb{R}\times\overline{\Omega}$ such that

$$\underline{u}_{\varepsilon_k}(t_k,x_k)=\underline{u}(t_k,x_k)-\varepsilon_k>\overline{u}(t_k,x_k)\ \text{ for all } k\in\mathbb{N}. \tag{2.3}$$

Since $\limsup_{t\in\mathbb{R},\ dist(x,\partial\Omega)\to+\infty}\big(\underline{u}(t,x)-\overline{u}(t,x)\big)\le 0$ and $\varepsilon^*>0$, the sequence $(dist(x_k,\partial\Omega))_{k\in\mathbb{N}}$ is bounded. Furthermore, since $\underline{u}\le\overline{u}$ on $\mathbb{R}\times\partial\Omega$ and $\underline{u}$, $\overline{u}$ are uniformly continuous, one has

$$\liminf_{k\to+\infty}\ dist(x_k,\partial\Omega)>0.$$

For each k, let y_k be a point on $\partial\Omega$ such that

$$|y_k-x_k|=dist(x_k,\partial\Omega).$$

Up to extraction of some subsequence, one can then assume that $y_k-x_k\to y$ as $k\to+\infty$, with $|y|=R>0$. Call B_R the open ball of $\mathbb{R}^N$ with centre 0 and radius R.

For each k, call

$$\underline{u}^k(t,x)=\underline{u}(t+t_k,x+x_k),\quad\text{and } \overline{u}^k(t,x)=\overline{u}(t+t_k,x+x_k).$$

Since the functions $\underline{u}$ and $\overline{u}$ are assumed to be uniformly continuous in $\mathbb{R}\times\overline{\Omega}$, of class $C^{1,\alpha}$ in t and $C^{2,\alpha}$ in x, in $\mathbb{R}\times\overline{\Omega}$, it follows that, up to extraction of some subsequence,

$$\underline{u}^k\to\underline{U}\text{ and }\overline{u}^k\to\overline{U}\text{ in }\mathbb{R}\times B_R\text{ as }k\to+\infty,\ \text{locally uniformly,}$$

and in C^1_{loc} in t and C^2_{loc} in x. Furthermore, $\underline{U}$ and $\overline{U}$ are still uniformly continuous in $\mathbb{R}\times B_R$ and can then be extended by continuity on $\mathbb{R}\times\partial B_R$. By uniform continuity of $\underline{u}$ and $\overline{u}$, and since $\underline{u}\le\overline{u}$ on $\mathbb{R}\times\partial\Omega$, one then gets that

$$\underline{U}(t,y)\le\overline{U}(t,y)\ \text{ for all } t\in\mathbb{R}. \tag{2.4}$$

Furthermore,

$$\underline{U}-\varepsilon^*\le\overline{U}\text{ in }\mathbb{R}\times\overline{B_R},$$

and passing to the limit in (2.3) yields $\underline{U}(0,0)-\varepsilon^*\geq\overline{U}(0,0)$. Thus,

$$\underline{U}(0,0)-\varepsilon^*=\overline{U}(0,0).$$

On the other hand, up to extraction of some subsequence, the functions $a^k_{ij}(t,x)=a_{ij}(t+t_k,x+x_k)$ and $b^k_i(t,x)=b_i(t+t_k,x+x_k)$ converge locally uniformly in $\mathbb{R}\times B_R$ to some continuous functions A_{ij} and B_i such that $A_{ij}(t,x)\xi_i\xi_j\geq c_0|\xi|^2$ for all $\xi\in\mathbb{R}^N$ and $(t,x)\in\mathbb{R}\times B_R$.

Lastly, one has

$$\begin{aligned}\partial_t\overline{u}^k-a^k_{ij}\partial_{ij}\overline{u}^k-b^k_i\partial_i\overline{u}^k-f(t+t_k,x+x_k,\overline{u}^k) &\geq \partial_t\underline{u}^k-a^k_{ij}\partial_{ij}\underline{u}^k-b^k_i\partial_i\underline{u}^k\\ &\qquad -f(t+t_k,x+x_k,\underline{u}^k)\\ &\geq \partial_t(\underline{u}^k-\varepsilon^*)-a^k_{ij}\partial_{ij}(\underline{u}^k-\varepsilon^*)-b^k_i\partial_i(\underline{u}^k-\varepsilon^*)\\ &\qquad -f(t+t_k,x+x_k,\underline{u}^k-\varepsilon^*)\end{aligned}$$

because $P\overline{u}\geq P\underline{u}$ and $f(t,x,s)$ is nonincreasing in $s\in(-\infty,\sup\underline{u}]$ for all $(t,x)\in\mathbb{R}\times\overline{\Omega}$. But since $\underline{u}$ and $\overline{u}$ are globally bounded and f is locally Lipschitz-continuous in s uniformly in (t,x), there exists then a constant $C\geq 0$ such that

$$\partial_t z^k-a^k_{ij}\partial_{ij}z^k-b^k_i\partial_i z^k+Cz^k\geq 0,$$

where

$$z^k=\overline{u}^k-\underline{u}^k+\varepsilon^*.$$

By passing to the limit as $k\to+\infty$ locally uniformly in $\mathbb{R}\times B_R$, it follows that

$$\partial_t z-A_{ij}\partial_{ij}z-B_i\partial_i z+Cz\geq 0\ \text{ in }\mathbb{R}\times B_R,$$

where $z=\overline{U}-\underline{U}+\varepsilon^*$. But z is continuous and nonnegative in $\mathbb{R}\times\overline{B_R}$ and $z(0,0)=0$. The strong maximum principle then implies that $z(t,x)=0$, namely $\overline{U}(t,x)=\underline{U}(t,x)-\varepsilon^*$ for all $t\leq 0$ and $x\in\overline{B_R}$. One gets a contradiction with (2.4) by choosing $x=y$ $(\in\partial B_R)$.

Therefore, $\varepsilon^*=0$ and the proof of Theorem 2.1 is complete. □

The next result is a variation of Theorem 2.1, for equations with Neumann type boundary conditions on parts of semi-infinite cylinders.

Theorem 2.3 *Assume here that Ω is a semi-infinite cylindrical domain*

$$\Omega=\omega\times(0,+\infty)=\{x=(x',x_N),\ x'=(x_1,\ldots,x_{N-1})\in\omega,\ x_N>0\},$$

where ω is a bounded open connected subset of $\mathbb{R}^{N-1}$, of class C^1, with outward unit normal denoted by ν.

Let $\underline{u}(t,x)=\underline{u}(t,x',x_N)$ and $\overline{u}(t,x)=\overline{u}(t,x',x_N)$ be two bounded uniformly continuous functions defined in $\mathbb{R}\times\overline{\Omega}=\mathbb{R}\times\overline{\omega}\times[0,+\infty)$, such that the partial derivatives $\partial_t\underline{u}$, $\partial_t\overline{u}$, $\partial_i\underline{u}$, $\partial_i\overline{u}$, $\partial_{ij}\underline{u}$, $\partial_{ij}\overline{u}$ exist and are of class $C^{0,\alpha}(\mathbb{R}\times\overline{\Omega})$. Assume that

$$P\underline{u}\leq P\overline{u}\ \ \textit{in }\mathbb{R}\times\Omega,$$

$$\underline{u}(t,x',0)\leq\overline{u}(t,x',0)\ \ \textit{for all }(t,x')\in\mathbb{R}\times\overline{\omega},$$

and

$$\limsup_{t\in\mathbb{R},\ x'\in\overline{\omega},\ x_N\to+\infty}\ \big(\underline{u}(t,x',x_N)-\overline{u}(t,x',x_N)\big)\leq 0.$$

Assume also that

$$\mu(x') \cdot \nabla_{x'} \underline{u}(t, x', x_N) \le \mu(x') \cdot \nabla_{x'} \overline{u}(t, x', x_N) \text{ for all } (t, x', x_N) \in \mathbb{R} \times \partial\omega \times (0, +\infty),$$

where $x' \mapsto \mu(x') \in \mathbb{R}^{N-1}$ is a continuous unit vector field defined in $\partial\omega$ such that $\mu(x') \cdot \nu(x') > 0$ on $\partial\omega$. Lastly, we assume that, for all $(t, x', x_N) \in \mathbb{R} \times \overline{\omega} \times [0, +\infty)$, $f(t, x', x_N, s)$ is nonincreasing in s for $s \in (-\infty, \sup \underline{u}]$.

Then

$$\underline{u}(t, x', x_N) \le \overline{u}(t, x', x_N) \quad \text{for all } (t, x', x_N) \in \mathbb{R} \times \overline{\omega} \times [0, +\infty).$$

Proof. We argue as in Theorem 2.1 : we define ε^*, assume $\varepsilon^* > 0$ and define (ε_k), $(t_k, x'_k, x_{N,k})$ as in Theorem 2.1, namely

$$\underline{u}_{\varepsilon_k}(t_k, x'_k, x_{N,k}) = \underline{u}(t_k, x'_k, x_{N,k}) - \varepsilon_k > \overline{u}(t_k, x'_k, x_{N,k}) \text{ for all } k \in \mathbb{N}. \tag{2.5}$$

Since $\limsup_{t\in\mathbb{R},\ x'\in\overline{\omega},\ x_N\to+\infty} \underline{u}(t, x', x_N) - \overline{u}(t, x', x_N) \le 0$, since $\underline{u}(t, x', 0) \le \overline{u}(t, x', 0)$ for all $(t, x') \in \mathbb{R} \times \overline{\omega}$, and since $\underline{u}$ and $\overline{u}$ are uniformly continuous, it follows that the sequence $(x_{N,k})_{k\in\mathbb{N}}$ is bounded from above and below by two positive constant. Up to extraction of some subsequence, one can then assume that

$$(x'_k, x_{N,k}) \to x_\infty = (x'_\infty, x_{N,\infty}) \in \overline{\omega} \times (0, +\infty) \text{ as } k \to +\infty.$$

Up to extraction of some subsequence, the functions

$$\underline{u}^k(t, x', x_N) = \underline{u}(t + t_k, x', x_N) \text{ and } \overline{u}^k(t, x', x_N) = \overline{u}(t + t_k, x', x_N)$$

converge locally uniformly in $\mathbb{R} \times \overline{\Omega}$, as well as in C^1_{loc} in t and C^2_{loc} in $x = (x', x_N)$, to two uniformly continuous functions $\underline{U}$ and $\overline{U}$ defined in $\mathbb{R} \times \overline{\Omega}$, such that

$$\underline{U} - \varepsilon^* \le \overline{U} \text{ in } \mathbb{R} \times \overline{\Omega},$$

$$\underline{U}(t, x', 0) \le \overline{U}(t, x', 0) \text{ for all } (t, x') \in \mathbb{R} \times \overline{\omega},$$

and $\underline{U}(0, x'_\infty, x_{N,\infty}) - \varepsilon^* \ge \overline{U}(0, x'_\infty, x_{N,\infty})$. Thus,

$$\underline{U}(0, x'_\infty, x_{N,\infty}) - \varepsilon^* = \overline{U}(0, x'_\infty, x_{N,\infty}).$$

Furthermore, $\mu(x') \cdot \nabla_{x'} \underline{U}(t, x', x_N) \le \mu(x') \cdot \nabla_{x'} \overline{U}(t, x', x_N)$ for all $(t, x', x_N) \in \mathbb{R} \times \partial\omega \times (0, +\infty)$. Lastly, as in the proof of Theorem 2.1, there exists a constant C such that

$$\partial_t z - A_{ij}\partial_{ij} z - B_i \partial_i z + Cz \ge 0 \quad \text{in } \mathbb{R} \times \omega \times (0, +\infty)$$

where $z = \overline{U} - \underline{U} + \varepsilon^*$. The functions A_{ij}, B_i are the uniformly local limits in $\mathbb{R} \times \overline{\Omega}$ (up to extraction of some subsequence) of $a_{ij}(t + t_k, x)$ and $b_i(t + t_k, x)$. The function z is nonnegative in $\mathbb{R} \times \overline{\Omega}$, it vanishes at $(0, x'_\infty, x_{N,\infty})$, with $x_{N,\infty} > 0$. The strong maximum priniciple then implies that either $z > 0$ in $\mathbb{R} \times \omega \times (0, +\infty)$, or $z \equiv 0$ in $(-\infty, t_0] \times \overline{\omega} \times [0, +\infty)$ for some $t_0 \in \mathbb{R}$. The latter case is impossible because $z(t, x', 0) \ge \varepsilon^* > 0$ for all $(t, x) \in \mathbb{R} \times \overline{\omega}$. Therefore, only the first case may occur, and then $x'_\infty \in \partial\omega$. But $\mu(x'_\infty) \cdot \nabla_{x'} z(0, x'_\infty, x_{N,\infty}) \ge 0$. The Hopf lemma yields then that

$$z(t, x', x_N) = 0 \text{ for all } (t, x', x_N) \in (-\infty, 0] \times \overline{\omega} \times [0, +\infty).$$

This is again ruled out because of the conditions on $\mathbb{R} \times \overline{\omega} \times \{0\}$.

One has then reached a contradiction. Therefore, $\varepsilon^* = 0$ and the proof of Theorem 2.3 is complete. □

The following theorem is a comparison principle, up to translation, between sub- and supersolutions defined in infinite cylindrical domains.

Theorem 2.4 *Assume here that Ω is an infinite cylindrical domain*

$$\Omega = \{x = (x', x_N),\ x' = (x_1, \ldots, x_{N-1}) \in \omega,\ x_N \in \mathbb{R}\},$$

where ω is either $\mathbb{R}^{N-1}$ or a bounded open connected subset of $\mathbb{R}^{N-1}$, of class C^1 with outward unit normal ν. Assume that all coefficients a_{ij}, b_i and f do not depend on the variable x_N. Let $\phi:\ \overline{\omega} \to \mathbb{R}$ be a continuous function.

Let $\underline{u}(t,x) = \underline{u}(t,x',x_N)$ and $\overline{u}(t,x) = \overline{u}(t,x',x_N)$ be two bounded uniformly continuous functions defined in $\mathbb{R}\times\overline{\Omega} = \mathbb{R}\times\overline{\omega}\times\mathbb{R}$, such that the partial derivatives $\partial_t\underline{u}$, $\partial_t\overline{u}$, $\partial_i\underline{u}$, $\partial_i\overline{u}$, $\partial_{ij}\underline{u}$, $\partial_{ij}\overline{u}$ exist and are of class $C^{0,\alpha}(\mathbb{R}\times\overline{\Omega})$. Assume that

$$P\underline{u} \le P\overline{u} \quad \text{in } \mathbb{R}\times\Omega,$$

$$\limsup_{t\in\mathbb{R},\ x'\in\overline{\omega},\ |x_N-\phi(x')|\to+\infty} \ \underline{u}(t,x',x_N) - \overline{u}(t,x',x_N) \ \le 0 \tag{2.6}$$

and there exist $a < b \in \mathbb{R}$ such that

$$\left\{\begin{array}{ll} \displaystyle\limsup_{t\in\mathbb{R},\ x'\in\overline{\omega},\ x_N-\phi(x')\to-\infty} |\underline{u}(t,x',x_N) - a| & = 0 \\ \displaystyle\limsup_{t\in\mathbb{R},\ x'\in\overline{\omega},\ x_N-\phi(x')\to+\infty} |\overline{u}(t,x',x_N) - b| & = 0. \end{array}\right. \tag{2.7}$$

If ω is bounded, one also assumes that

$$\mu(x')\cdot\nabla_{x'}\underline{u}(t,x',x_N) \le \mu(x')\cdot\nabla_{x'}\overline{u}(t,x',x_N) \text{ for all } (t,x',x_N) \in \mathbb{R}\times\partial\omega\times\mathbb{R},$$

where $x' \mapsto \mu(x') \in \mathbb{R}^{N-1}$ is a continuous unit vector field defined in $\partial\omega$ such that $\mu(x')\cdot\nu(x') > 0$ on $\partial\omega$. Lastly, one assumes the existence of $\delta > 0$ such that, for all $(t,x') \in \mathbb{R}\times\overline{\omega}$, $f(t,x',s)$ is nonincreasing in s for all $s \in (-\infty, a+\delta]$, and for all $s \in [b-\delta, +\infty)$.

Then the set

$$I = \{\tau \in \mathbb{R},\ \forall s \ge \tau,\ \overline{u}(t,x',x_N+s) \ge \underline{u}(t,x',x_N) \text{ for all } (t,x',x_N) \text{ in } \mathbb{R}\times\overline{\omega}\times\mathbb{R}\}$$

is not empty. Furthermore, if $\tau^* := \inf I > -\infty$, then

$$\forall\ y \in \mathbb{R}, \quad \inf_{t\in\mathbb{R},\ x'\in\overline{\omega}} \overline{u}(t,x',\phi(x')+\tau^*+y) - \underline{u}(t,x',\phi(x')+y) \ = \ 0. \tag{2.8}$$

Proof. Throughout the proof, we will use the notations

$$\left\{\begin{array}{lcl} \Omega^+(y) & = & \{(x',x_N)\in\Omega,\ x_N > y+\phi(x'),\ x'\in\omega\} \\ \Omega^-(y) & = & \{(x',x_N)\in\Omega,\ x_N < y+\phi(x'),\ x'\in\omega\} \\ \Gamma(y) & = & \{(x',x_N)\in\Omega,\ x_N = y+\phi(x'),\ x'\in\omega\} \end{array}\right.$$

and $w^y(t,x',x_N) = w(t,x',x_N+y)$ for any function w defined in $\mathbb{R}\times\overline{\Omega}$ and for any $y\in\mathbb{R}$.

Under the assumptions of Theorem 2.4, there exists $A \ge 0$ such that

$$\overline{u} \ge b - \frac{\delta}{2} \text{ in } \mathbb{R}\times\overline{\Omega^+(A)}$$

and

$$\underline{u} \le a+\delta \text{ in } \mathbb{R}\times\overline{\Omega^-(-A)}.$$

Even if it means decreasing δ, one can assume without loss of generality that $\delta < (b-a)/2$.

Consider first the case where $\omega = \mathbb{R}^{N-1}$. Choose any $y \geq 2A$. One especially has that

$$\underline{u} \leq \overline{u}^y \text{ in } \mathbb{R} \times \Gamma(-A).$$

It is then straightforward to check that all assumptions of Theorem 2.1 are satisfied with the domain $\Omega^-(-A)$ and the functions $\underline{u}$ and $\overline{u}^y$: notice indeed that $P\overline{u}^y = P\overline{u}$ because of the invariance of the coefficients a_{ij}, b_i and f with respect to the variable x_N, and that

$$\limsup_{t\in\mathbb{R},\ x\in\Omega^-(-A),\ dist(x,\partial\Omega^-(-A))\to+\infty} \underline{u}(t,x) - \overline{u}^y(t,x) \ \leq\ 0$$

because of (2.6) and (2.7). Therefore,

$$\underline{u} \leq \overline{u}^y \text{ in } \mathbb{R} \times \overline{\Omega^-(-A)}.$$

Similarly, Theorem 2.1 can be applied to the domain $\Omega^+(-A)$, with the functions $\underline{U} = -\overline{u}^y$, $\overline{U} = -\underline{u}$ and the nonlinearity $g(t,x',s) = -f(t,x',-s)$. Indeed, notice that $\underline{U} \leq -b+\delta$ in $\mathbb{R} \times \overline{\Omega^+(-A)}$ and g is nonincreasing in s for $s \in (-\infty, -b+\delta]$. Therefore, $\underline{U} \leq \overline{U}$ in $\mathbb{R} \times \overline{\Omega^+(-A)}$, namely

$$\underline{u} \leq \overline{u}^y \text{ in } \mathbb{R} \times \overline{\Omega^+(-A)}.$$

To sum up,

$$\underline{u} \leq \overline{u}^y \text{ in } \mathbb{R} \times \overline{\Omega} = \mathbb{R} \times \mathbb{R}^N \text{ for all } y \geq 2A.$$

Thus, $I \neq \emptyset$ and $\tau^* = \inf I \leq 2A$.

Assume now that $\tau^* > -\infty$. Then, $\underline{u} \leq \overline{u}^{\tau^*}$ in $\mathbb{R} \times \mathbb{R}^N$. It only remains to prove (2.8). Let us first show that

$$\inf_{t\in\mathbb{R},\ x'\in\mathbb{R}^{N-1},\ -A\leq x_N-\phi(x')\leq A-\tau^*} \overline{u}^{\tau^*}(t,x',x_N) - \underline{u}(t,x',x_N) \ =\ 0. \tag{2.9}$$

Assume not. Since the functions $\underline{u}$ and $\overline{u}$ are uniformly continuous, there exists then $\eta_0 > 0$ such that

$$\overline{u}^{\tau^*-\eta}(t,x',x_N) \geq \underline{u}(t,x',x_N) \text{ for all } (t,x') \in \mathbb{R} \times \mathbb{R}^{N-1},\ -A \leq x_N - \phi(x') \leq A - \tau^* \tag{2.10}$$

and for all $\eta \in [0,\eta_0]$. Since $\overline{u}^{\tau^*} \geq b - \delta/2$ in $\mathbb{R} \times \overline{\Omega^+(A-\tau^*)}$, one can also assume that η_0 is small enough so that $\overline{u}^{\tau^*-\eta} \geq b-\delta$ in $\mathbb{R}\times\overline{\Omega^+(A-\tau^*)}$ for all $\eta \in [0,\eta_0]$. As in the first step of the proof of this theorem, one can then apply Theorem 2.1 to $\Omega^-(-A)$ and $\Omega^+(A-\tau^*)$. Together with (2.10), one finally gets that

$$\overline{u}^{\tau^*-\eta} \geq \underline{u} \text{ in } \mathbb{R} \times \mathbb{R}^N \text{ for all } \eta \in [0,\eta_0].$$

That contradicts the minimality of τ^*, and eventually the claim (2.9) is proved.

Because of (2.9), there exists then a sequence of points $(t_k, x'_k, x_{N,k})$ such that

$$\overline{u}^{\tau^*}(t_k, x'_k, x_{N,k}) - \underline{u}(t_k, x'_k, x_{N,k}) \to 0 \text{ as } k \to +\infty,$$

with $-A \leq x_{N,k} - \phi(x'_k) \leq A - \tau^*$. Up to extraction of some subsequence, the functions

$$z^k(t,x',x_N) = \overline{u}^{\tau^*}(t+t_k, x'+x'_k, x_N+x_{N,k}) - \underline{u}(t+t_k, x'+x'_k, x_N+x_{N,k})$$

converge locally uniformly in $\mathbb{R}\times\mathbb{R}^N$, and in C^1_{loc} in t and C^2_{loc} in x, to a nonnegative function z such that $z(0,0,0)=0$ and

$$\partial_t z - A_{ij}(t,x')\partial_{ij}z - B_i(t,x')\partial_i z - Cz \geq 0 \ \text{ in } \mathbb{R}\times\mathbb{R}^N,$$

where C is a constant, A_{ij}, B_i are bounded, and $A_{ij}(t,x')\xi_i\xi_j \geq c_0|\xi|^2$ for all $\xi\in\mathbb{R}^N$ and $(t,x')\in\mathbb{R}\times\mathbb{R}^{N-1}$. The maximum principle then implies that $z(t,x',x_N)=0$ for all $t\leq 0$ and $(x',x_N)\in\mathbb{R}^N$. Up to extraction of some subsequence, one can also assume that

$$x_{N,k}-\phi(x'_k)\to l\in\mathbb{R} \text{ as } k\to+\infty.$$

Now, if y is any real number, one has that

$$\overline{u}(t_k,x'_k,\phi(x'_k)+\tau^*+y)-\underline{u}(t_k,x'_k,\phi(x'_k)+y)=z^k(0,0,y+\phi(x'_k)-x_{N,k})\to z(0,0,y+l)=0$$

as $k\to+\infty$. That completes the proof of Theorem 2.4 in the case where $\omega=\mathbb{R}^{N-1}$.

Consider now the case where ω is bounded. The continuous function ϕ is then bounded in $\overline{\omega}$. Therefore, there exists $A\geq 0$ such that

$$\overline{u}\geq b-\frac{\delta}{2} \text{ in } \mathbb{R}\times\overline{\omega}\times[A,+\infty)$$

and

$$\underline{u}\leq a+\delta \text{ in } \mathbb{R}\times\overline{\omega}\times(-\infty,-A].$$

One can assume without loss of generality that $\delta<(b-a)/2$. Choose any $y\geq 2A$. One especially has that $\underline{u}\leq\overline{u}^y$ in $\mathbb{R}\times\overline{\omega}\times\{-A\}$. It is then straightforward to check that all assumptions of Theorem 2.3 are satisfied with the functions $\underline{U}=-\overline{u}^{y-A}$ and $\overline{U}=-\underline{u}^{-A}$ and with the nonlinearity $g(t,x',s)=-f(t,x',-s)$. Therefore,

$$\underline{U}\leq\overline{U} \text{ in } \mathbb{R}\times\overline{\omega}\times[0,+\infty),$$

that is

$$\underline{u}\leq\overline{u}^y \text{ in } \mathbb{R}\times\overline{\omega}\times[-A,+\infty).$$

Similarly, Theorem 2.3 can be applied with the functions $\underline{U}(t,x',x_N)=\underline{u}(t,x',-x_N-A)$, $\overline{U}(t,x',x_N)=\overline{u}^y(t,x',-x_N-A)$, up to a change of sign in the coefficients a_{iN} for $i\neq N$ and b_N. As a consequence, $\underline{u}\leq\overline{u}^y$ in $\mathbb{R}\times\overline{\omega}\times(-\infty,-A]$. To sum up,

$$\underline{u}\leq\overline{u}^y \text{ in } \mathbb{R}\times\overline{\Omega}=\mathbb{R}\times\overline{\omega}\times\mathbb{R} \text{ for all } y\geq 2A.$$

Thus, $I\neq\emptyset$ and $\tau^*=\inf I\leq 2A$.

Assume now that $\tau^*>-\infty$. Then, $\underline{u}\leq\overline{u}^{\tau^*}$ in $\mathbb{R}\times\overline{\omega}\times\mathbb{R}$. It only remains to prove (2.8). Let us first show that

$$\inf_{t\in\mathbb{R},\ x'\in\overline{\omega},\ -A\leq x_N\leq A-\tau^*} \overline{u}^{\tau^*}(t,x',x_N)-\underline{u}(t,x',x_N) \ = \ 0. \tag{2.11}$$

Assume not. Since the functions $\underline{u}$ and $\overline{u}$ are uniformly continuous, there exists then $\eta_0>0$ such that

$$\overline{u}^{\tau^*-\eta}(t,x',x_N)\geq\underline{u}(t,x',x_N) \ \text{ for all } (t,x')\in\mathbb{R}\times\overline{\omega},\ -A\leq x_N\leq A-\tau^* \text{ and } \eta\in[0,\eta_0]. \tag{2.12}$$

Since $\overline{u}^{\tau^*} \geq b - \delta/2$ in $\mathbb{R} \times \overline{\omega} \times [A - \tau^*, +\infty)$, one can also assume that η_0 is small enough so that $\overline{u}^{\tau^*-\eta} \geq b - \delta$ in $\mathbb{R} \times \overline{\omega} \times [A - \tau^*, +\infty)$ for all $\eta \in [0, \eta_0]$. As above, one can then apply Theorem 2.3 twice and conclude that

$$\overline{u}^{\tau^*-\eta}(t, x', x_N) \geq \underline{u}(t, x', x_N) \text{ for all } (t, x') \in \mathbb{R} \times \overline{\omega},\ x_N \in (-\infty, -A] \cup [A - \tau^*, +\infty)$$

and for all $\eta \in [0, \eta_0]$. Together with (2.12), one finally gets that $\overline{u}^{\tau^*-\eta} \geq \underline{u}$ in $\mathbb{R} \times \overline{\omega} \times \mathbb{R}$ for all $\eta \in [0, \eta_0]$. That contradicts the minimality of τ^*, and eventually the claim (2.11) is proved.

Because of (2.11), there exists then a sequence of points $(t_k, x'_k, x_{N,k})$ such that

$$\overline{u}^{\tau^*}(t_k, x'_k, x_{N,k}) - \underline{u}(t_k, x'_k, x_{N,k}) \to 0 \text{ as } k \to +\infty,$$

with $-A \leq x_{N,k} \leq A - \tau^*$. Since $\overline{\omega}$ is compact, one can then assume that $(x'_k, x_{N,k}) \to (x'_\infty, x_{N,\infty}) \in \overline{\omega} \times [-A, A - \tau^*]$. Up to extraction of another subsequence, one can also assume that the functions

$$z^k(t, x', x_N) = \overline{u}^{\tau^*}(t + t_k, x', x_N) - \underline{u}(t + t_k, x', x_N)$$

converge locally uniformly in $\mathbb{R} \times \overline{\omega} \times \mathbb{R}$, and in C^1_{loc} in t and C^2_{loc} in x, to a nonnegative function z such that $z(0, x'_\infty, x_{N,\infty}) = 0$ and

$$\partial_t z - A_{ij}(t, x')\partial_{ij} z - B_i(t, x')\partial_i z - Cz \geq 0 \text{ in } \mathbb{R} \times \omega \times \mathbb{R},$$

where C is a constant, A_{ij}, B_i are bounded, and $A_{ij}(t, x')\xi_i\xi_j \geq c_0|\xi|^2$ for all $\xi \in \mathbb{R}^N$ and $(t, x') \in \mathbb{R} \times \mathbb{R}^{N-1}$. Furthermore, $\mu(x') \cdot \nabla_{x'} z(t, x', x_N) \geq 0$ for all $(t, x', x_N) \in \mathbb{R} \times \partial\omega \times \mathbb{R}$. The maximum principle and Hopf lemma then imply that $z(t, x', x_N) = 0$ for all $t \leq 0$ and $(x', x_N) \in \overline{\omega} \times \mathbb{R}$. Up to extraction of some subsequence, one can also assume that $\phi(x'_k) \to L \in \mathbb{R}$ as $k \to +\infty$.

Now, if y is any real number, one has that

$$\overline{u}(t_k, x'_k, \phi(x'_k) + \tau^* + y) - \underline{u}(t_k, x'_k, \phi(x'_k) + y) = z^k(0, x'_k, \phi(x'_k) + y) \to z(0, x'_\infty, L + y) = 0$$

as $k \to +\infty$. The proof of Theorem 2.4 is now complete. $\square$

Remark 2.5 *Notice that Theorem 2.4 does not work in general if $a = -\infty$ and $b = +\infty$. For instance, consider the ellptic equation $u'' = 0$ in $\mathbb{R}$ with $u(-\infty) = -\infty$, $u(+\infty) = +\infty$ and take $\overline{u}(x) = x$ and $\underline{u}(x) = 2x$. In the whole real line $\mathbb{R}$, it is clear that one cannot compare any translate of $\overline{u}$ to $\underline{u}$.*

A consequence of Theorem 2.4 is the monotonicity result:

Theorem 2.6 *Under all assumptions of Theorem 2.4, if $u = \underline{u} = \overline{u}$ solves $Pu = 0$ in $\mathbb{R} \times \Omega$, then $a < u(t, x) < b$ for all $(t, x) \in \mathbb{R} \times \overline{\omega}$ and u is increasing in x_N.*

Proof. We will actually only prove that u is increasing in x_N. That clearly implies that $a < u < b$ in $\mathbb{R} \times \overline{\Omega}$.

Set $\underline{u} = \overline{u} = u$. By Theorem 2.4, the set

$$I = \{\tau \in [0, +\infty),\ \forall s \geq \tau,\ u^s \geq u \text{ in } \mathbb{R} \times \overline{\Omega}\}$$

is not empty. Set $\tau^* = \inf I$ and suppose that $\tau^* > 0$. As a consequence, formula (2.8) is valid. Choose $y = 0$ in this formula. There exists then a sequence $(t_k, x'_k) \in \mathbb{R} \times \overline{\omega}$ such that

$$u^{\tau^*}(t_k, x'_k, \phi(x'_k)) - u(t_k, x'_k, \phi(x'_k)) \to 0 \text{ as } k \to +\infty.$$

Consider first the case where $\omega = \mathbb{R}^{N-1}$. As in the proof of Theorem 2.4, the functions

$$u_k(t, x', x_N) = u(t + t_k, x' + x'_k, x_N + \phi(x'_k))$$

converge, up to extraction of some subsequence, locally uniformly in $\mathbb{R} \times \mathbb{R}^N$, and in C^1_{loc} in t and C^2_{loc} in x, to a function U. Because of the uniformity of the limits in (2.7), it follows that the function U is such that

$$U(0, 0, x_N) \to b \text{ (resp. } \to a) \text{ as } x_N \to +\infty \text{ (resp. as } x_N \to -\infty). \tag{2.13}$$

On the other hand, the functions

$$\begin{array}{rcl} v_k & = & u(t + t_k, x' + x'_k, x_N + \phi(x'_k) + \tau^*) - u(t + t_k, x' + x'_k, x_N + \phi(x'_k)) \\ & = & u_k^{\tau^*}(t, x', x_N) - u_k(t, x', x_N) \end{array}$$

converge to $V = U^{\tau^*} - U$ and the function V is nonnegative, vanishes at $(0, 0, 0)$, and it satisfies a parabolic equation of the type

$$\partial_t V - A_{ij}(t, x')\partial_{ij} V - B_i(t, x')\partial_i V - C(t, x', x_N)V = 0 \text{ in } \mathbb{R} \times \mathbb{R}^N,$$

where C is a bounded function. From the strong maximum principle, one concludes that

$$V(t, x', x_N) = 0 \text{ for all } t \leq 0 \text{ and } (x', x_N) \in \mathbb{R}^N.$$

In particular, $U(0, 0, x_N + \tau^*) = U(0, 0, x_N)$ for all $x_N \in \mathbb{R}$. The positivity of τ^* and of $b - a$ then contradicts (2.13). As a consequence, $\tau^* = 0$. Therefore,

$$u(t, x', x_N + s) \geq u(t, x', x_N) \text{ for all } (t, x', x_N) \in \mathbb{R} \times \mathbb{R}^N \text{ and } s \geq 0.$$

The same arguments as before also imply that the inequality is strict everywhere in $\mathbb{R} \times \mathbb{R}^N$ for all $s > 0$.

The case where ω is bounded can be treated with the same type of arguments. The proof is left to the reader. $\square$

3 Conical-shaped fronts: monotonicity, uniqueness and further qualitative properties

In this section, we prove various monotonicity, uniqueness and other qualitative results for the solutions u of equations of the type (1.1) under various conical conditions or more general conditions at infinity. We shall apply the general comparison principles proved in Section 2.

Throughout this section, the function f is always assumed to be Lipschitz-continuous in $\mathbb{R}$ and $f(0) = f(1) = 0$. We will chiefly be concerned here with nonlinearities of the type (1.2) or (1.4), but some of the results hold for more general nonlinearities f which are nonincreasing in some neighbourhoods of 0 and 1.

Most of the results below are borrowed from [F. Hamel and R. Monneau, Solutions of semilinear elliptic equations in $\mathbb{R}^N$ with conical-shaped level sets, *Comm. Part. Diff. Equations* **25** (2000), pp. 769–819] and [F. Hamel, R. Monneau and J.-M. Roquejoffre, Existence and qualitative properties of multidimensional conical bistable fronts, *Disc. Cont. Dyn. Systems* (2005), to appear].

3.1 Monotonicity in cones of directions, properties of the level sets

We here list some properties satisfied by the solutions u of (1.1) and such that

$$\begin{cases} \displaystyle\limsup_{y-\phi(x)\to+\infty} |u(x,y)-1| &= 0, \\ \displaystyle\limsup_{y-\phi(x)\to-\infty} |u(x,y)| &= 0 \end{cases}$$

for some (globally) Lipschitz function $\phi : \mathbb{R}^{N-1} \to \mathbb{R}$. The above limits mean that u converges to 1 (resp. 0) as $y-\phi(x) \to +\infty$ (resp. as $y-\phi(x) \to -\infty$) uniformly in $x \in \mathbb{R}^{N-1}$. We will see that the function u is then monotone in some cones of directions around the vertical axis, and that the level sets of u will all have the same Lipschitz norm.

Before that, let us prove a few basic lemmas.

Lemma 3.1 *Assume that f is positive in $(-\infty,0)$ and negative in $(1,+\infty)$. Let u be a bounded solution of (1.1). Then $0 \le u \le 1$ in $\mathbb{R}^N$.*

Proof. Let $M = \sup_{\mathbb{R}^N} u$. Let (x_n, y_n) be a sequence in $\mathbb{R}^N$ such that $u(x_n,y_n) \to M$ as $n \to +\infty$. Call

$$u_n(x,y) = u(x+x_n, y+y_n).$$

Up to extraction of some subsequence, the functions u_n converge in $C^2_{loc}(\mathbb{R}^N)$ to a classical solution u_∞ of (1.1), namely

$$\Delta u_\infty - c\partial_y u_\infty + f(u_\infty) = 0 \text{ in } \mathbb{R}^N,$$

and $u_\infty(0,0) = M = \max_{\mathbb{R}^N} u_\infty$. Therefore, $f(M) \ge 0$. Since f is negative in $(1,+\infty)$, one gets that $M \le 1$. Similarly, one can easily prove that $m := \inf_{\mathbb{R}^N} u \ge 0$. $\square$

Lemma 3.2 *Assume that f is negative in $(0,\theta_1)$ and positive in $(\theta_2,1)$, for some $0 < \theta_1 \le \theta_2 < 1$. Let $0 \le u \le 1$ be a solution of (1.1) such that*

$$\liminf_{y-\phi(x)\to+\infty} u(x,y) > \theta_2, \qquad \limsup_{y-\phi(x)\to-\infty} u(x,y) < \theta_1, \tag{3.1}$$

for some Lipschitz function $\phi : \mathbb{R}^{N-1} \to \mathbb{R}$. Then $0 < u < 1$ in $\mathbb{R}^N$ and

$$\liminf_{y-\phi(x)\to+\infty} u(x,y) = 1, \qquad \limsup_{y-\phi(x)\to-\infty} u(x,y) = 0. \tag{3.2}$$

Proof. Since $f(0) = f(1) = 0$, $0 \le u \le 1$ in $\mathbb{R}^N$ and u is not identically equal to 0 or 1 because of (3.1), the strong maximum principle then yields $0 < u < 1$ in $\mathbb{R}^N$. Assume now that there exists $\varepsilon > 0$ and a sequence $(x_n, y_n) \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that

$$y_n - \phi(x_n) \to +\infty \text{ and } u(x_n,y_n) \le 1-\varepsilon.$$

Up to extraction of some subsequence, the functions $u_n(x,y) = u(x+x_n, y+y_n)$ converge in $C^2_{loc}(\mathbb{R}^N)$ to a solution u_∞ of (1.1) such that $u_\infty(0,0) \le 1-\varepsilon$. On the other hand, the assumption (3.1) and the fact that $y_n - \phi(x_n) \to +\infty$ imply that

$$m_\infty := \inf_{\mathbb{R}^N} u_\infty > \theta_2,$$

whence $\theta_2 < m_\infty \leq 1 - \varepsilon < 1$. Let (x'_n, y'_n) be a sequence such that $u_\infty(x'_n, y'_n) \to m_\infty$. Up to extraction of some subsequence, the functions $v_n(x, y) = u_\infty(x + x'_n, y + y'_n)$ converge in $C^2_{loc}(\mathbb{R}^N)$ to a solution v_∞ of (1.1) such that $m_\infty = v_\infty(0, 0) = \min_{\mathbb{R}^N} v_\infty$. Therefore, $f(m_\infty) \leq 0$, which contradicts the positivity of f on $(\theta_2, 1)$. Thus,

$$\liminf_{y - \phi(x) \to \infty} u(x, y) = 1.$$

The uniform limit of u to 0 as $y - \phi(x) \to -\infty$ can be proved the same way. □

The preceding lemmas provide sufficient conditions for the function u to converge to 0 and 1 uniformly as $y - \phi(x) \to -\infty$ and $+\infty$ respectively. For instance, if f a cubic nonlinearity $f(s) = s(1-s)(s-\theta)$ with $0 < \theta < 1$ and if u is bounded solution of (1.1) satisfying (3.1) with $\theta_1 = \theta_2 = \theta$, then $0 < u < 1$ in $\mathbb{R}^N$ and (3.2) holds.

In the following theorem, which is one of the main results in this section, we deal with solutions u of (1.1) satisfying (3.2) for some Lipschitz function ϕ.

Theorem 3.3 *Assume that f is of class $C^1([0,1])$, and that f is nonincreasing in $[0, \delta]$ and in $[1-\delta, 1]$, for some $\delta > 0$. Let $0 \leq u \leq 1$ be a solution of (1.1) satisfying (3.2) for some Lipschitz function ϕ.*

Then, for each $\lambda \in (0,1)$, the level set $\{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R},\ u(x, y) = \lambda\}$ is a Lipschitz graph $\{y = \phi_\lambda(x),\ x \in \mathbb{R}^{N-1}\}$ and u satisfies

$$\liminf_{y - \phi_\lambda(x) \to +\infty} u(x, y) = 1, \qquad \limsup_{y - \phi_\lambda(x) \to -\infty} u(x, y) = 0. \tag{3.3}$$

Furthermore, all functions ϕ_λ have the same Lipschitz norm $\|\phi_\lambda\|_{Lip} = \cot\alpha$ with $\alpha \in (0, \pi/2]$, and $\|\phi_\lambda\|_{Lip} \leq \|\phi\|_{Lip}$. Lastly, the function u is decreasing in any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $\tau_y < -\cos\alpha$, and

$$\forall \lambda \in (0,1), \qquad \inf_{x \in \mathbb{R}^{N-1}} u_y(x, \phi_\lambda(x)) > 0. \tag{3.4}$$

Proof. Write $\|\phi\|_{Lip} = \cot\alpha_0$, where $\alpha_0 \in (0, \pi/2]$. Choose any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $\tau_y < -\cos\alpha_0$ and call

$$(X, Y) = (\tau_y x - \tau_x y, -\tau_x \cdot x - \tau_y y) \in \mathbb{R}^{N-1} \times \mathbb{R}.$$

The function $v(X, Y) = u(x, y)$ is such that $0 \leq v \leq 1$ and

$$\liminf_{Y - \psi(X) \to +\infty} v(X, Y) = 1, \qquad \limsup_{Y - \psi(X) \to -\infty} v(X, Y) = 0$$

for some globally Lipschitz function ψ (because $\cot\alpha_0$ is the Lipschitz norm of ϕ, and $\tau_y < -\cos\alpha_0$). Theorem 2.6 in Section 2 then implies that v is increasing in the variable Y, namely u is decreasing in the direction τ.

Choosing $\tau = (0, -1)$ means that u is increasing in the variable y. Therefore, because of (3.2), for each $\lambda \in (0,1)$, the level set $\{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R},\ u(x, y) = \lambda\}$ is the graph

$$\{y = \phi_\lambda(x),\ x \in \mathbb{R}^{N-1}\}$$

of a globally Lipschitz function ϕ_λ, whose Lipschitz norm is such that

$$\|\phi_\lambda\|_{Lip} \leq \|\phi\|_{Lip}.$$

In other words, $\|\phi_\lambda\|_{Lip} = \cot\alpha_\lambda$ with $\alpha_\lambda \in [\alpha_0, \pi/2]$.

Let $\lambda \in (0,1)$ be fixed. Because of (3.2), the quantity $\sup_{x\in\mathbb{R}^{N-1}} |\phi(x) - \phi_\lambda(x)|$ is finite, and the function u then satisfies the limits (3.3). The same arguments as above then imply that the function u is decreasing in any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1}\times\mathbb{R}$ such that $\tau_y < -\cos\alpha_\lambda$. It especially follows that the Lipschitz norm $\cot\alpha_{\lambda'}$ of the graph $\{y = \phi_{\lambda'}(x)\}$ of any level set $\{u(x,y) = \lambda'\}$ is such that $\cot\alpha_{\lambda'} \leq \cot\alpha_\lambda$. Since λ was arbitrary in $(0,1)$, one concludes that $\|\phi_\lambda\|_{Lip} = \cot\alpha_\lambda$ does not depend on λ. In other words, $\alpha := \alpha_\lambda$ does not depend on λ.

Notice that, by continuity, u is nonincreasing in any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1}\times\mathbb{R}$ such that $\tau_y \leq -\cos\alpha$.

In particular, since the nonnegative function $v = u_y$ satisfies $\Delta v - cv_y + f'(u)v = 0$ in $\mathbb{R}^N$ and $v \not\equiv 0$, the strong maximum principle yields $u_y > 0$ in $\mathbb{R}^N$.

Let us now prove (3.4). Assume by contradiction that there exist $\lambda \in (0,1)$ and a sequence $(x_n) \in \mathbb{R}^{N-1}$ such that $u_y(x_n, \phi_\lambda(x_n)) \to 0$ as $n \to +\infty$. Let

$$u_n(x,y) = u_n(x + x_n, y + \phi_\lambda(x_n)) \ \text{ and } \ \phi_n(x) = \phi_\lambda(x + x_n) - \phi_\lambda(x_n)$$

(notice that the functions ϕ_n are uniformly Lipschitz continuous). Up to extraction of some subsequence, the functions u_n (resp. ϕ_n) converge in $C^2_{loc}(\mathbb{R}^N)$ (resp. locally uniformly in $\mathbb{R}^{N-1}$) to a function u_∞ (resp. ϕ_∞) such that u_∞ solves (1.1) and (3.2) with ϕ_∞ instead of ϕ (because of the limits (3.3) for u). The same arguments as above then imply that $\partial_y u_\infty > 0$ in $\mathbb{R}^N$. But $\partial_y u_\infty(0,0) = 0$. One has then reached a contradiction. Therefore, (3.4) follows. $\square$

Remark 3.4 *Under the assumptions of Theorem 3.3, if $N = 2$ and if the function ϕ is of class C^1 and $\tan\beta \leq \phi'(x) \leq \tan\gamma$ for all $x \in \mathbb{R}$, with $-\pi/2 < \beta \leq \gamma < \pi/2$, then u is decreasing in any direction $(\cos\varphi, \sin\varphi)$ such that $\gamma - \pi < \varphi < \beta$. The proof of this fact uses the same type of arguments as in the beginning of the proof of Theorem 3.3.*

The following proposition provides some lower and upper exponential bounds below and above any level curve, under the condition of strict stability of the zeroes 0 and 1.

Proposition 3.5 *Under the assumptions of Theorem 3.3, if one further assumes that $f'(0) < 0$, then, for each $\lambda \in (0,1)$, there exist $0 < \beta \leq \gamma$ such that*

$$\lambda\, e^{\gamma(y-\phi_\lambda(x))} \leq u(x,y) \leq \lambda\, e^{\beta(y-\phi_\lambda(x))}$$

for all $(x,y) \in \mathbb{R}^{N-1}\times\mathbb{R}$ such that $y \leq \phi_\lambda(x)$. Furthermore, γ can be chosen independently of λ.

Similarly, if $f'(1) < 0$, then, for each $\lambda \in (0,1)$, there exist $0 < \beta' \leq \gamma'$ such that

$$(1-\lambda)e^{-\gamma'(y-\phi_\lambda(x))} \leq 1 - u(x,y) \leq (1-\lambda)e^{-\beta'(y-\phi_\lambda(x))}$$

for all $(x,y) \in \mathbb{R}^{N-1}\times\mathbb{R}$ such that $y \geq \phi_\lambda(x)$. Furthermore, γ' can be chosen independently of λ.

Proof. Let us first observe that, since u is (at least) of class $C^2(\mathbb{R}^N)$ and $u_y > 0$, it follows from the implicit function theorem that ϕ_λ is of class C^2 as well. A straightforward calculation leads to

$$\partial_{x_ix_j}\phi_\lambda(x) = -\frac{\partial_{x_ix_j}u + \partial_{x_i}\phi_\lambda(x)\partial_{x_jy}u + \partial_{x_j}\phi_\lambda(x)\partial_{x_iy}u + \partial_{x_i}\phi_\lambda(x)\partial_{x_j}\phi_\lambda(x)\partial_y^2 u}{u_y}$$

for all $x \in \mathbb{R}^{N-1}$ and $1 \leq i,j \leq N-1$, where the function u and its derivatives are taken at $(x,\phi_\lambda(x))$. On the other hand, the function u is globally bounded in C^2 from standard elliptic estimates. Therefore, since $\nabla\phi_\lambda$ is bounded (by $\cot\alpha$) and $\inf_{x\in\mathbb{R}^{N-1}} u_y(x,\phi_\lambda(x)) > 0$, it follows that $D^2\phi_\lambda$ is bounded as well.

Let us now turn to the proof of the exponential behavior far away from a given level set. First of all, it follows from standard elliptic estimates and Harnack inequality that $|\nabla u|/u$ is globally bounded in $\mathbb{R}^N$. Call

$$\gamma = \sup_{(x,y)\in\mathbb{R}^N} \frac{u_y(x,y)}{u(x,y)}.$$

The real number γ is positive since $u_y > 0$ in $\mathbb{R}^N$. It immediately follows that

$$\forall\lambda\in(0,1),\quad \forall y\leq\phi_\lambda(x),\quad u(x,y)\geq\lambda\ e^{\gamma(y-\phi_\lambda(x))}.$$

Let $\eta\in(0,1)$ be fixed small enough so that $f(s)\leq f'(0)s/2$ for all $s\in[0,\eta]$. One can assume that $\eta\leq\delta$, so that f is nonincreasing in $(-\infty,\eta)$ (even if it means extending f by $f(s)=f'(0)s$ for $s\leq 0$). The function $\overline{u}(x,y)=\eta\ e^{\beta(y-\phi_\eta(x))}$ satisfies

$$\Delta\overline{u}-c\overline{u}_y+f(\overline{u})\leq\left(\beta^2+\beta^2|\nabla\phi_\eta(x)|^2-\beta\Delta\phi_\eta(x)-c\beta+\frac{f'(0)}{2}\right)\overline{u}\leq 0\ \text{ in } \{y\leq\phi_\eta(x)\}$$

for $\beta>0$ small enough (remember that $\nabla\phi_\eta$ and $\Delta\phi_\eta$ are bounded). Let $\beta>0$ be as such. It then follows from Theorem 2.1 in Section 2 that

$$u(x,y)\leq\overline{u}(x,y)=\eta\ e^{\beta(y-\phi_\eta(x))}\ \text{ for all } (x,y)\in\mathbb{R}^{N-1}\times\mathbb{R} \text{ such that } y\leq\phi_\eta(x). \tag{3.5}$$

Let now λ be any number in $(0,1)$. One claims that there exists $\beta_\lambda>0$ such that

$$u(x,y)\leq\lambda\ e^{\beta_\lambda(y-\phi_\lambda(x))} \tag{3.6}$$

for all (x,y) such that $y\leq\phi_\lambda(x)$. Otherwise, there is a sequence of points (x_n,y_n) such that $y_n\leq\phi_\lambda(x_n)$ and

$$u(x_n,y_n)>\lambda\ e^{(y_n-\phi_\lambda(x_n))/n}.$$

Up to extraction of some subsequence, three cases may occur:

Case 1: $y_n-\phi_\lambda(x_n)\to-\infty$. As already underlined, $\sup_{x\in\mathbb{R}^{N-1}}|\phi_\lambda(x)-\phi_\eta(x)|<+\infty$ and one then gets a contradiction with (3.5).

Case 2: $y_n-\phi_\lambda(x_n)\to h<0$. Then $\liminf_{n\to+\infty}u(x_n,y_n)\geq\lambda$. On the other hand, since $\inf_{x\in\mathbb{R}^{N-1}}u_y(x,\phi_\lambda(x))>0$, $u_y>0$ and u_{yy} is bounded in $\mathbb{R}^N$, it follows from assumption that $\limsup_{n\to+\infty}u(x_n,y_n)<\lambda$. Therefore, Case 2 is ruled out too.

Case 3: $y_n-\phi_\lambda(x_n)\to 0$ as $n\to+\infty$. One gets a contradiction by using the same arguments as in Case 2.

As a consequence, the claim (3.6) is proved. The proof of the exponential lower and upper bound of $1-u$ above a given level set is similar and it is left to the reader. □

The following theorem states the same monotonicity properties as Theorem 3.3 when f is of the bistable type (1.4), under some assumptions which are of different nature from (3.1) or (3.2). This result will then be used in Section 4 to get the existence of bistable conical fronts.

Theorem 3.6 *Assume that $f \in C^1([0,1])$ is of the bistable type (1.4), that $\displaystyle\int_0^1 f \geq 0$, and f is extended in $\mathbb{R}$ so that $f > 0$ in $(-\infty, 0)$ and $f < 0$ in $(1, +\infty)$. Let u be a nonzero bounded solution of (1.1) with $c \geq 0$, and such that*

$$\inf\ u < \theta, \quad u_y \geq 0 \text{ in } \mathbb{R}^N, \quad u(x,y) = \tilde{u}(|x|, y), \quad \text{and } \partial_{|x|}\tilde{u} \geq 0 \text{ in } \mathbb{R}^N. \tag{3.7}$$

Then $0 < u < 1$ in $\mathbb{R}^N$ and u satisfies all conclusions of Theorem 3.3 and Proposition 3.5.

Proof. First, since f is positive in $(-\infty, 0)$ and negative in $(1, +\infty)$, one can prove as in Lemma 3.1 that

$$0 \leq \inf_{\mathbb{R}^N} u \leq \sup_{\mathbb{R}^N} u \leq 1.$$

We then claim that

$$\sup u > \theta.$$

Assume not. Then $u \leq \theta$ in $\mathbb{R}^N$ and the strong elliptic maximum principle yields that $u < \theta$ in $\mathbb{R}^N$, because $f(\theta) = 0$ and $\inf u < \theta$. Let φ_R and λ_R be the principal eigenfunction and eigenvalue of

$$\begin{cases} -\Delta\varphi_R &= \lambda_R\varphi_R & \text{in } B_R \\ \varphi_R &> 0 & \text{in } B_R \\ \varphi_R &= 0 & \text{on } \partial B_R, \end{cases}$$

where $B_R \subset \mathbb{R}^N$ is the open euclidean ball with centre 0 and radius $R > 0$. Let $R > 0$ be chosen large enough so that $\lambda_R \leq f'(\theta)/2$ (this is possible since $f'(\theta) > 0$ and $\lambda_R \to 0$ as $R \to +\infty$). Choose now $\eta > 0$ small enough so that

$$u < \theta - \eta\varphi_R \text{ in } \overline{B_R} \text{ and } f(\theta - \eta\varphi_R) \leq -\eta\varphi_R f'(\theta)/2 \text{ in } B_R.$$

The function $v := \theta - \eta\varphi_R$ then satisfies

$$\Delta v + f(v) \leq \eta\lambda_R\varphi_R - \eta\varphi_R f'(\theta)/2 \leq 0 \text{ in } B_R$$

and $v = \theta$ on ∂B_R. Let z_0 be any vector in $\mathbb{R}^N$. From the local uniform continuity of u, there exists $\kappa > 0$ such that

$$u(\cdot + tz_0) < v \text{ in } \overline{B_R} \text{ for all } t \in [0, \kappa].$$

Call

$$t^* = \sup\ \{t \in [0, +\infty),\ u(\cdot + t'z_0) < v \text{ in } \overline{B_R} \text{ for all } t' \in [0, t]\}.$$

One has $0 < \kappa \leq t^* \leq +\infty$. Assume $t^* < +\infty$. Then, $u(\cdot + t^*z_0) \leq v$ in $\overline{B_R}$ and there exists $z^* \in \overline{B_R}$ such that $u(z^* + t^*z_0) = v(z^*)$. Since $v = \theta$ on ∂B_R and $u < \theta$ in $\mathbb{R}^N$, it follows that $z^* \in B_R$. On the other hand,

$$\Delta u(\cdot + t^*z_0) + f(u(\cdot + t^*z_0)) = cu_y(\cdot + t^*z_0) \geq 0 \geq \Delta v + f(v) \text{ in } B_R.$$

Hence, there exists a bounded function b such that the function $w := v - u(\cdot + t^*z_0)$ satisfies $\Delta w + bw \leq 0$ in B_R. Since w is nonnegative and vanishes at the point $z^* \in B_R$, the strong maximum principle yields $w \equiv 0$ in $\overline{B_R}$. This is impossible because $v = \theta$ on ∂B_R and $u < \theta$ in $\mathbb{R}^N$. As a consequence, $t^* = +\infty$. Since $z_0 \in \mathbb{R}^N$ was arbitrary, one gets that

$$u(z) < v(0) < \theta \text{ for all } z \in \mathbb{R}^N.$$

As a consequence, $\sup u < \theta$. Since f is negative in $(0,\theta)$, one can then prove as in Lemma 3.1 that $\sup u \leq 0$, whence $u \equiv 0$ in $\mathbb{R}^N$, which is impossible by assumption.

Therefore, the claim $\sup u > \theta$ is proved. The strong maximum principle then yields $0 < u < 1$ in $\mathbb{R}^N$, because $0 \leq\not\equiv u \leq\not\equiv 1$ and $f(0) = f(1) = 0$.

Let us now prove that $u(x,y) \to 1$ (resp. $u(x,y) \to 0$) as $y \to +\infty$ (resp. $y \to -\infty$) locally uniformly in $x \in \mathbb{R}^{N-1}$. Since $0 < u < 1$ and u is nondecreasing with respect to the variable y, there exist two functions $0 \leq u_\pm(x) \leq 1$ such that $u(x,y) \to u_\pm(x)$ as $y \to \pm\infty$, for all $x \in \mathbb{R}^{N-1}$. From standard elliptic estimates, the functions $u(x, y + y_0)$ converge to $u_\pm(x)$ as $y_0 \to \pm\infty$ in $C^2_{loc}(\mathbb{R}^N)$ and the functions $u_\pm$ satisfy

$$\Delta_x u_\pm + f(u_\pm) = 0 \ \text{ in } \mathbb{R}^{N-1}.$$

Notice that $0 \leq u_- < 1$ in $\mathbb{R}^{N-1}$ and that u_- can be written as $u_-(x) = \tilde{u}_-(|x|)$ with $\tilde{u}_-'(r) \geq 0$ for $r \geq 0$ by (3.7). Call $l = \tilde{u}_-(+\infty) \in [0,1]$. From standard elliptic estimates, $\tilde{u}_-'(r) \to 0$ as $r \to +\infty$ and l is a zero of f, namely $l = 0$, $l = \theta$ or $l = 1$. If $l = 0$, then $u_- \equiv 0$, which is the desired result. If $l = 1$, multiply the equation

$$\tilde{u}_-''(r) + \frac{N-2}{r}\tilde{u}_-'(r) + f(\tilde{u}_-(r)) = 0, \ r > 0$$

by $\tilde{u}_-'(r)$ and integrate on $(0,+\infty)$. It follows that

$$\int_{u_-(0)}^1 f(s)ds = -\int_0^{+\infty} \frac{(N-2)(\tilde{u}_-'(r))^2}{r} dr \leq 0.$$

But $0 \leq u_-(0) < 1$ and the assumptions on the profile of f ($f < 0$ on $(0,\theta)$, $f > 0$ on $(\theta,1)$ and $\int_0^1 f > 0$) lead to a contradiction. If $l = \theta$, then $0 \leq u_- \leq \theta$ in $\mathbb{R}^{N-1}$ and the arguments in the first part of the proof of this theorem yield $u_- \equiv 0$ or $u_- \equiv \theta$. The latter is impossible because $\inf_{\mathbb{R}^N} u < \theta$ whence $\inf_{\mathbb{R}^{N-1}} u_- < \theta$. One concludes that

$$u_- \equiv 0 \text{ in } \mathbb{R}^{N-1}.$$

With similar arguments, one can prove that $u_+ \equiv 1$ in $\mathbb{R}^{N-1}$.

Furthermore, the nonnegative function u_y satisfies an elliptic equation with continuous coefficients, and it is not identically 0. Therefore, $u_y > 0$ in $\mathbb{R}^N$ from the strong maximum principle.

From the above results, each level set of u, $\{(x,y) \in \mathbb{R}^N,\ u(x,y) = \lambda\}$, for any $\lambda \in (0,1)$, is a graph $\{y = \phi_\lambda(x),\ x \in \mathbb{R}^{N-1}\}$. We shall now prove property (3.1) with $\theta_1 = \theta_2 = \theta$ and $\phi = \phi_\theta$ (which would then imply (3.2)). To do so, we will prove the nondegeneracy property (3.4). Take first $\lambda \in (0,\theta)$. Assume that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{N-1}$ such that $u_y(x_n, \phi_\lambda(x_n)) \to 0$ as $n \to +\infty$. Let e be any fixed unit vector in $\mathbb{R}^{N-1}$. Since u only depends on $|x|$ and y, one can assume that $x_n = r_n e$ with $r_n \geq 0$. Furthermore, $r_n \to +\infty$ because u_y is continuous and positive in $\mathbb{R}^N$. From standard elliptic estimates, the functions

$$u_n(x,y) = u(x + x_n, y + \phi_\lambda(x_n))$$

converge in $C^2_{loc}(\mathbb{R}^N)$, up to extraction of some subsequence, to a solution u_∞ of (1.1) such that $0 \leq u_\infty \leq 1$, $u_\infty(0,0) = \lambda$, $\partial_y u_\infty \geq 0$ and $\partial_y u_\infty(0,0) = 0$. Therefore, $\partial_y u_\infty \equiv 0$ from the strong maximum principle. On the other hand, since $r_n \to +\infty$ and u depends on $|x|$ and y only, the function u_∞ eventually depends on $x \cdot e$ only. Namely, $u_\infty(x,y) = v(x \cdot e)$ and v satisfies

$$v''(\xi) + f(v(\xi)) = 0, \ \xi \in \mathbb{R}. \tag{3.8}$$

Furthermore, $v' \geq 0$ in $\mathbb{R}$ because $\partial_{|x|}\tilde{u}(|x|, y) \geq 0$ in $\mathbb{R}^N$ and $r_n \to +\infty$. Call $l_\pm = v(\pm\infty) \in [0, 1]$. Standard elliptic estimates yield $f(l_\pm) = 0$ and $v'(\xi) \to 0$ as $|\xi| \to +\infty$. Moreover, $0 \leq l_- \leq \lambda < \theta$ and $0 < \lambda \leq l_+ \leq 1$. Therefore, $l_- = 0$ and $l_+ = \theta$ or 1. In both cases, multiply (3.8) by v' and integrate over $\mathbb{R}$. It follows that $\displaystyle\int_0^{l_+} f = 0$, which is impossible due to the profile of f. That shows that

$$\inf_{x \in \mathbb{R}^{N-1}} u_y(x, \phi_\lambda(x)) > 0 \text{ for all } \lambda \in (0, \theta).$$

Notice that the same result holds similarly with $\lambda \in (\theta, 1)$.

We then claim that the same nondegeneracy property holds good for $\lambda = \theta$ as well. Assume not and let e be a given unit vector of $\mathbb{R}^{N-1}$. There exists then a sequence $r_n \to +\infty$ such that the functions $u_n(x, y) = u(x + r_n e, y + \phi_\theta(r_n e))$ converge in $C^2_{loc}(\mathbb{R}^N)$, up to extraction of some subsequence, to a function $u_\infty(x, y) = v(x \cdot e)$. The function v satisfies (3.8) in $\mathbb{R}$ and $0 \leq v \leq 1$, $v' \geq 0$, $v(0) = \theta$. Since $f(v(\pm\infty)) = 0$ and $\displaystyle\int_{v(-\infty)}^{v(+\infty)} f = 0$, it follows that $v(\pm\infty) = \theta$, namely $v \equiv \theta$. In other words, the functions u_n converge locally uniformly to the constant θ. Fix now any $\lambda \in (0, \theta)$. It then follows that $\phi_\theta(r_n e) - \phi_\lambda(r_n e) \to +\infty$ as $n \to +\infty$ and, for any compact set $K \subset \mathbb{R}^N$,

$$\limsup_{n \to +\infty} \max_{(x,y) \in K} u(x + r_n e, y + \phi_\lambda(r_n e)) \leq \theta.$$

As a consequence, the functions $w_n(x, y) = u(x + r_n e, y + \phi_\lambda(r_n e))$ converge in $C^2_{loc}(\mathbb{R}^N)$, up to extraction of some subsequence, to a function w_∞ satisfying (1.1) and $0 \leq w_\infty \leq \theta$ in $\mathbb{R}^N$. Furthermore, $w_\infty(0, 0) = \lambda$ and $\partial_y w_\infty \geq 0$. Since $c \geq 0$, one has $\Delta w_\infty + f(w_\infty) \geq 0$ and then $w_\infty \equiv 0$, using the arguments of the beginning of the proof of this theorem. This gives a contradiction.

Therefore, one has proved that $\inf_{x \in \mathbb{R}^{N-1}} u_y(x, \phi_\theta(x)) > 0$. Actually, the above arguments imply that

$$\inf_{\lambda \in [\lambda_1, \lambda_2],\ x \in \mathbb{R}^{N-1}} u_y(x, \phi_\lambda(x)) > 0 \quad \text{for all} \quad 0 < \lambda_1 \leq \lambda_2 < 1.$$

Moreover, given $\lambda \in (0, 1)$, the implicit function theorem implies that ϕ_λ is of class C^2. Since $|\nabla u|$ is globally bounded in $\mathbb{R}^N$ from standard elliptic estimates, and $\nabla \phi_\lambda(x) = -\nabla_x u(x, \phi_\lambda(x))/u_y(x, \phi_\lambda(x))$ for all $x \in \mathbb{R}^{N-1}$, it then follows that $\nabla \phi_\lambda$ is globally bounded in $\mathbb{R}^{N-1}$. In other words, each function ϕ_λ is globally Lipschitz-continuous.

We shall prove now that u satisfies (3.1) with $\theta_1 = \theta_2 = \theta$ and $\phi = \phi_\theta$. Let m be the positive number defined by

$$m := \inf_{\lambda \in [\theta/2, (1+\theta)/2],\ x \in \mathbb{R}^{N-1}} u_y(x, \phi_\lambda(x)) > 0.$$

The mean value theorem yields

$$\forall x \in \mathbb{R}^{N-1}, \quad \frac{1+\theta}{2} - \theta = u(x, \phi_{(1+\theta)/2}(x)) - u(x, \phi_\theta(x)) \geq m \times (\phi_{(1+\theta)/2}(x) - \phi_\theta(x)).$$

Hence, $0 \leq \phi_{(1+\theta)/2}(x) - \phi_\theta(x) \leq (1 - \theta)/(2m)$ for all $x \in \mathbb{R}^{N-1}$. Therefore,

$$u(x, y) \geq \frac{1+\theta}{2} \quad \text{for all } x \in \mathbb{R}^{N-1} \text{ and } y \geq \phi_\theta(x) + (1 - \theta)/(2m)$$

(because $\partial_y u > 0$ in $\mathbb{R}^N$). Similarly, one can prove that $u(x, y) \leq \theta/2$ for all $x \in \mathbb{R}^{N-1}$ and $y \leq \phi_\theta(x) - \theta/(2m)$. As a consequence, the function u satisfies (3.1) with $\theta_1 = \theta_2 = \theta$ and $\phi = \phi_\theta$. Lemma 3.2 implies that u satisfies (3.2) with $\phi = \phi_\theta$, and then all conclusions of Theorem 3.3 and

Proposition 3.5 hold (in particular, (3.2) holds for any function ϕ_λ with $\lambda \in (0,1)$). The proof of Theorem 3.6 is now complete. $\square$

Remark 3.7 *The conclusion of Theorem 3.6 holds if, instead of $c \geq 0$ and $u_y \geq 0$, one assumes that $c \leq 0$ and $u_y \leq 0$. Similarly, instead of $\inf u < \theta$ and $cu_y \geq 0$, one could have assumed that $\sup u > \theta$, $u \not\equiv 1$ and $cu_y \leq 0$.*

3.2 Uniqueness of the speed and asymptotic behavior along the level sets

In this section, we will see how to relate the speed c in (1.1) to a simple angle which is determined by the level sets of u, and to the unique speed of the one-dimensional problem (1.6). Actually, it is well-known that, if f is nonincreasing in $[0,\delta]$ and $[1-\delta,1]$ for some $\delta > 0$, then the equation

$$U'' - c_0 U' + f(U), \quad U(-\infty) = 0 \leq U \leq U(+\infty) = 1, \quad U(0) = \frac{1}{2} \tag{3.9}$$

has at most one solution $c_0 = c(f)$ and U (the normalization of U at 0 is made to fix the solution among all possible shifts). Notice however that this problem does not always have a solution.

In the sequel, $\hat{x}$ denotes the vector $x/|x|$ for $x \neq 0$. The main result of this section is the following

Theorem 3.8 *Assume that $f \in C^1([0,1])$ is nonincreasing in $[0,\delta]$ and $[1-\delta,1]$ for some $\delta > 0$. Let $0 \leq u \leq 1$ be a solution of (1.1).*

1) *If (3.2) is satisfied for some radially symmetric Lipschitz function $\phi : \mathbb{R}^{N-1} \to \mathbb{R}$, then the conclusion of Theorem 3.3 holds. Moreover, problem (3.9) has a, unique, solution $(c(f), U)$ and*

$$c = \frac{c(f)}{\sin \alpha} \tag{3.10}$$

where $\alpha \in (0, \pi/2]$ and $\cot \alpha$ denotes the Lipschitz norm of all level graphs ϕ_λ of u.

2) *If $N = 2$ and if (3.2) is satisfied for some Lipschitz function $\phi : \mathbb{R} \to \mathbb{R}$, then the same conclusion as in part 1) holds.*

3) *If (3.2) is satisfied for some Lipschitz function $\phi : \mathbb{R}^{N-1} \to \mathbb{R}$ which is of class C^1 for large $|x|$ and such that*

$$|\hat{x} \cdot \nabla \phi(x)| \to \cot \alpha \quad \text{as } |x| \to +\infty, \quad \text{for some } \alpha \in (0, \pi/2],$$

then the conclusion of Theorem 3.3 holds. Moreover, problem (3.9) has a, unique, solution $(c(f), U)$, $c = c(f)/\sin \alpha$ and the Lipschitz norm of all level graphs ϕ_λ of u is equal to $\cot \alpha$. In particular, if $\alpha = \pi/2$, then u is planar, it only depends on the variable y and it is unique up to translation.

Proof. 1) The assumption (3.2) is satisfied and one can then apply Theorem 3.3. Denote by $\cot \alpha$ the Lipschitz norm $\|\phi_\lambda\|_{Lip}$ of all level graphs of u, with $\alpha \in (0, \pi/2]$.

If $\alpha = \pi/2$, then the level sets of u are parallel hyperplanes which are orthogonal to the direction y and the function u is a function of y only, which then satisfies (3.9). By uniqueness, u is then a translate of U and $c = c(f) = c(f)/\sin(\pi/2)$.

Let us now assume in the sequel that $\alpha \in (0, \pi/2)$. Let e be a given unit direction of $\mathbb{R}^{N-1}$ and let $\tilde{\varphi} : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by

$$\tilde{\varphi}(k+t) = \phi_{1/2}(ke) + t(\phi_{1/2}((k+1)e) - \phi_{1/2}(ke))$$

for all $k \in \mathbb{N}$ and $t \in [0,1)$. Lastly, let $\tilde{\phi}(x) = \tilde{\varphi}(|x|)$ for all $x \in \mathbb{R}^{N-1}$. This function $\tilde{\phi}$ is clearly Lipschitz continuous and its Lipschitz norm $\|\tilde{\phi}\|_{Lip}$, denoted by $\cot\beta$ with $\beta \in (0,\pi/2]$, is such that

$$\|\tilde{\phi}\|_{Lip} = \cot\beta \le \|\phi_{1/2}\|_{Lip} = \cot\alpha.$$

In other words, $\beta \ge \alpha$. Furthermore, one has

$$\sup_{x\in\mathbb{R}^{N-1}} |\phi_{1/2}(x) - \phi(x)| < +\infty$$

because of (3.3) and (3.2). Therefore, $\sup_{t\ge 0} |\tilde{\varphi}(t) - \phi(te)| < +\infty$ and $\sup_{x\in\mathbb{R}^{N-1}} |\tilde{\phi}(x) - \phi(x)| < +\infty$ by radial symmetry of $\tilde{\phi}$ and ϕ. As a consequence,

$$\sup_{x\in\mathbb{R}^{N-1}} |\phi_{1/2}(x) - \tilde{\phi}(x)| < +\infty$$

and the function u satisfies (3.3) with $\tilde{\phi}$ as well as with $\phi_{1/2}$.

Therefore, the arguments used in the proof of Theorem 3.3 imply that the function u is decreasing in any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $\tau_y < -\cos\beta$. If β were strictly larger than α, each level graph of u would then have a Lipschitz norm less than or equal to $\cot\beta$, and then strictly less than $\cot\alpha$, which is impossible due to the definition of α. Therefore, $\beta = \alpha$ and the functions $\tilde{\phi}$ and $\phi_{1/2}$ have the same Lipschitz norm, namely $\cot\alpha$.

By construction of $\tilde{\phi}$, there exists then a sequence of integers $(k(n))_{n\in\mathbb{N}}$ such that

$$|\tilde{\phi}((k(n)+1)e) - \tilde{\phi}(k(n)e)| = |\phi_{1/2}((k(n)+1)e) - \phi_{1/2}(k(n)e)| \to \cot\alpha \tag{3.11}$$

as $n \to +\infty$. Up to extraction of some subsequence, two cases may occur:

Case 1: $\tilde{\phi}((k(n)+1)e) - \tilde{\phi}(k(n)e) = \phi_{1/2}((k(n)+1)e) - \phi_{1/2}(k(n)e) \to \cot\alpha$ as $n \to +\infty$. Up to extraction of another subsequence, the functions

$$u_n(x,y) = u(x + k(n)e, y + \phi_{1/2}(k(n)e))$$

converge in $C^2_{loc}(\mathbb{R}^N)$ to a solution v of (1.1) such that $v(0,0) = v(e, \cot\alpha) = 1/2$. By passage to the limit, the function v is nonincreasing in any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $\tau_y \le -\cos\alpha$. It especially follows that

$$v(te, t\cot\alpha) = 1/2 \text{ for all } t \in [0,1].$$

For any $t \in (0,1]$, the function $w(x,y) = v(x,y) - v(x+te, y+t\cot\alpha)$ is nonpositive, it vanishes at $(0,0)$, and it satisfies an equation of the type $\Delta w - cw_y + b(x,y)w = 0$ in $\mathbb{R}^N$, for some bounded function b (because f is Lipschitz continuous). The strong maximum principle implies that $w(x,y) = 0$ for all $(x,y) \in \mathbb{R}^N$. Since $t \in (0,1]$ was arbitrary, one concludes that v is constant in the direction $(e, \cot\alpha)$.

Furthermore, one has that $k(n) \to +\infty$: if not, one could have assumed that the sequence $(k(n))$ was bounded (up to extraction of some subsequence), and then the function u itself would have been constant along the direction $(e, \cot\alpha)$. Since $\alpha \in (0,\pi/2)$ here, one gets a contradiction with the fact that u satisfies (3.2) with the radial function $\phi(x)$.

As a consequence, $k(n) \to +\infty$ as $n \to +\infty$. Since ϕ is radial in (3.2), there exists then a globally Lipschitz function $\phi_\infty : \mathbb{R} \to \mathbb{R}$ such that

$$\liminf_{y-\phi_\infty(x\cdot e)\to+\infty} v(x,y) = 1, \qquad \limsup_{y-\phi_\infty(x\cdot e)\to-\infty} v(x,y) = 0.$$

Since $v(te, t\cot\alpha) = 1/2$ for all $t \in \mathbb{R}$, one gets that $\sup_{x\in\mathbb{R}^{N-1}} |\phi_\infty(x\cdot e) - (x\cdot e)\cot\alpha| < +\infty$, whence

$$\liminf_{y-(x\cdot e)\cot\alpha\to+\infty} v(x,y) = 1, \qquad \limsup_{y-(x\cdot e)\cot\alpha\to-\infty} v(x,y) = 0. \tag{3.12}$$

The arguments used in the beginning of the proof of Theorem 3.3, relying on the comparison principles stated in Section 2, then imply that the function v is increasing in any direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1}\times\mathbb{R}$ such that $\tau_y > 0$ and $\tau_x \cdot e = 0$. Fix any such $\tau_x \in \mathbb{R}^{N-1}$ such that $\tau_x\cdot e = 0$ and consider the directions $\tau_\pm = (\tau_x, \pm\tau_y)$ with $\tau_y > 0$. The function v is increasing in both directions τ_+ and $-\tau_-$. Letting $\tau_y \to 0^+$ implies that v is constant in the direction $(\tau_x, 0)$. Therefore, v does not depend in the directions of $\mathbb{R}^{N-1}$ which are orthogonal to e. On the other hand, one has already got that v was constant in the direction $(e, \cot\alpha)$. In other words, there exists a function $v_0 : \mathbb{R} \to [0,1]$ such that

$$v(x,y) = v_0((-x\cdot e)\cos\alpha + y\sin\alpha)$$

for all $(x,y) \in \mathbb{R}^N$. As a consequence, the function $v_0 = v_0(\xi)$ satisfies

$$v_0'' - c\sin\alpha\ v_0' + f(v_0) = 0, \quad 0 \le v_0 \le 1 \quad \text{in } \mathbb{R}$$

together with $v_0(-\infty) = 0$, $v_0(+\infty) = 1$ (because of (3.12)). The uniqueness result for the above equation yields $c\sin\alpha = c(f)$ and $v_0 = U$.

Case 2: $\tilde\phi((k(n)+1)e) - \tilde\phi(k(n)e) = \phi_{1/2}((k(n)+1)e) - \phi_{1/2}(k(n)e) \to -\cot\alpha$ as $n \to +\infty$. This case can be treated similarly and leads to the same conclusion that $c\sin\alpha = c(f)$.

2) The case $N = 2$ with assumption (3.2) without radial symmetry of ϕ is just an adaptation of the previous proof in part 1). Let $\cot\alpha$ denote the Lipschitz norm of all level graphs ϕ_λ of u, with $\alpha \in (0, \pi/2]$. Let $\tilde\phi : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\tilde\phi(k+t) = \phi_{1/2}(k) + t(\phi_{1/2}(k+1) - \phi_{1/2}(k))$$

for all $k \in \mathbb{Z}$ and $t \in [0,1)$. As in part 1), one can prove that $\sup_{x\in\mathbb{R}} |\phi_{1/2}(x) - \tilde\phi(x)| < +\infty$ and that $\|\tilde\phi\|_{Lip} = \|\phi_{1/2}\|_{Lip} = \cot\alpha$. Therefore, there exists a sequence of integers $(k(n))_{n\in\mathbb{N}}$ in $\mathbb{Z}$ such that (3.11) holds with $e = 1$. Notice that the $k(n)$ may be negative, because $\tilde\phi$ is not radially symmetric anymore. Two cases may occur:

Case 1: $\tilde\phi(k(n)+1) - \tilde\phi(k(n)) = \phi_{1/2}(k(n)+1) - \phi_{1/2}(k(n)) \to \cot\alpha$ as $n \to +\infty$. As in part 1), up to extraction of another subsequence, the functions $u_n(x,y) = u(x+k(n), y+\phi_{1/2}(k(n)))$ converge in $C^2_{loc}(\mathbb{R}^N)$ to a solution v of (1.1) in dimension $N = 2$, which is constant in the direction $(1, \cot\alpha)$, and such that $v(x,+\infty) = 1$, $v(x,-\infty) = 0$ for all x. In other words, $v(x,y) = v_0(-x\cos\alpha + y\sin\alpha)$ with $v_0(+\infty) = 1$ and $v_0(+\infty) = 1$. One concludes as above that $c\sin\alpha = c(f)$.

Case 2: $\tilde\phi(k(n)+1) - \tilde\phi(k(n)) = \phi_{1/2}(k(n)+1) - \phi_{1/2}(k(n)) \to -\cot\alpha$ as $n \to +\infty$. This case is treated similarly.

Notice that in both cases, the sequence $(k(n))$ may here be bounded. Actually, if it is, then the function u itself is planar, of the type $v_0(\pm x\cos\alpha + y\sin\alpha)$. But, even when $\alpha < \pi/2$, there is no contradiction, because the function ϕ in (3.2) was not assumed to be radially symmetric.

3) First observe that the conclusions of Theorem 3.3 and Proposition 3.5 hold. Fix now any unit direction $\tau = (\tau_x, \tau_y) \in \mathbb{R}^{N-1}\times\mathbb{R}$ such that $\tau_y < -\cos\alpha$. Choose a set of vectors $(\tau^1, \cdots, \tau^{N-1})$ such that $(\tau^1, \cdots, \tau^{N-1}, \tau)$ is an orthonormal basis of $\mathbb{R}^N$ and define the new cartesian coordinates $X_i = \tau^i\cdot(x,y)$ $(1 \le i \le N-1)$, $Y = \tau\cdot(x,y)$. Let us note $X = (X_1, \cdots, X_{N-1})$. The function $\tilde u(X,Y) = u(x,y)$ satisfies

$$\Delta\tilde u - c\tilde\tau\cdot\nabla\tilde u + f(\tilde u) = 0 \text{ in } \mathbb{R}^N = \{(X,Y),\ X \in \mathbb{R}^{N-1},\ Y \in \mathbb{R}\},$$

where $\tilde{\tau}$ is the constant vector $\tilde{\tau} = (\tau_y^1, \cdots, \tau_y^{N-1}, \tau_y)$. Besides, because of the assumptions made here in part 3), it is easy to see that there exists a Lipschitz function $X \mapsto \tilde{\phi}(X)$ such that $\tilde{u}$ satisfies (3.2) in the variables (X, Y) with the function $\tilde{\phi}$ (notice that the set $\{Y = \tilde{\phi}(X)\}$ is not necessarily equal to the set $\{y = \phi(x)\}$ but we can choose a real number R large enough such that $\{Y = \tilde{\phi}(X),\ |X| \geq R\}$ is a subset of $\{y = \phi(x)\}$). Theorem 2.6 can be applied in the variables (X, Y) and it implies that the function $\tilde{u}$ is increasing in Y. This means that the function u is decreasing in any unit direction τ such that $\tau_y < -\cos\alpha$.

Because of these monotonicity properties, the Lipschitz norm $\cot\alpha'$ (with $\alpha' \in (0, \pi/2]$) of the level graphs ϕ_λ of u is such that $\cot\alpha' \leq \cot\alpha$, namely $\alpha' \geq \alpha$. But

$$\sup_{x \in \mathbb{R}^{N-1}} |\phi_\lambda(x) - \phi(x)| < +\infty \text{ for all } \lambda \in (0,1),$$

because of (3.3) and (3.2). Since $|\hat{x}\cdot\nabla\phi(x)| \to \cot\alpha$ as $|x| \to +\infty$, it follows that $\cot\alpha' = \|\phi_\lambda\|_{Lip} \geq \cot\alpha$ for all $\lambda \in (0,1)$. Thus, $\alpha' = \alpha$ and $\|\phi_\lambda\|_{Lip} = \cot\alpha$ for all $\lambda \in (0,1)$.

Let us now prove the formula for the speed c. Call $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^{N-1}$. Since $|\hat{x} \cdot \nabla\phi| \to \cot\alpha$ as $|x| \to +\infty$, either $\hat{x} \cdot \nabla\phi \to \cot\alpha$ as $|x| \to +\infty$ in dimensions $N \geq 3$ (resp. $\phi'(x) \to \cot\alpha$ as $x \to +\infty$ in dimension $N = 2$), or $\hat{x} \cdot \nabla\phi \to -\cot\alpha$ as $|x| \to +\infty$ in dimensions $N \geq 3$ (resp. $\phi'(x) \to -\cot\alpha$ as $x \to +\infty$ in dimension $N = 2$). Assume here that the limit is $+\cot\alpha$ (the other $-\cot\alpha$ could be treated similarly). Consider the sequences $x_n = ne_1$, $y_n = \phi(ne_1)$ and define the functions

$$u_n(x, y) = u(x + x_n, y + y_n).$$

By standard elliptic estimates, up to extraction of some subsequence, the sequence (u_n) converges in $C^2_{loc}(\mathbb{R}^N)$ to a solution u_∞ of (1.1). We now claim that

$$\begin{cases} \liminf\limits_{y - x_1\cot\alpha \to +\infty,\ (x_2, \cdots, x_{N-1}) \in \mathbb{R}^{N-2}} u_\infty(x, y) = 1, \\ \limsup\limits_{y - x_1\cot\alpha \to -\infty,\ (x_2, \cdots, x_{N-1}) \in \mathbb{R}^{N-2}} u_\infty(x, y) = 0. \end{cases} \tag{3.13}$$

Let us prove the formula when $y + x_1 \cot\alpha \to +\infty$ (the proof of the other one is similar). Let $\varepsilon > 0$. Since u satisfies the asymptotic conditions (3.2), there exists a real number y_0 such that $u(x, y) \geq 1 - \varepsilon$ if $y \geq y_0 + \phi(x)$. Fix any point (x, y) such that $y \geq y_0 + 1 + x_1\cot\alpha$ with $(x_2, \cdots, x_{N-1}) \in \mathbb{R}^{N-2}$. From the finite increment theorem, we have that $\phi(x + x_n) - \phi(x_n) = \nabla\phi(x_n + t_n x)\cdot x$ with some $t_n \in [0, 1]$. From the assumption made on ϕ, and since $x_n = (n, 0, \cdots, 0)$, one gets that $\phi(x + x_n) - \phi(x_n) \to \cot\alpha\ x_1$ as $n \to +\infty$. This implies that

$$y + y_n = y + \phi(x_n) \geq y_0 + \phi(x + x_n) \text{ and } u_n(x, y) \geq 1 - \varepsilon \text{ for } n \text{ large enough.}$$

The limit as $n \to +\infty$ gives $u_\infty(x, y) \geq 1 - \varepsilon$. Therefore, $\inf_{y \geq y_0 + 1 + x_1\cot\alpha} u_\infty(x, y) \geq 1 - \varepsilon$, which implies the desired result.

In the new coordinates $X_1 = x_1 \sin\alpha + y\cos\alpha$, $X_2 = x_2, \cdots, X_{N-1} = x_{N-1}$, $Y = -x_1\cos\alpha + y\sin\alpha$, the function $\tilde{u}(X, Y) = u(x, y)$ satisfies the equation

$$\Delta\tilde{u} - c\cos\alpha\ \tilde{u}_{X_1} - c\sin\alpha\ \tilde{u}_Y + f(\tilde{u}) = 0 \text{ in } \mathbb{R}^N$$

together with $\liminf_{Y \to +\infty,\ X \in \mathbb{R}^{N-1}} \tilde{u}(X, Y) = 1$ and $\limsup_{Y \to -\infty,\ X \in \mathbb{R}^{N-1}} \tilde{u}(X, Y) = 0$ by (3.13). With the same arguments as in part 1), one concludes that the function $\tilde{u}$ depends on Y only, and then that $c\sin\alpha = c(f)$. That completes the proof of Theorem 3.8. $\square$

The next result is the follow-up of Theorem 3.8. It is concerned with the planar behaviour of the function u along its level sets, at infinity. Denote by sgn the function defined on $\mathbb{R}$ by $\text{sgn}(\xi) = -1$ if $\xi < 0$, $\text{sgn}(0) = 0$ and $\text{sgn}(\xi) = 1$ if $\xi > 0$. Under the assumption that f is nonincreasing in $[0, \delta]$ and in $[1 - \delta, 1]$ for some $\delta > 0$, we denote by $(c(f), U)$ the unique (if any) solution of (3.9). We also recal that

$$\text{sgn}(c(f)) = \text{sgn}\left(\int_0^1 f\right).$$

Proposition 3.9 *Under the same assumptions and notations as in Theorem 3.8, and assuming that $c(f) \neq 0$, one has:*

1bis) *In case* 1) *of Theorem 3.8, then*

$$\hat{x} \cdot \nabla\phi_\lambda(x) \to -\text{sgn}(c(f)) \cot\alpha \text{ as } |x| \to +\infty,$$

for all $\lambda \in (0, 1)$. Furthermore, for all unit vector $e \in \mathbb{R}^{N-1}$ and all $\lambda \in (0, 1)$,

$$u(x + re, y + \phi_\lambda(re)) \to U\left(\text{sgn}(c(f)) \cos\alpha\ x \cdot e + y \sin\alpha + U^{-1}(\lambda)\right) \text{ as } r \to +\infty, \text{ in } C^2_{loc}(\mathbb{R}^N).$$

2bis) *In case* 2) *of Theorem 3.8, then either u is a planar front $u(x, y) = U(\pm x \cos\alpha + y \sin\alpha + \tau)$ (for some $\tau \in \mathbb{R}$), or $\phi'_\lambda(x) \to \mp\text{sgn}(c(f)) \cot\alpha$ as $x \to \pm\infty$ for all $\lambda \in (0, 1)$ and*

$$u(x + r, y + \phi_\lambda(r)) \to U(\pm\text{sgn}(c(f))\ x \cos\alpha + y \sin\alpha + U^{-1}(\lambda)) \text{ as } r \to \pm\infty, \text{ in } C^2_{loc}(\mathbb{R}^2).$$

3bis) *In case* 3) *of Theorem 3.8: if $N \geq 3$, then the conclusion of part* 1bis) *holds and $\hat{x} \cdot \nabla\phi(x) \to -\text{sgn}(c(f)) \cot\alpha$ as $|x| \to +\infty$; if $N = 2$, then the conclusion of part* 2bis) *holds.*

Proof. 1bis) Let $\lambda \in (0, 1)$ be fixed. With the same notations as in Theorem 3.8, we first recall that $\|\phi_\lambda\|_{Lip} = \cot\alpha$ and $c = c(f)/\sin\alpha$. If $\alpha = \pi/2$, then, as already underlined, the function $u = u(y)$ is the unique solution U of (3.9), up to shift in y. The conclusion of Proposition 3.9 follows immediately. We then assume in the following that $0 < \alpha < \pi/2$. One has

$$-\cot\alpha \leq m := \liminf_{|x| \to +\infty} \ \hat{x} \cdot \nabla\phi_\lambda(x) \leq M := \limsup_{|x| \to +\infty} \ \hat{x} \cdot \nabla\phi_\lambda(x) \leq \cot\alpha.$$

We shall prove that $m = M = -\text{sgn}(c(f)) \cot\alpha$.

Assume first that $|M| < \cot\alpha$. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence such that $|x_n| \to +\infty$ and $\hat{x}_n \cdot \nabla\phi_\lambda(x_n) \to M$ as $n \to +\infty$, and call

$$u_n(x, y) = u(x + x_n, y + \phi_\lambda(x_n)) \text{ and } \phi_n(x) = \phi_\lambda(x + x_n) - \phi_\lambda(x_n).$$

From standard elliptic estimates, the functions u_n converge in $C^2_{loc}(\mathbb{R}^N)$, up to extraction of some subsequence, to a solution u_∞ of (1.1). One can also assume that $\hat{x}_n \to e \in \mathbb{R}^{N-1}$ as $n \to +\infty$. Furthermore, from the arguments in the proof of Proposition 3.5, the function ϕ_λ has bounded derivatives up to second order. Thus, the functions ϕ_n converge in $C^1_{loc}(\mathbb{R}^{N-1})$, up to extraction of a subsequence, to a C^1 function ϕ_∞ such that $\|\phi_\infty\|_{Lip} \leq \cot\alpha$,

$$\forall x \in \mathbb{R}^{N-1}, \quad -\cot\alpha \leq e \cdot \nabla\phi_\infty(x) \leq M,$$

and $e \cdot \nabla\phi_\infty(0) = M$. Furthermore, the function u_∞ solves (3.2) with ϕ_∞ and $u_\infty(x, \phi_\infty(x)) = \lambda$ for all $x \in \mathbb{R}^{N-1}$. Call

$$\gamma = \arctan(M).$$

The same arguments as in the proof of Case 1) of Theorem 3.8 imply that the function u_∞ can be written as a function of $x \cdot e$ and y only, namely

$$u_\infty(x,y) = v(x \cdot e, y).$$

Therefore, $\phi_\infty(x) = \tilde{\phi}(x \cdot e)$. On the other hand, since $-\cot\alpha = \tan(\alpha - \pi/2) \le \tilde{\phi}' \le M = \tan\gamma$, it then follows from Remark 3.4 that the function v is decreasing in any direction $(\cos\varphi, \sin\varphi)$ such that $\gamma - \pi < \varphi < \alpha - \pi/2$. By continuity,

$$\rho \cdot \nabla v \le 0 \text{ in } \mathbb{R}^2 \text{ for } \rho = (\cos(\gamma - \pi), \sin(\gamma - \pi)) = -(\cos\gamma, \sin\gamma).$$

But since $\tilde{\phi}'(0) = M = \tan\gamma$ and $\{y = \tilde{\phi}(x \cdot e)\}$ is a level curve of v, one concludes that $\rho \cdot \nabla v(0,0) = 0$. The nonpositive function $z := \rho \cdot \nabla v$ satisfies an elliptic equation with continuous coefficients, and $z(0,0) = 0$. It follows from the strong maximum principle that $z \equiv 0$ in $\mathbb{R}^2$. In other words, $v(x,y) = v_0(-x\sin\gamma + y\cos\gamma)$. By uniqueness for problem (3.9), one concludes, in both cases $\gamma \ge 0$ or $\gamma \le 0$, that

$$c\sin(\pi/2 - |\gamma|) = c(f) = c\sin\alpha. \tag{3.14}$$

But $-\cot\alpha < \tan\gamma < \cot\alpha$ from our assumption, whence $\pi/2 \ge \pi/2 - |\gamma| > \alpha$ (> 0). Since $c(f)$ is here assumed to be not zero, one gets a contradiction with (3.14).

Similarly, we cannot have $|m| < \cot\alpha$.

Therefore, either $m = M = -\cot\alpha$, or $m = M = \cot\alpha$, or $-\cot\alpha = m = -M$.

If $c(f) > 0$ and $m = M = \cot\alpha$, then $0 < u(x,y) < 1$ and $\limsup_{y\to-\infty,\ x\in\mathbb{R}^{N-1}} u(x,y) = 0$ because of (3.3). Furthermore, $0 < U(y) < 1$, $U(+\infty) = 1$ and

$$\Delta U - cU_y + f(U) = U'' - cU' + f(U) = (c(f) - c)U' < 0 \tag{3.15}$$

because $U' > 0$, $c = c(f)/\sin\alpha$, $\alpha \in (0, \pi/2)$ and $c(f) > 0$. Theorem 2.4 of Section 2 applied in $\mathbb{R}^N$ to $\underline{u} = u$, $\overline{u} = U$ and $\phi = 0$ yields the existence of $\tau^* \in \mathbb{R}$ such that $u(x,y) \le U(y + \tau^*)$ in $\mathbb{R}^N$ and $\inf_{x\in\mathbb{R}^{N-1}} U(\tau^*) - u(x,0) = 0$ –notice that the critical shift τ^* is finite because $U(-\infty) = 0$. If $u(x_0, 0) = U(\tau^*)$ for some $x_0 \in \mathbb{R}^{N-1}$, then $u(x,y) = U(y + \tau^*)$ from the string elliptic maximum principle. This is impossible because of the strict inequality in (3.15). Otherwise, $\limsup_{|x|\to+\infty} u(x,0) = U(\tau^*) > 0$, which is impossible because of (3.3) and the assumption that $m = M = \cot\alpha > 0$. Therefore, the case where $c(f) > 0$ and $m = M = \cot\alpha$ is ruled out.

Similarly, one can prove that the case where $c(f) < 0$ and $m = M = -\cot\alpha$ is ruled out too.

Assume now that $-\cot\alpha = m = -M$. Arguing as in the first part of the proof, there exists a sequence $(x_{1,n})_{n\in\mathbb{N}}$ in $\mathbb{R}^{N-1}$ such that

$$|x_{1,n}| \to +\infty,\ \hat{x}_{1,n} \cdot \nabla\phi_\lambda(x_{1,n}) \to M = \cot\alpha,\ \hat{x}_{1,n} \to e \in \mathbb{R}^{N-1}$$

and

$$u(x + x_{1,n}, y + \phi_\lambda(x_{1,n}) \to U(-(x \cdot e)\cos\alpha + y\sin\alpha + U^{-1}(\lambda)) \text{ in } C^2_{loc}(\mathbb{R}^N)$$

as $n \to +\infty$. Reminding that ϕ is radially symmetric and $\sup_{x\in\mathbb{R}^{N-1}} |\phi(x) - \phi_\lambda(x)| < +\infty$ because of (3.2) and (3.3), there is then a sequence $(\rho_{1,n})_{n\in\mathbb{N}} \in \mathbb{R}_+$ going to $+\infty$ such that

$$\sup_{n\in\mathbb{N}} \max_{|x|\le\rho_{1,n}} \phi(x + x_{1,n}) - \phi(x_{1,n}) - |x|\cot\alpha < +\infty.$$

Similarly, there are two sequences $(x_{2,n})_{n\in\mathbb{N}} \in \mathbb{R}^{N-1}$ and $(\rho_{2,n})_{n\in\mathbb{N}} \in \mathbb{R}_+$ such that $|x_{2,n}|$ and $\rho_{2,n}$ converge to $+\infty$, and

$$\sup_{n\in\mathbb{N}} \max_{|x|\le\rho_{2,n}} x + \phi(x_{2,n}) - \phi(x_{2,n}) + |x|\cot\alpha < +\infty.$$

Therefore, there are some sequences $(x_n)_{n\in\mathbb{N}}$, $(\tilde{x}_n)_{n\in\mathbb{N}} \in \mathbb{R}^{N-1}$ and $(R_n)_{n\in\mathbb{N}}$, $(\tilde{R}_n)_{n\in\mathbb{N}} \in \mathbb{R}_+$ such that $|x_n|$, $|\tilde{x}_n|$, R_n, $\tilde{R}_n$ converge to $+\infty$, and

$$\phi(x_n) = \min_{|x|\le R_n} \phi(x_n + x), \quad \phi(\tilde{x}_n) = \max_{|x|\le \tilde{R}_n} \phi(\tilde{x}_n + x).$$

Now, the functions $u(x + x_n, y + \phi(x_n))$ (resp. $u(x + \tilde{x}_n, y + \phi(\tilde{x}_n))$) converge, up to extraction of some subsequence, to a solution $0 < u_\infty < 1$ (resp. $0 < \tilde{u}_\infty < 1$) of (1.1) such that

$$\limsup_{y\to-\infty,\ x\in\mathbb{R}^{N-1}} u_\infty(x,y) = 0 \text{ (resp. } \liminf_{y\to+\infty,\ x\in\mathbb{R}^{N-1}} \tilde{u}_\infty(x,y) = 1).$$

If $c(f) > 0$, then, arguing as in the case where $m = M = \cot\alpha$, there exists $\tau^* \in \mathbb{R}$ such that $u_\infty(x,y) \le U(y+\tau^*)$ and $\inf_{x\in\mathbb{R}^{N-1}} U(\tau^*) - u_\infty(x,0) = 0$. Let $(x_{0,n})$ be a sequence in $\mathbb{R}^{N-1}$ such that $u_\infty(x_{0,n},0) \to U(\tau^*)$. Up to extraction of some subsequence, the functions $u_\infty(x + x_{0,n}, y)$ converge to a solution $u_{\infty,\infty}(x,y)$ of (1.1) such that $u_{\infty,\infty}(x,y) \le U(y+\tau^*)$ with equality at $(0,0)$. The strong maximum principle implies that $u_{\infty,\infty}(x,y) \equiv U(y+\tau^*)$, which is impossible because of the strict inequality in (3.15). Similarly, one gets a contradiction if $c(f) < 0$, by comparing $\tilde{u}_\infty$ and U.

All the above results imply that

$$m = M = -\mathrm{sgn}(c(f))\cot\alpha.$$

Choose now any unit vector $e \in \mathbb{R}^{N-1}$, any $\lambda \in (0,1)$ and any sequence $(r_n) \to +\infty$. As in the proof of part 1) of Theorem 3.8, one can prove that the functions $u(x + r_n e, y + \phi_\lambda(r_n e))$ converge, up to extraction of some subsequence, to $U\left(\mathrm{sgn}(c(f))\cos\alpha\ x\cdot e + y\sin\alpha + U^{-1}(\lambda)\right)$ in $C^2_{loc}(\mathbb{R}^N)$. By uniqueness of the limit, one concludes that the whole family $u(x + re, y + \phi_\lambda(re))$ converges to $U\left(\mathrm{sgn}(c(f))\cos\alpha\ x\cdot e + y\sin\alpha + U^{-1}(\lambda)\right)$ in $C^2_{loc}(\mathbb{R}^N)$ as $r \to +\infty$.

2bis) Let $\lambda \in (0,1)$ be given. With the same arguments as in the proof of case 1bis), one gets that $|\phi'_\lambda(x)| \to \cot\alpha$ as $|x| \to +\infty$. If $\phi'_\lambda(x) \to \cot\alpha$ as $|x| \to +\infty$, then, as in the beginning of the proof of Theorem 3.3, one can apply the monotonicity result (Theorem 2.6) of Section 2 and get that u is decreasing in any direction $(\cos\varphi, \sin\varphi)$ such that $-\pi/2 - \alpha < \varphi < \pi/2 - \alpha$. By passing to the limit $\varphi \to -\pi/2 - \alpha$ and $\varphi \to \pi/2 - \alpha$, one concludes that u is constant in the direction $(\sin\alpha, \cos\alpha)$. In other words, u is a planar solution of the type $v_0(-x\cos\alpha + y\sin\alpha)$ such that $v_0(-\infty) = 0$ and $v_0(+\infty) = 1$. By uniqueness for (3.9), v_0 is then a translte of U. If $\phi'_\lambda(x) \to -\cot\alpha$, one can prove similarly that u is then a translate of $U(x\cos\alpha + y\sin\alpha)$.

The case where $c(f) > 0$ and $\phi'_\lambda(x) \to \pm\cot\alpha$ as $x \to \pm\infty$ can be ruled out, as was the case where $c(f) > 0$ and $m = M = \cot\alpha$ in 1bis). Similarly, the case where $c(f) < 0$ and $\phi'_\lambda(x) \to \mp\cot\alpha$ as $x \to \pm\infty$ can be ruled out.

One concludes that $\phi'_\lambda(x) \to \mp\mathrm{sgn}(c(f))\cot\alpha$ as $x \to \pm\infty$ if u is not planar. The planar behavior of u along its level sets follows as in 1bis).

3bis) First, as already underlined in Theorem 3.8, u only depends on y if $\alpha = \pi/2$, and the conclusion holds. Assume in the following that $\alpha \in (0, \pi/2)$. If $N \ge 3$, then $\hat{x}\cdot\nabla\phi(x)$ has a constant sign for $|x|$ large. The case where $c(f) > 0$ and $\hat{x}\cdot\nabla\phi(x) \to \cot\alpha$ as $|x| \to +\infty$ can be ruled out, as was the case where $c(f) > 0$ and $m = M = \cot\alpha$ in 1bis). Similarly, the case where $c(f) < 0$ and $\hat{x}\cdot\nabla\phi(x) \to -\cot\alpha$ as $|x| \to +\infty$ can be ruled out. Therefore, $\hat{x}\cdot\nabla\phi(x) \to -\mathrm{sgn}(c(f))\cot\alpha$ as $|x| \to +\infty$. The arguments used in Theorem 3.8 and in case 1bis) above can be applied and the conclusion of part 1bis) follows.

If $N = 2$, then one can argue as in case 2bis) and the conclusion follows. $\square$

If the nonlinearity f is of the combustion type (1.2), then problem (3.9) does have a (unique) solution, and the speed $c(f)$ is positive. Therefore, for the model presented in Section 1, we can see that the speed $c = c(f)/\sin\alpha$ of the non-planar flame (for $\alpha < \pi/2$) is greater than the speed $c_0 = c(f)$ of the planar flame. Furthermore, the angle α is all the smaller as the speed c is larger. That is physically meaningful since the curvature of the flame tip increases with the speed of the outgoing fuel flow. It is worth noticing that formula (3.10) has been known for a long time and had been formally derived from the planar behavior of the flame, far away from its centre, along the directions $(\pm\sin\alpha, -\cos\alpha)$. This formula had been used in experiments to find the planar speed $c_0 = c(f)$: indeed, the vertical speed c of the gases at the exit of the Bunsen burner being known, one can measure the angle α and the one-dimensional speed $c_0 = c(f)$ is then given by the formula $c(f) = c\sin\alpha$ (*see* [16], [38], [51], [55]).

Furthermore, one can easily derive heuristically this formula (3.10) from the asymptotic planar behavior of the solution along its level sets. Indeed, if the medium were quiescent, the flame front would move with speed c downwards and with speed $c(f)$ in the directions which are asymptotically orthogonal to its level sets (*see* Figure 1). Since the angle between the vertical direction and the level sets is asymptotically equal to α, the speed $c(f)$ is then nothing else than the projection of the speed c on the directions which are orthogonal to the level sets.

To complete this subsection, we prove the formula (3.10) for the speed c under some conditions which are somehow weaker than those of Theorem 3.8, in the sense that we assume that the limits 0 and 1 are satisfied in some strict sub-cones only.

Theorem 3.10 *Assume that $f \in C^1([0,1])$ is nonincreasing in $[0,\delta]$ and $[1-\delta,1]$, negative in $(0,\delta]$ and positive in $[1-\delta,1)$, for some $\delta > 0$. Assume that there exists a, unique, solution $(c(f),U)$ of problem (3.9). Let $0 \le u \le 1$ be a solution of (1.1) such that*

$$\left\{\begin{array}{lll}\forall\ \beta\in(\alpha,\pi), & \displaystyle\liminf_{y+|x|\cot\beta\to+\infty} & u(x,y)=1\\ \forall\ \beta\in(0,\alpha), & \displaystyle\limsup_{y+|x|\cot\beta\to-\infty} & u(x,y)=0\end{array}\right. \tag{3.16}$$

for some $\alpha \in (0,\pi)$. Then $c = c(f)/\sin\alpha$.

Here, in (3.16), the limits are only uniform in strict subcones, while they were uniform far above or below a given graph in Theorem 3.8. In particular, one does not know a priori whether, given $a \le b \in (0,1)$, the region $\{a \le u \le b\}$ has a finite width or not in the y direction.

Remark 3.11 *The arguments used in the proof of Lemma 3.2 imply that the conditions (3.16) are immediately satisfied if one only assumes that $\liminf_{y+|x|\cot\beta\to+\infty}\ u(x,y) > \theta_2$ for all $\beta \in (\alpha,\pi)$ and $\limsup_{y+|x|\cot\beta\to-\infty}\ u(x,y) < \theta_1$ for all $\beta \in (0,\alpha)$, where $0 < \theta_1 \le \theta_2 < 1$ and f is assumed to be negative in $(0,\theta_1)$ and positive in $(\theta_2,1)$.*

Let us now turn to the proof of Theorem 3.10.

Proof. Assume first that $N = 2$ and call

$$\gamma = 1 - \frac{\delta}{2}.$$

Fix temporarily $n \in \mathbb{N}$, and an angle β such that

$$0 < \beta < \min(\alpha, \pi - \alpha).$$

By (3.16), we have $u(x,y) \to 1$ as $y \to +\infty$ and $u(x,y) \to 0$ as $y \to -\infty$ for any $x \in \mathbb{R}$. Since u is continuous, we can therefore define the functions

$$\phi_-(x) = \min\left\{y \in \mathbb{R},\ u(x,y) = \frac{\delta}{2}\right\}, \quad \text{and } \phi_+(x) = \max\{y \in \mathbb{R},\ u(x,y) = \gamma\}.$$

For any $x_0 \in \mathbb{R}$, let us define the set

$$\begin{array}{rcl} \mathcal{A}_{x_0} & = & \{(x,y) \in \mathbb{R}^2,\ x_0 - n \le x \le x_0,\ y \ge \phi_+(x_0) + \cot(\alpha-\beta)|x - x_0|\} \\ & \cup & \{(x,y) \in \mathbb{R}^2,\ x_0 \le x \le x_0 + n,\ y \ge \phi_+(x_0) - \cot(\alpha+\beta)|x - x_0|\}. \end{array}$$

We claim that there exists $x_n \ge n/2$ such that

$$u(x,y) \ge 1 - \delta \text{ for all } (x,y) \in \mathcal{A}_{x_n}.$$

Assume not. First, because of the first assumption in (3.16) applied to any angle in (α, π), there exists $y_0' \in \mathbb{R}$ such that

$$\forall\ |x| \le n/2,\ \forall y \ge y_0' - \cot(\alpha-\beta)x, \quad u(x,y) > \gamma. \tag{3.17}$$

Besides, by the second assumption in (3.16), applied to the angle $\alpha - \beta/2$, we have $u(x, y_0' - \cot(\alpha - \beta)x) \to 0$ as $x \to +\infty$. Hence, there exists a real number $x_0 > n/2$ such that $u(x_0, y_0' - \cot(\alpha - \beta)x_0) \le \gamma$, whence

$$y_0 := \phi_+(x_0) \ge y_0' - \cot(\alpha-\beta)x_0. \tag{3.18}$$

Because of our assumption on the sets $\mathcal{A}_x$ for $x \ge n/2$, there exists then a point (x_1, y_1') in $\mathcal{A}_{x_0}$ such that $u(x_1, y_1') < 1 - \delta$. By definition of $\mathcal{A}_{x_0}$ and of $\phi_+(x_1)$, it follows that the point $(x_1, y_1) := (x_1, \phi_+(x_1))$ is in $\mathcal{A}_{x_0}$. Now, if $x_1 \ge x_0$, then $x_1 > n/2$. If $x_1 \le x_0$, then

$$y_1 \ge \phi_+(x_0) + \cot(\alpha-\beta)\ |x_1 - x_0| = \phi_+(x_0) + \cot(\alpha-\beta)\ (x_0 - x_1) \ge y_0' - \cot(\alpha-\beta)x_1$$

because of (3.18). Thus, $|x_1| > n/2$ because of (3.17) and $u(x_1, y_1) = \gamma$. But $|x_1 - x_0| \le n$ and $x_0 > n/2$. Therefore, $x_1 > n/2$. On the other hand, since $u(x_0, y) \ge \gamma = 1 - \delta/2$ for any $y \ge y_0$ and since $(x_1, y_1') \in \mathcal{A}_{x_0}$ satisfies $u(x_1, y_1') < 1 - \delta$, it easily follows from the definition of $\mathcal{A}_{x_0}$ and Finite Increment Theorem that

$$|x_1 - x_0| \ge \eta := \frac{\delta}{2} \times (\|\nabla u\|_\infty \sin(\alpha - \beta))^{-1} > 0,$$

where $\|\nabla u\|_\infty = \sup_{(x,y)\in\mathbb{R}^2} |\nabla u(x,y)| < +\infty$. By induction, there exists a sequence of points $(x_k, y_k) = (x_k, \phi_+(x_k))$ such that $(x_k, y_k) \in \mathcal{A}_{x_{k-1}}$, $x_k > n/2$ and $|x_k - x_{k-1}| \ge \eta$ for all $k \in \mathbb{N}^*$. Since $|x_k - x_{k-1}| \ge \eta > 0$ and $x_k > n/2$ for all k, there is an infinite number of k's such that $x_k > x_{k-1}$. For such k's, we actually have $x_k \ge x_{k-1} + \eta$ and

$$\begin{array}{rcl} y_k & \ge & y_{k-1} - \cot(\alpha+\beta)\ |x_k - x_{k-1}| \\ & = & y_{k-1} - \cot(\alpha+\beta)\ (x_k - x_{k-1}) \\ & \ge & y_{k-1} + \eta' - \cot\left(\alpha + \dfrac{\beta}{2}\right)(x_k - x_{k-1}), \end{array}$$

where $\eta' = [\cot(\alpha + \beta/2) - \cot(\alpha+\beta)]\eta > 0$. On the other hand, if $x_k < x_{k-1}$, we have

$$y_k \ge y_{k-1} + \cot(\alpha-\beta)\ |x_k - x_{k-1}| \ge y_{k-1} - \cot\left(\alpha + \frac{\beta}{2}\right)\ (x_k - x_{k-1})$$

because $(x_k, y_k) \in \mathcal{A}_{x_{k-1}}$. Call $N(k)$ the number of l's in $\{1, \cdots, k\}$ such that $x_l > x_{l-1}$. By an immediate induction, we deduce that

$$y_k \geq y_0 + \eta' N(k) - \cot\left(\alpha + \frac{\beta}{2}\right)(x_k - x_0).$$

Hence, since we noticed that $N(k) \to +\infty$ as $k \to +\infty$, it follows that $y_k + \cot(\alpha + \beta/2)x_k \to +\infty$ as $k \to +\infty$. Since each x_k is nonnegative, the first assumption in (3.16) implies that $u(x_k, y_k) \to 1$ as $k \to +\infty$, which is impossible because $u(x_k, y_k) = \gamma = 1 - \delta/2$.

Therefore, for each $\beta \in (0, \min(\alpha, \pi - \alpha))$, there exists $x_n = x_n^\beta \geq n/2$ such that $u(x, y) \geq 1 - \delta$ for all (x, y) in $\mathcal{A}_{x_n}$. Let now $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers in $(0, \min(\alpha, \pi - \alpha))$ such that $\beta_n \to 0$ as $n \to +\infty$. For each $n \in \mathbb{N}$, there exists a real x_n such that $u(x, y) \geq 1 - \delta$ for all $(x, y) \in \mathcal{A}_{x_n}$, where $\mathcal{A}_{x_n}$ is defined with the angle β_n. Set $y_n = \phi_+(x_n)$. We have $u(x_n, y_n) = \gamma = 1 - \delta/2$ for all n. Define the functions

$$u_n(x, y) = u(x + x_n, y + y_n) \text{ in } \mathbb{R}^2.$$

Up to extraction of some subsequence, the functions u_n converge in $C^2_{loc}(\mathbb{R}^2)$ to a solution $0 \leq \overline{u} \leq 1$ of (1.1) such that $\overline{u}(0, 0) = \gamma$ and

$$\forall\, x \in \mathbb{R},\ \forall\, y \geq -x\cot\alpha, \quad \overline{u}(x, y) \geq 1 - \delta.$$

The same arguments as in the proof of Lemma 3.2, using the positivity of f in $[1 - \delta, 1)$, imply that

$$\liminf_{y+x\cot\alpha\to+\infty} \overline{u}(x, y) = 1.$$

Assume now that $c < c(f)/\sin\alpha$, where $(c(f), U)$ denotes the unique solution of (3.9). The function $\underline{u}(x, y) = U(x\cos\alpha + y\sin\alpha)$ then satisfies

$$\Delta\underline{u} - c\underline{u}_y + f(\underline{u}) = (c(f) - c\sin\alpha)U'(x\cos\alpha + y\sin\alpha) > 0 \tag{3.19}$$

because of our assumption and $U' > 0$. Furthermore, $0 \leq \underline{u} \leq 1$ and $\limsup_{y+x\cot\alpha\to-\infty} \underline{u}(x, y) = 0$. Since $U(+\infty) = 1$ and $\overline{u}(0, 0) = 1 - \delta \in (0, 1)$, one concludes from Theorem 2.4 that there exists $\tau^* \in \mathbb{R}$ such that

$$\underline{u}(x, y) = U(x\cos\alpha + y\sin\alpha) \leq \overline{u}(x, y + \tau^*)$$

and $\inf_{y=-x\cot\alpha,\ x\in\mathbb{R}} \overline{u}(x, y + \tau^*) - U(0) = 0$. Therefore, there exists a sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ such that $\overline{u}(x_n, -x_n\cot\alpha + \tau^*) \to U(0)$ and such that the functions $\overline{u}(x + x_n, y - x_n\cot\alpha)$ converge to a solution $0 \leq \overline{u}_\infty \leq 1$ of (1.1) such that

$$\overline{u}_\infty(x, y + \tau^*) \geq U(x\cos\alpha + y\sin\alpha)$$

and $\overline{u}_\infty(0, 0) = U(0)$. Since $U(x\cos\alpha + y\sin\alpha)$ is a subsolution of (1.1), the strong maximum principle implies that $\overline{u}_\infty(x, y + \tau^*) \equiv U(x\cos\alpha + y\sin\alpha)$ in $\mathbb{R}^2$. This is impossible because of the strict inequality in (3.19).

As a consequence, one has proved that $c \geq c(f)/\sin\alpha$. Similarly, by considering some sets of the type

$$\begin{aligned} \mathcal{A}'_{x_0} \quad = \quad & \{(x, y) \in \mathbb{R}^2,\ x_0 - n \leq x \leq x_0,\ y \leq \phi_-(x_0) + \cot(\alpha + \beta)|x - x_0|\} \\ \cup \quad & \{(x, y) \in \mathbb{R}^2,\ x_0 \leq x \leq x_0 + n,\ y \leq \phi_-(x_0) - \cot(\alpha - \beta)|x - x_0|\}, \end{aligned}$$

passing to the limit and arguing by contradiction, it follows that $c \leq c(f)/\sin\alpha$.

As a conclusion, the formula $c = c(f)/\sin\alpha$ holds in dimension $N = 2$.

Consider now the general case $N \geq 3$. Let SO_{N-1} be the group of rotations in $\mathbb{R}^{N-1}$. For any $\rho \in SO_{N-1}$, the function

$$u_\rho(x, y) = u(\rho(x), y)$$

is also a solution of (1.1) in $\mathbb{R}^N$. Since the function u is globally lipschitz-continuous in $\mathbb{R}^N$, the function

$$v(x, y) = \min_{\rho \in SO_{N-1}} u_\rho(x, y)$$

is globally lipschitz-continuous as well and it satisfies $\Delta v - v_y + f(v) \leq 0$ in $\mathbb{R}^N$ in the sense of distributions. By definition of v, there exists then a globally lipschitz-continuous function $\tilde{v}$ defined in $\mathbb{R}_+ \times \mathbb{R}$ such that $v(x, y) = \tilde{v}(r, y)$ where $r = |x|$. Define $w(x, y) = \tilde{v}(|x|, y)$ for $(x, y) \in \mathbb{R}^2$. The function w is globally lipschitz-continuous in $\mathbb{R}^2$ and it solves

$$\Delta w + \frac{N-2}{x} w_x - c w_y + f(w) \leq 0 \text{ in } \mathbb{R}^* \times \mathbb{R}$$

in the sense of distributions. Furthermore, since the function u fulfills the asymptotic conditions (3.16) in $\mathbb{R}^N$, it is easy to see that the function w satisfies the analogous conditions (3.16) in $\mathbb{R}^2$.

Henceforth, with the same arguments as above, for any sequence $\beta_n \to 0$, $\beta_n \in (0, \min(\alpha, \pi - \alpha))$, there exists a sequence of points $(x_n, y_n) \in \mathbb{R}^2$ such that $x_n \geq n/2$, $y_n = \max\{y \in \mathbb{R},\ w(x_n, y) = 1 - \delta/2\}$ and $w \geq 1 - \delta$ in $\mathcal{A}_{x_n}$, where

$$\begin{array}{rcl} \mathcal{A}_{x_n} & = & \{(x, y) \in \mathbb{R}^2,\ x_n - n \leq x \leq x_n,\ y \geq y_n + \cot(\alpha - \beta_n)\ |x - x_n|\} \\ & \cup & \{(x, y) \in \mathbb{R}^2,\ x_n \leq x \leq x_n + n,\ y \geq y_n - \cot(\alpha + \beta_n)\ |x - x_n|\}. \end{array}$$

Since the function w is globally lipschitz-continuous in $\mathbb{R}^2$, it follows from Arzela-Ascoli's theorem that the functions $u_n(x, y) = w(x + x_n, y + y_n)$ converge locally uniformly in $\mathbb{R}^2$ to a lipschitz-continuous function $0 \leq \overline{u} \leq 1$, up to extraction of some subsequence. We have $\overline{u}(0, 0) = 1 - \delta/2$ and $\overline{u}(x, y) \geq 1 - \delta$ for all $y \geq -x\cot\alpha$. Since $x_n \to +\infty$ and w is globally lipschitz-continuous, the terms $\frac{N-2}{x+x_n} w_x(x + x_n, y + y_n)$ converge locally to 0. Hence, in the sense of distributions, the function $\overline{u}$ satisfies

$$\Delta\overline{u} - c\overline{u}_y + f(\overline{u}) \leq 0 \text{ in } \mathbb{R}^2.$$

By adapting the arguments of Section 2 and those used above, one gets that

$$\liminf_{y + x\cot\alpha \to +\infty} \overline{u}(x, y) = 1,$$

$\overline{u}(x, y + \tau^*) \geq U(x\cos\alpha + y\sin\alpha)$ and $\inf_{x\in\mathbb{R}} \overline{u}(x, -x\cot\alpha + \tau^*) = U(0)$, for some $\tau^* \in \mathbb{R}$. Up to extraction of another subsequence, one would then get a contradiction if one had assumed that $c < c(f)/\sin\alpha$. Therefore, $c \geq c(f)/\sin\alpha$. One can get the other inequality similarly. That completes the proof of Theorem 3.10. □

3.3 Global curvature of conical fronts

The results of the previous subsection lead to the following

Theorem 3.12 1) *Assume that $f \in C^1([0,1])$ is nonincreasing in $[0,\delta]$ and $[1-\delta,1]$ for some $\delta > 0$. Let $0 \le u \le 1$ be a solution of (1.1). If (3.2) is satisfied for some Lipschitz function $\phi : \mathbb{R}^{N-1} \to \mathbb{R}$ which is of class C^1 for large $|x|$ and such that*

$$\hat{x} \cdot \nabla\phi(x) \to -\cot\alpha \quad \textit{as } |x| \to +\infty, \quad \textit{for some } \alpha \in \Big(0, \frac{\pi}{2}\Big) \cup \Big(\frac{\pi}{2}, \pi\Big),$$

then

$$\operatorname{sgn}\Big(\frac{\pi}{2} - \alpha\Big) = \operatorname{sgn}(c(f)).$$

2) *Under the assumptions of Theorem 3.10 with $\alpha \ne \pi/2$, then* $\operatorname{sgn}(\pi/2-\alpha) = \operatorname{sgn}(c(f))$.

Proof. To prove part 2), assume for instance that $\alpha > \pi/2$ and $c(f) \ge 0$. The function $U(y)$ is then a supersolution of (1.1) since

$$\Delta U - cU_y + f(U) = (c(f) - c(f)/\sin\alpha)U' \le 0 \text{ in } \mathbb{R}^N.$$

Furthermore, $U(+\infty) = 1$ and $\limsup_{y \to -\infty,\, x \in \mathbb{R}^{N-1}} u(x,y) = 0$ because of (3.16) and $\alpha > \pi/2$. Therefore, comparing u and U and applying Theorem 2.4 (as in the case where $c(f) > 0$ and $m = M = \cot\alpha$ in part 1bis) of Proposition 3.9), it follows that

$$U(y + \tau^*) \ge u(x,y) \text{ in } \mathbb{R}^2,$$

for some $\tau^* \in \mathbb{R}$, together with $\inf_{x \in \mathbb{R}^{N-1}} U(\tau^*) - u(x,0) = 0$. If $U(y+\tau^*)$ and u touch somewhere, they will then be equal from the strong maximum principle. This is impossible since u cannot be independent of x because of (3.16) and $\alpha > \pi/2$. Thus, $\limsup_{|x| \to +\infty} u(x,0) = U(\tau^*) > 0$, this is again impossible because of (3.16). One has then reached a contradiction. Similarly, the case where $\alpha < \pi/2$ and $c(f) \le 0$ can be ruled out.

Part 1) can be proved the same way as part 2), and the proof of Theorem 3.12 is complete. □

This result shows that the global curvature of the conical fronts solving (1.1) is related to the sign of the speed $c(f)$. For instance, in Figure 1, for a combustion nonlinearity f satisfying (1.2), one has $c_0 = c(f) > 0$, and then the angle α cannot be larger than or equal to $\pi/2$. Thus, despite its simplicity, the mathematical model which was presented in Section 1 to describe premixed Bunsen flames is robust enough and physically meaningful: there is no flame which points inside the Bunsen burner. Using the terminology of Haragus and Scheel [32], there is no exterior corner for (1.1) if $c(f) > 0$.

3.4 Uniqueness of the profile of the fronts, and a non-existence result in dimensions $N \ge 3$, under conditions (1.3)

In this subsection, we focus on the solutions (c,u) of (1.1) under the conditions (1.3), for some angle $\alpha \in (0,\pi)$. Conditions (1.3) are of the type (3.2) with, for instance, $\phi(x) = -|x|\cot\alpha$. Therefore, all results in the previous subsections hold under the conditions (1.3). In the sequel, we call $(c(f),U)$ the unique (if any) solution of (3.9) under the assumption that f is nonincreasing in $[0,\delta]$ and $[1-\delta,1]$ for some $\delta > 0$.

However, it turns out that there is a main difference between dimension $N = 2$ and higher dimensions $N \ge 3$. In Section 4 below, we state some existence results in dimension $N = 2$ under the conditions (1.3), for some special nonlinearities. Here, we prove the uniqueness and evenness (in x) of the solutions up to shift in dimension $N = 2$, as well as a non-existence result in dimensions $N \ge 3$ provided $\alpha \ne \pi/2$.

Theorem 3.13 *Assume that $f \in C^1([0,1])$ is nonincreasing in $[0,\delta]$ and $[1-\delta,1]$ for some $\delta>0$. Let $0 \le u \le 1$ be a solution of (1.1) in dimension $N=2$, and satisfying (1.3) for some $\alpha \in (0,\pi)$. Then, there exists $x_0 \in \mathbb{R}$ such that $u(x_0+x,y)=u(x_0-x,y)$ for all $(x,y)\in\mathbb{R}^2$ and*

$$\mathrm{sgn}(u_x(x,y)) = \mathrm{sgn}(\pi/2-\alpha) \text{ for all } x > x_0 \text{ and } y \in \mathbb{R}.$$

Furthermore, there are two real numbers $\tau_\pm \in \mathbb{R}$ such that $u(x+r, y-|r|\cot\alpha) \to U(\pm x\cos\alpha + y\sin\alpha+\tau_\pm)$ in $C^2_{loc}(\mathbb{R}^2)$ as $r\to\pm\infty$. Lastly, if u_1 and u_2 are two such solutions, then there exists $(a,b)\in\mathbb{R}^2$ such that $u_1(x,y)=u_2(x+a,y+b)$ for all $(x,y)\in\mathbb{R}^2$.

Proof. As already underlined, the results of the previous sections apply. In particular, from Theorems 3.3 and 3.8, $0<u<1$ in $\mathbb{R}^2$, and u is decreasing in any unit direction $(\tau_x,\tau_y)\in\mathbb{R}^2$ such that $\tau_y < -|\cos\alpha|$ and all level graphs ϕ_λ of u have their Lipschitz norm equal to $|\cot\alpha|$ (consider both cases $\alpha \le \pi/2$ and $\alpha \ge \pi/2$). Moreover, problem (3.9) then has a solution, and $c = c(f)/\sin\alpha$. Notice that, if $\alpha=\pi/2$, then u only depends on y and it is then equal, up to translation, to the unique solution U of (3.9). All conclusions of Theorem 3.13 follow in this case.

One can then assume in the sequel of the proof that $\alpha \neq \pi/2$. Let us now prove the limiting behavior along the directions $(\pm\sin\alpha,-\cos\alpha)$. Consider first the case where $\alpha<\pi/2$. By continuity, u is nonincreasing in the two directions $(\pm\sin\alpha,-\cos\alpha)$. Therefore, there exist two functions $U_\pm : \mathbb{R}\to[0,1]$ such that

$$u(x+r,y-|r|\cot\alpha) \to U_\pm(\pm x\cos\alpha + y\sin\alpha)$$

as $r\to\pm\infty$. From standard elliptic estimates, the convergence holds in $C^2_{loc}(\mathbb{R}^2)$ and the limiting functions satisfy (1.1). Furthermore, it follows from (1.3) that $U_\pm(+\infty)=1$ and $U_\pm(-\infty)=0$. In other words, the functions $U_\pm$ solve (3.9) with the speed $c\sin\alpha$. From the uniqueness result for problem (3.9), one gets that there exist two real numbers $t_\pm$ such that $U_\pm(s)=U(s+t_\pm)$ for all $s\in\mathbb{R}$.

The proof of the asymptotic behavior of u in the directions $(\pm\sin\alpha,-\cos\alpha)$ is similar in the case $\alpha>\pi/2$.

Let us now prove the evenness of the solutions in the variable x, up to shift. One first considers the case $\alpha<\pi/2$. Under the previous notations, call

$$x_0 = \frac{t_- - t_+}{2\cos\alpha}$$

and let us show that u is symmetric with respect to the line $\{x=x_0\}$. Let $a<x_0$ be fixed and define

$$H=\{(x,y)\in\mathbb{R}^2,\ x<a\} \text{ and } v(x,y)=u(2a-x,y).$$

The function v is obtained from u by symmetry with respect to the line $\partial H=\{x=a\}$. One shall now prove that $u>v$ in H. From the limiting conditions (1.3), there exists $A>0$ such that $u\ge 1-\delta$ in $H\cap\{y>A-|x|\cot\alpha\}$ and $v\le\delta$ in $H\cap\{y<-A-|x|\cot\alpha\}$, where $\delta>0$ was chosen so that f is nonincreasing in $(-\infty,\delta]$ and $[1-\delta,+\infty)$ (extend f by 0 outside the interval $[0,1]$). One can asusme that $\delta\le 1/2$ without loss of generality. Call now

$$u^\tau(x,y)=u(x,y+\tau)$$

and choose any $\tau\ge 2A$. Notice that $u^\tau(a,y)>v(a,y)$ for all $y\in\mathbb{R}$ since $u_y>0$ in $\mathbb{R}^2$ and $\tau>0$. Observe also that $u^\tau\ge u$ on $\partial H\cap\{y=-A-|x|\cot\alpha\}$ since $\tau\ge 2A$. Since both u and v satisfy

(1.1) and (1.3), it is easy to check that Theorem 2.1 of Section 2 can be applied to $(\underline{u}, \overline{u}) = (v, u^\tau)$ in the set $\Omega_- = \{x < a,\ y < -A - |x| \cot\alpha\}$. Therefore,

$$v \le u^\tau \text{ in } \overline{\Omega_-}.$$

Similarly, one has $v \le u^\tau$ in the set $\{x \le a,\ y \ge -A - |x| \cot\alpha\}$. As a consequence, $v \le u^\tau$ in $\overline{H}$ for all $\tau \ge 2A$.

Call now

$$\tau^* = \inf\ \{\tau > 0,\ v \le u^\tau \text{ in } \overline{H}\}$$

and assume that $\tau^* > 0$. By continuity, the function $z = u^{\tau^*} - v$ is nonnegative in H. Furthermore, it is positive on ∂H (since $u_y > 0$ in $\mathbb{R}^2$ and $\tau^* > 0$) and it satisfies an equation of the type

$$\Delta z - c\partial_y z + b(x, y)z = 0 \text{ in } H$$

for some bounded function b. The strong maximum principle yields $z > 0$ in $\overline{H}$. On the other hand, it follows from the previous results that

$$z(x + r, y - |r| \cot\alpha) \to U(-x\cos\alpha + y\sin\alpha + t_- + \tau^*) - U(-x\cos\alpha + y\sin\alpha + 2a\cos\alpha + t_+)$$

in $C^2_{loc}(\mathbb{R}^2)$ as $r \to -\infty$. The assumption made on a means that $2a\cos\alpha + t_+ < t_- < t_- + \tau^*$ (one here uses the positivity of $\cos\alpha$, because $0 < \alpha < \pi/2$). Since U is increasing, one especially gets that

$$\liminf_{y_0\in[y_1,y_2],\ x\to-\infty} z(x, y_0 - |x| \cot\alpha) > 0 \text{ for all } y_1 \le y_2\mathbb{R}.$$

It then follows that there exists $\tau_* \in (0, \tau^*)$ such that, for all $\tau \in [\tau_*, \tau^*]$,

$$u^\tau > v \text{ in } \{x \le a,\ -A - |x| \cot\alpha \le y \le A - |x| \cot\alpha\} \text{ and on } \partial H.$$

Let τ be any real number in $[\tau_*, \tau^*]$. Since u is increasing in y, it follows that

$$u^\tau \ge 1 - \delta \text{ in } \{x \le a,\ y \ge A - |x| \cot\alpha\}$$

and Theorem 2.1 yields

$$v \le u^\tau \text{ in the set } \{x \le a,\ y \ge A - |x| \cot\alpha\}.$$

Similarly, $v \le u^\tau$ in the set $\{x \le a,\ y \le -A - |x| \cot\alpha\}$. One concludes that $v \le u^\tau$ in $\overline{H}$ for all $\tau \in [\tau_*, \tau^*]$, which contradicts the minimality of τ^*.

As a consequence, $\tau^* = 0$ and $v \le u$ in $\overline{H}$. Call $w = u - v$. The function w is nonnegative in $\overline{H}$ and it vanishes on ∂H. Furthermore,

$$w(x + r, y - |r| \cot\alpha) \to U(-x\cos\alpha + y\sin\alpha + t_-) - U(-x\cos\alpha + y\sin\alpha + 2a\cos\alpha + t_+) > 0$$

in $C^2_{loc}(\mathbb{R}^2)$ as $r \to -\infty$ (the positivity of the limit holds since U is increasing and $2a\cos\alpha + t_+ < t_-$). The strong maximum principle and Hopf lemma imply that $w > 0$ in H and $w_x < 0$ on ∂H, whence

$$u(x, y) > u(2a - x, y) \text{ for all } x < a \text{ and } y \in \mathbb{R}, \quad u_x(a, y) < 0 \text{ for all } y \in \mathbb{R},$$

for all $a < x_0$.

Similarly, by using the same sliding method in y, one can prove that, for all $a' > x_0$,

$$u(x, y) < u(2a' - x, y) \text{ for all } x < a' \text{ and } y \in \mathbb{R}, \quad u_x(a', y) > 0 \text{ for all } y \in \mathbb{R}.$$

Passing to the limits $a \to x_0$ (from below) and $a' \to x_0$ (from above) yields that $u(x_0+x,y) = u(x_0-x,y)$ for all $(x,y) \in \mathbb{R}^2$.

The case where $\alpha > \pi/2$ can be treated with the same type of arguments: the function u is symmetric with respect to some line $\{x = x_0\}$, but this time one has $u(x,y) < u(2a-x,y)$ for all $x < a < x_0$ and $u(x,y) > u(2a'-x,y)$ for all $x < a'$ and $a' > x_0$.

Let us now show the uniqueness of the solutions u, up to shift. Let u_1 and u_2 be two such solutions. First, up to shift in x, one can then assume that u_1 and u_2 are both even in x. Therefore,

$$u_i(x+r, y-|r|\cot\alpha) \to U(\pm x\cos\alpha + y\sin\alpha + t_i)$$

locally in (x,y) as $r \to \pm\infty$, for some $t_i \in \mathbb{R}$, $i = 1,\ 2$. Up to shift in y, one can assume that $t_1 = t_2 = 0$. Let us now show that $u_1 = u_2$. Call $u_2^t(x,y) = u_2(x,y+t)$. From (1.3) and Theorem 2.4 applied to $(\underline{u},\overline{u}) = (u_1,u_2)$ and $\phi(x) = -|x|\cot\alpha$, there exists $\tau^* \in \mathbb{R}$ such that $u_2^\tau \geq u_1$ for all $\tau \geq \tau^*$ and

$$\inf_{y=y_0-|x|\cot\alpha} u_2^{\tau^*}(x,y) - u_1(x,y) = 0 \text{ for all } y_0 \in \mathbb{R}.$$

If $u_2^{\tau^*}$ and u_1 touch somewhere, then they are identically equal from the strong maximum principle, and we get the desired result. Otherwise, one has in particular

$$u_2^{\tau^*}(x,-|x|\cot\alpha) - u_1(x,-|x|\cot\alpha) \to 0 \ \text{ as } |x| \to +\infty.$$

But the limit is equal to $U(\tau^*) - U(0)$ because of the previous arguments. Therefore, $\tau^* = 0$ (since U is increasing), and $u_2 \geq u_1$.

By reserving the roles of u_2 and u_1, one concludes that either u_2 and u_1 are identically equal up to shift, or $u_1 \geq u_2$. In all cases, one concludes that u_1 and u_2 are equal up to shift, and actually, because of the limiting behavior in the directions $(\pm\sin\alpha,\cos\alpha)$, the shift is equal to 0. That completes the proof of Theorem 3.13. □

The next theorem is a non-existence result in dimensions $N \geq 3$, under the conditions (1.3) with $\alpha \neq \pi/2$ and under a sign assumption on f.

Theorem 3.14 *Assume that $f \in C^1([0,1])$ is nonincreasing in $[0,\delta]$ and $[1-\delta,1]$ for some $\delta > 0$. Assume that f is either nonnegative or nonpositive in $[0,1]$. If $N \geq 3$ and if $0 \leq u \leq 1$ is a solution of (1.1) satisfying (1.3) for some $\alpha \in (0,\pi)$, then $\alpha = \pi/2$.*

Proof. Let us argue by contradiction and assume that $0 \leq u \leq 1$ is a solution of (1.1) satisfying (1.3) for some $\alpha \in (0,\pi/2) \cup (\pi/2,\pi)$. From Theorems 3.8 and 3.12, there exists then a, unique, solution $(c(f),U)$ of (3.9). Furthermore, $\mathrm{sgn}(\pi/2-\alpha) = \mathrm{sgn}(c(f))$ and $c = c(f)/\sin\alpha$. It is also known that $c(f)$ has the same sign as $\displaystyle\int_0^1 f$.

We only consider here the case where $\alpha < \pi/2$, the other case being dealt with similarly. One then has that $c(f) > 0$ and $f \geq 0$ in $[0,1]$.

Set

$$\underline{u}(x,y) = U(\sin\alpha\ (y-\phi(r))), \quad r = |x| \tag{3.20}$$

where ϕ is a function, to be chosen later, of class C^2 in $\mathbb{R}^+$ such that $\phi(0) = \phi'(0) = 0$. A straightforward calculation shows that

$$\begin{aligned}\Delta\underline{u} - c\underline{u}_y + f(\underline{u}) \ &= \ \sin\alpha\left[c(f)\sin\alpha\ (1+\phi'^2(r)) - \phi''(r) - \frac{N-2}{r}\phi'(r) - c\right]U'(\sin\alpha\ (y-\phi(r))) \\ &\quad + \left[1-\sin^2\alpha\ (1+\phi'^2(r))\right]f(\underline{u}) \ \text{ in } \mathbb{R}^N.\end{aligned}$$

We want the function $\underline{u}$ to be a subsolution of (1.1). Since $c = c(f)/\sin\alpha$ and $f \geq 0$, it suffices that $|\phi'| \leq \cot\alpha$ and

$$\phi'' + \frac{N-2}{r}\phi' - c(f)\sin\alpha\,\phi'^2 + c(f)\cos\alpha\cot\alpha = 0 \ \text{ in } \ \mathbb{R}_+, \quad \phi(0) = \phi'(0) = 0. \tag{3.21}$$

We claim that there exists a $C^2(\mathbb{R}_+)$ solution ϕ of (3.21) such that $-\cot\alpha < \phi'(r) < 0$ for all $r > 0$, and $\phi(r) + r\cot\alpha \to +\infty$ as $r \to +\infty$.[1] In $\mathbb{R}^{N-1}$, let B_R be the open ball centered at the origin and with radius $R > 0$. Let w_R be the unique solution of the Dirichlet problem

$$\begin{cases} \Delta w_R - c(f)^2\cos^2\alpha\ w_R &= 0 \text{ in } B_R \\ w_R &= 1 \text{ on } \partial B_R. \end{cases}$$

Since the constants 0 and 1 are respectively strict sub- and supersolutions of this problem, we have $0 < w_R < 1$ in B_R. Using a moving plane method, we get that the function w_R is radial, $w_R = w_R(r) = w_R(|x|)$ and $w_R'(r) > 0$ for all $0 < r \leq R$. Let us now define the function

$$z_R(x) = \frac{w_R(|x|)}{w_R(0)} \ \text{ in } B_R.$$

This function $z_R = z_R(r)$ satisfies $z_R \geq z_R(0) = 1$ in B_R, $z_R'(0) = 0$ and $z_R'(r) > 0$ for all $0 < r \leq R$. From Harnack inequality, the functions $(z_R)_{R\geq 1}$ are locally bounded in $\mathbb{R}^{N-1}$. From standard elliptic estimates and Sobolev injections, there exists a radial function $z = z(r)$ defined in $\mathbb{R}^{N-1}$ such that $z_R \to z$ in $C^2_{loc}(\mathbb{R}^{N-1})$ for some sequence $R = R_n \to +\infty$. The function z satisfies $z \geq z(0) = 1$, $z'(0) = 0$, $z'(r) \geq 0$ in $\mathbb{R}_+$ and

$$z'' + \frac{N-2}{r}z' - c(f)^2\cos^2\alpha\ z = 0 \ \text{ in } \mathbb{R}_+.$$

The function

$$\phi(r) = -\frac{\ln z(r)}{c(f)\sin\alpha}$$

is such that $\phi(0) = \phi'(0) = 0$, $\phi(r) \leq 0$, $\phi'(r) \leq 0$ in $\mathbb{R}_+$ and it satisfies

$$\phi'' + \frac{N-2}{r}\phi' - c(f)\sin\alpha\,\phi'^2 + c(f)\cos\alpha\cot\alpha = 0 \ \text{ in } \mathbb{R}_+,$$

that is to say equation (3.21). Notice also that ϕ is of class $C^\infty(0,+\infty)$.

Let us now prove that $-\cot\alpha < \phi'(r) < 0$ for all $r > 0$. Suppose first that there exists $r_0 > 0$ such that $\phi'' \geq 0$ in $[r_0,+\infty)$. By (3.21), the function ϕ'' cannot be identically 0 in $[r_0,+\infty)$ (otherwise, by (3.21), ϕ' should be a nonzero constant, this is impossible because of the term $(N-2)/r\ \phi'$). Hence, the function ϕ' has a limit $\phi'(+\infty)$ such that $\phi'(r_0) < \phi'(+\infty) \leq 0$. By (3.21), the function ϕ'' has then a limit $\phi''(+\infty)$, which turns out to be 0 since $\phi'(+\infty)$ is finite. Finally, equation (3.21) at $+\infty$ gives

$$\phi'(+\infty) = -\cot\alpha.$$

Since $\phi'(r_0) < \phi'(+\infty)$ and $\phi'(0) = 0$, there exists then a real number $r_1 > 0$ such that $\phi'(r_1) < -\cot\alpha$ and $\phi''(r_1) = 0$. This is in contradiction with equation (3.21) at the point r_1. Hence, we

[1] Notice that, in dimension $N = 2$, the function $\phi(r) = -(1/c(f)\sin\alpha)\ln\cosh(c(f)r\cos\alpha)$ is the unique solution of (3.21) such that $-\cot\alpha < \phi' \leq 0$ in $\mathbb{R}^+$. Besides, $\phi(r) + r\cot\alpha \to (\ln 2)/(c(f)\sin\alpha)$ as $r \to +\infty$. The situation is then very different in dimension 2 from higher dimensions.

just proved that for any $r_0 > 0$, there exists a real $r_1 \geq r_0$ such that $\phi''(r_1) < 0$. Let us now assume that there exists a real $r_0 \geq 0$ such that $\phi''(r_0) \geq 0$. First of all, from equation (3.21) at the point 0, and since $\phi'(0) = 0$, we have $\phi''(0) = -c(f)\cos^2\alpha/((N-1)\sin\alpha) < 0$. In particular, we have $r_0 > 0$. From the previous arguments, there exists $r_1 > r_0$ such that $\phi''(r_1) < 0$. Hence, there exists a real number $r_2 \in (0, r_1)$ such that $\phi''(r_2) \geq 0$ and $\phi'''(r_2) = 0$. On the other hand, we have

$$\phi''' + \frac{N-2}{r}\phi'' - \frac{N-2}{r^2}\phi' - 2c(f)\sin\alpha\ \phi'\phi'' = 0 \text{ in } (0, +\infty).$$

At the point r_2, we have $\phi''(r_2) \geq 0$ and $\phi'(r_2) \leq 0$ (by definition of z and ϕ), and all terms in the previous equality (at $r = r_2$) are nonnegative. But $\phi''(r_2)$ and $\phi'(r_2)$ cannot be both 0 because of (3.21). Hence, we have reached a contradiction. That proves that $\phi'' < 0$ in $\mathbb{R}_+$, whence $\phi'(r) < 0$ for all $r > 0$. Furthermore, it follows from (3.21), together with the negativity of ϕ'' and the nonpositivity of ϕ' in $\mathbb{R}_+$, that

$$c(f)\sin\alpha\ {\phi'}^2 < c(f)\cos\alpha\cot\alpha \text{ in } \mathbb{R}_+,$$

whence, eventually, $\phi' > -\cot\alpha$ in $\mathbb{R}_+$.

Suppose now that the integral $\displaystyle\int_0^{+\infty}(\phi'(r) + \cot\alpha)dr$ is finite. By (3.21), we get

$$0 = -\int_1^{+\infty}\Big[\phi'' + \frac{N-2}{r}(\phi' + \cot\alpha) - \frac{N-2}{r}\cot\alpha - c(f)\sin\alpha\ (\phi' + \cot\alpha)^2 + 2c(f)\cos\alpha\ (\phi' + \cot\alpha)\Big]\ dr.$$

In the right hand side, all the integrals converge but $\displaystyle\int_1^{+\infty}\frac{N-2}{r}\cot\alpha\ dr$ (notice that we here use the fact that $N \geq 3$ and $\alpha \neq \pi/2$). We get a contradiction. As a conclusion,

$$\int_0^{+\infty}(\phi'(r) + \cot\alpha)dr = +\infty$$

and $\phi(r) + r\cot\alpha \to +\infty$ as $r \to +\infty$.

The function $0 < \underline{u} < 1$ defined by (3.20) is then a subsolution of (1.1) and it is such that

$$\limsup_{y+|x|\cot\alpha\to-\infty}\underline{u}(x,y) = 0$$

because $\phi(r) \geq -r\cot\alpha$ in $\mathbb{R}_+$ and $U(-\infty) = 0$. Since $0 < u < 1$ satisfies (1.1) and (1.3), Theorem 2.4 yields the existence of $\tau^* \in \mathbb{R}$ such that $u(x, y + \tau^*) \geq \underline{u}(x,y)$ in $\mathbb{R}^N$ and

$$\forall\ y_0 \in \mathbb{R},\quad \inf_{x\in\mathbb{R}^{N-1}} u(x, -|x|\cot\alpha + \tau^* + y_0) - \underline{u}(x, -|x|\cot\alpha + y_0) = 0. \tag{3.22}$$

Because of (1.3), one can choose y_0 such that $u(x,y) \geq 1/2$ for all $y \geq -|x|\cot\alpha + \tau^* + y_0$. But one has

$$\underline{u}(x, -|x|\cot\alpha + y_0) = U(\sin\alpha\ (-|x|\cot\alpha + y_0 - \phi(|x|)) \to 0 \text{ as } |x| \to +\infty$$

because $\phi(r) + r\cot\alpha \to +\infty$ as $r \to +\infty$ and $U(-\infty) = 0$. Therefore, (3.22) implies that $u(x, y+\tau^*)$ and $\underline{u}(x,y)$ touch somewhere. The strong maximum principle then yields $u(x, y + \tau^*) \equiv \underline{u}(x,y)$ in $\mathbb{R}^N$. But this is impossible, for instance for $y = -|x|\cot\alpha + y_0$ as $|x| \to +\infty$.

That completes the proof of Theorem 3.14. □

4 Existence results in dimensions 2 and higher, for combustion-type or bistable nonlinearities

After the qualitative properties in Section 3, we are here concerned with the existence of solutions of (1.1) under some conical conditions at infinity. We first quote an existence result for combustion-type nonlinearities, and we then prove the existence of solutions in the case where the function f is bistable.

4.1 Combustion nonlinearities

We quote here without proof a result from [A. Bonnet and F. Hamel, Existence of non-planar solutions of a simple model of premixed Bunsen flames, *SIAM J. Math. Anal.*, **31** (1999), pp. 80–118] and [F. Hamel, R. Monneau and J.-M. Roquejoffre, Stability of conical fronts in a combustion model, *Ann. Sci. Ecole Normale Supérieure* **37** (2004), pp. 469–506].

Theorem 4.1 *If f satisfies (1.2), if $N = 2$ and $\alpha \in (0, \pi/2]$, then there exists a solution (c, u) of (1.1) with the conical conditions (1.3).*

The proof is rather lenghty and involves several techniques: first, one solves equivalent problems in bounded rectangles such that the ratio between the y-length and the x-length approaches $\cot\alpha$ as the size of the rectangles goes to infinity. One imposes Dirichlet conditions 0 and 1 respectively on the lower and upper sides, and oblique Neumann boundary conditions on the vertical sides. By proving some a priori estimates, one passes to the limit in the whole plane $\mathbb{R}^2$. Furthermore, by using a sliding method, one can prove that the solutions are decreasing in any unit direction $(\tau_x, \tau_y) \in \mathbb{R}^2$ such that $\tau_y < -\cos\alpha$. The difficulty is to show the asymptotic conditions (1.3) at infinity. One especially makes several uses of the sliding method in several orthogonal directions and one proves that the weaker conditions (3.16) are fulfilled. One then uses some results on related free boundary problems to get (1.3). Notice finally that all qualitative properties stated in Section 3 hold (especially, u is unique and even in x up to shift, and $c = c(f)/\sin\alpha$, where $c(f) > 0$ is the unique speed for problem (3.9) –problem (3.9) is known to have a solution for a combustion nonlinearity f satisfying (1.2)).

4.2 Bistable nonlinearities

The following result, from [F. Hamel, R. Monneau and J.-M. Roquejoffre, Existence and qualitative properties of multidimensional conical bistable fronts, *Disc. Cont. Dyn. Systems* (2005), to appear], states the existence of cylindrically symmetric solutions of (1.1) and (3.3) in any dimension $N \geq 2$ for a bistable nonlinearity f satisfying (1.4).

Theorem 4.2 *Assume that f satisfies (1.4) and $\int_0^1 f > 0$. In any dimension $N \geq 2$ and for any $\alpha \in (0, \pi/2]$, there exists a solution (c, u) of (1.1) and (3.3) of the type $u(x, y) = \tilde{u}(|x|, y)$, and any level graph ϕ_λ satisfies*

$$\hat{x} \cdot \nabla\phi_\lambda(x) \to -\cot\alpha \ \text{ as } |x| \to +\infty.$$

Furthermore, $\partial_{|x|}\tilde{u}(|x|, y) > 0$ for all $x \neq 0$ and $y \in \mathbb{R}$.

Remark 4.3 *From the results of the Section 3, u is decreasing in any unit direction $(\tau_x, \tau_y) \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that $\tau_y < -\cos\alpha$, and u is asymptotically planar along its level sets. Furthermore,*

$c = c(f)/\sin\alpha$. *Notice that the assumption* $\displaystyle\int_0^1 f > 0$ *implies that* $c(f) > 0$*, and that the condition* $\alpha \leq \pi/2$ *is then necessary (Theorem 3.12). If one assumes* $\displaystyle\int_0^1 f > 0$*, then the conclusion of Theorem 4.2 holds with* $\alpha \in [\pi/2, \pi)$*, but* $\partial_{|x|}\tilde{u}(|x|, y) < 0$ *for all* $x \neq 0$ *and* $y \in \mathbb{R}$*. The case where* $\displaystyle\int_0^1 f = 0$ *is mentionned at the end of this section.*

This subsection is devoted to the proof of Theorem 4.2.

Proof. Notice that when $\alpha = \pi/2$, the function $u(x, y) = U(y)$ satisfies all properties of Theorem 4.2, where U is the unique solution of (3.9). Therefore, one can assume in the sequel that $\alpha < \pi/2$. The proof is based on the existence of solutions of some approximate problems in bounded cylinders. Then, a passage to the limit in an infinite cylinder in the direction y gives the existence of a solution of (1.1) with suitable conditions on the boundary of the cylinder. A second passage to the limit in the whole space $\mathbb{R}^N$ provides the existence of a solution u of (1.1) satisfying (3.3).

First, let R and L be two positive real numbers and call

$$\Omega^{R,L} = B_R \times (-L, L),$$

where B_R is the open euclidean ball of $\mathbb{R}^{N-1}$ with radius R and center 0. Call

$$c = \frac{c(f)}{\sin\alpha},$$

where $c(f)$ is the unique planar front velocity, for problem (3.9). From [9], we know that, for all $a > 0$, there exists a unique solution (c_a, u_a) of

$$u_a'' - c_a u_a' + f(u_a) = 0 \ \text{ in } [-a, a], \quad u_a(-a) = 0,\ u_a(0) = \theta,\ u_a(a) = 1, \tag{4.1}$$

where u_a is of class $C^2([-a, a])$, $0 < u_a < 1$ in $(-a, a)$, $u_a' > 0$ in $[-a, a]$. Furthermore, as $a \to +\infty$, $c_a \to c(f)$, and $u_a \to U(\cdot + U^{-1}(\theta))$ in $C^{2,\beta}_{loc}(\mathbb{R})$ for all $0 \leq \beta < 1$. Consider now the following problem

$$\left\{\begin{array}{rcll} \Delta u - cu_y + f(u) & = & 0 & \text{in } \Omega^{R,L} \\ u(x, y) & = & u_L(y) & \text{on } \partial\Omega^{R,L}, \end{array}\right. \tag{4.2}$$

where u_L is the solution of (4.1) in the interval $[-L, L]$ (namely with $a = L$). The constant function 0 is clearly a subsolution of this problem. On the other hand, the function $u_L(x, y) = u_L(y)$ satisfies

$$\Delta u_L - c\partial_y u_L + f(u_L) = (c_L - c)u_L' < 0 \ \text{ in } \Omega_{R,L}$$

for L large enough (indeed $u_L' > 0$ in $[-L, L]$ and $c_L - c \to c(f) - c(f)/\sin\alpha < 0$ as $L \to +\infty$). In the sequel, one assumes that $L > 0$ is large enough so that $c_L - c < 0$. There exists then a classical solution $u^{R,L}$ of (4.2) such that

$$0 \leq u^{R,L}(x, y) \leq u_L(y) \ \text{ for all } (x, y) \in \overline{\Omega_{R,L}}.$$

The strong maximum principle then yields

$$0 < u^{R,L}(x, y) < u_L(y)\ (< 1) \ \text{ for all } (x, y) \in \Omega^{R,L}. \tag{4.3}$$

We now claim that $u^{R,L}$ is then unique and increasing in the variable y. The proof is based on a sliding method, which we detail here. For $\lambda \in (0, 2L)$, call $\Omega_\lambda = B_R \times (-L, -L+\lambda)$ and

$$u_\lambda(x,y) = u^{R,L}(x, y+2L-\lambda).$$

Both functions $u^{R,L}$ and u_λ are then defined and continuous (at least) in $\overline{\Omega_\lambda}$ and they are of class C^2 in Ω_λ. From the boundary conditions of $u^{R,L}$ at $y = \pm L$, it follows that $u^{R,L} < u_\lambda$ in $\overline{\Omega_\lambda}$ for $\lambda > 0$ small enough. Set

$$\lambda^* = \sup\{\lambda \in (0, 2L),\ u^{R,L} < u_\mu \text{ in } \overline{\Omega_\mu} \text{ for all } \mu \in (0,\lambda)\} > 0$$

and assume that $\lambda^* < 2L$. One has $u^{R,L} \le u_{\lambda^*}$ in $\overline{\Omega_{\lambda^*}}$ and there exists $(x,y) \in \overline{\Omega_{\lambda^*}}$ such that

$$u^{R,L}(x,y) = u_{\lambda^*}(x,y) = u^{R,L}(x, y+2L-\lambda^*).$$

If $(x,y) \in \partial B_R \times [-L, -L+\lambda^*]$, then $u_L(y) = u_L(y+2L-\lambda^*)$, which is impossible because u_L is increasing and $2L - \lambda^* > 0$. If $y = -L$ and $x \in B_R$, then $0 = u_L(-L) = u^{R,L}(x,-L) = u^{R,L}(x, L-\lambda^*)$, which is impossible because $L - \lambda^* > -L$, whence $u^{R,L}(x, L-\lambda^*) > 0$ by (4.3). Similarly, the case where $y = -L+\lambda^*$ and $x \in B_R$ is impossible. Therefore, $(x,y) \in \Omega_{\lambda^*}$. The function

$$z = u_{\lambda^*} - u^{R,L}$$

is nonnegative and continuous in $\overline{\Omega_{\lambda^*}}$, and of class C^2 in Ω_{λ^*}. Furthermore, z satisfies an inequation of the type

$$\Delta z - c z_y + \zeta(x,y) z \le 0 \ \text{ in } \Omega_{\lambda^*}$$

for some bounded function ζ. Since z vanishes at the interior point $(x,y) \in \Omega_{\lambda^*}$, the strong maximum principle then yields $z \equiv 0$ in $\overline{\Omega_{\lambda^*}}$. But one can check as above that $z > 0$ on $\partial\Omega_{\lambda^*}$. One has then reached a contradiction. As a consequence, $\lambda^* = 2L$ and u is then increasing in the variable y in $\overline{\Omega^{R,L}}$.

If v is another solution of (4.2) satisfying (4.3), then slide v in y and compare it with $u^{R,L}$. One can prove as above that $v \ge u^{R,L}$ in $\overline{\Omega^{R,L}}$. Reversing the roles or v and $u^{R,L}$ implies that $u^{R,L}$ is actually unique.

Let us now prove that the function $u^{R,L}$ only depends on $|x|$ and y, namely $u^{R,L}(x,y) = \tilde{u}^{R,L}(|x|, y)$, and that

$$\partial_{|x|}\tilde{u}^{R,L}(|x|, y) > 0 \ \text{ for all } 0 < |x| \le R \text{ and } y \in (-L, L).$$

To do so, fix a unit vector e in $\mathbb{R}^{N-1}$ and, for $a \in [0,R)$, call $\omega_a = \{x \in B_R,\ x \cdot e > a\}$. Let now u_a be the function defined in $\overline{\omega_a} \times [-L, L]$ by

$$u_a(x,y) = u^{R,L}(x + 2(a - x\cdot e)e, y).$$

The function u_a is the orthogonal reflection of the function u with respect to the hyperplane $H_a = \{x \in \mathbb{R}^{N-1},\ x \cdot e = a\}$. The function u_a is still a solution of $\Delta u_a - c\partial_y u_a + f(u_a) = 0$ in $\omega_a \times (-L, L)$. Furthermore, because of (4.3) and since $u^{R,L}$ and u_L are increasing in y, it is easy to prove, with the same sliding method as above, that

$$u_a \le u^{R,L} \ \text{ in } \overline{\omega_a} \times [-L, L].$$

Moreover, if $a > 0$ and $(x,y) \in (\partial\omega_a \backslash H_a) \times (-L, L)$, one has $(x + 2(a - x\cdot e)e, y) \in \Omega^{R,L}$, whence

$$u_a(x,y) = u^{R,L}(x + 2(a - x\cdot e)e, y) < u_L(y) = u^{R,L}(x,y).$$

The strong maximum principle then yields $u_a < u^{R,L}$ in $\omega_a \times (-L, L)$. But since $u_a = u^{R,L}$ on $(B_R \cap H_a) \ \times (-L, L)$, it follows from Hopf lemma that

$$e \cdot \nabla_x u_a < e \cdot \nabla_x u \ \text{ on } (B_R \cap H_a) \ \times (-L, L).$$

Owing to the definition of u_a, one has $e \cdot \nabla_x u_a = -e \cdot \nabla_x u^{R,L}$, whence $e \cdot \nabla_x u^{R,L} > 0$ on $(B_R \cap H_a) \times (-L, L)$. On the other hand, the case $a = 0$ implies that $u_a \le u$ in $\overline{\omega_0} \times [-L, L]$. By choosing $-e$ instead of e, one gets that

$$u^{R,L}(x, y) = u^{R,L}(x - 2(x \cdot e)e, y) \ \text{ for all } (x, y) \in \overline{\Omega_{R,L}}.$$

Since e was an arbitrary unit vector in $\mathbb{R}^{N-1}$, one concludes that

$$u^{R,L} = \tilde{u}^{R,L}(|x|, y)$$

only depends on $|x|$ and y. The monotonicity in $|x|$ follows from the above arguments. Notice furthermore that, since u_L is a supersolution of (4.2) and $u^{R,L} < u_L$ in $\Omega_{R,L}$ with equality on $\partial\Omega_{R,L}$, the Hopf lemma actually implies that $\partial_{|x|}\tilde{u}(R, y) > 0$ for all $y \in (-L, L)$.

Next, one shall pass to the limit as $L \to +\infty$. From standard elliptic estimates and diagonal extraction process, there exists a sequence $(L_n)_{n\in\mathbb{N}} \to +\infty$ such that $u^{R,L_n} \to u^R$ in $C^{2,\beta}_{loc}(\overline{B_R} \times \mathbb{R})$ for all $0 \le \beta < 1$, where u^R solves

$$\left\{ \begin{array}{rcll} \Delta u^R - c\partial_y u^R + f(u^R) & = & 0 & \text{in } B_R \times \mathbb{R} \\ u^R(x, y) & = & U(y) & \text{on } \partial B_R \times \mathbb{R}. \end{array} \right. \tag{4.4}$$

Furthermore, $0 \le u^R \le U(y)$ in $\overline{B_R} \times \mathbb{R}$ because of (4.3) and because $u_L \to U$ in $C^2_{loc}(\mathbb{R})$ as $L \to +\infty$. Since $u^R = U(y) > 0$ on $\partial B_R \times \mathbb{R}$, the strong maximum principle then yields $u^R > 0$ in $\overline{B_R} \times \mathbb{R}$. Similarly,

$$u^R(x, y) < U(y) \ \text{ for all } (x, y) \in B_R \times \mathbb{R} \tag{4.5}$$

because U is a strict supersolution of (4.4).

By passage to the limit, the function u^R is nondecreasing in y and since u^R is increasing in y on $\partial B_R \times \mathbb{R}$, it follows from the strong maximum principle that u^R is increasing in y in the whole cylinder $\overline{B_R} \times \mathbb{R}$. Similarly, the function u^R is a function of $|x|$ and y only, namely

$$u^R(x, y) = \tilde{u}^R(|x|, y) \text{ in } \overline{B_R} \times \mathbb{R}$$

and $\tilde{u}^R$ is nondecreasing in $|x|$. Let e be a given unit direction of $\mathbb{R}^{N-1}$. Under the same notations as above, one then has

$$u^R(x + 2(a - x \cdot e)e, y) \le u^R(x, y) \ \text{ for all } (x, y) \in \overline{\omega_a} \times \mathbb{R}$$

and for all $0 \le a < R$. Furthermore, if $a > 0$, the above inequality is strict on $(\partial\omega_a \backslash H_a) \times \mathbb{R}$ because of (4.4) and (4.5). The strong maximum principle and the Hopf lemma then imply that $u^R(x + 2(a - x \cdot e)e, y) < u^R(x, y)$ in $\omega_a \times \mathbb{R}$ and $e \cdot \nabla u^R > 0$ on $(B_R \cap H_a) \times \mathbb{R}$, provided $a > 0$. Therefore, one concludes as above that $\partial_{|x|}\tilde{u}^R > 0$ for all $0 < |x| < R$, and also for $|x| = R$.

From the monotonicity of u^R in y, there exist two functions $u^R_\pm$ defined in $\overline{B_R}$ such that $u^R(x, y) \to u^R_\pm(x) = \tilde{u}^R_\pm(|x|)$ as $y \to \pm\infty$. Furthermore, the convergence holds in $C^{2,\beta}_{loc}(\overline{B_R})$ (for all $0 \le \beta < 1$) from standard elliptic estimates. The functions $u^R_\pm$ satisfy

$$\Delta u^R_\pm + f(u^R_\pm) = 0 \ \text{ in } \overline{B_R}$$

and $0 \le u^R_- \le u^R_+ \le 1$ in $\overline{B_R}$. Since $u^R(x,y) \le U(y)$ in $\overline{B_R} \times \mathbb{R}$ and $U(-\infty) = 0$, one immediately gets that $u^R_- \equiv 0$ in $\overline{B_R}$. On the other hand, $u^R_+(x) = U(+\infty) = 1$ for all $x \in \partial B_R$. The function $v(r) := \tilde{u}^R_+(r)$ satisfies

$$v''(r) + \frac{N-2}{r} v'(r) + f(v(r)) = 0, \quad 0 < r \le R,$$

$0 \le v \le 1$, $v' \ge 0$ in $[0,R]$, $v'(0) = 0$ and $v(R) = 1$. Multiply the above equation by v' and integrate in $[0,R]$. It follows that

$$\frac{1}{2} v'(R)^2 + \int_{v(0)}^{1} f(s) ds \le 0,$$

whence $\displaystyle\int_{v(0)}^{1} f \le 0$. It follows from the profile of f that $v(0) = 1$ (remember that f satisfies (1.4) and $\displaystyle\int_0^1 f > 0$). Consequently, $v \equiv 1$ and $u^R_+ \equiv 1$ in $\overline{B_R}$.

Let now (R_n) be a sequence converging to $+\infty$ and let $u_n = u^{R_n}$. Up to a shift in the y variable, one can assume that $u_n(0,0) = \theta/2$, where $\theta \in (0,1)$ was given in (1.4). From standard elliptic estimates, the functions u_n converge in $C^{2,\beta}_{loc}(\mathbb{R}^N)$ (for all $0 \le \beta < 1$), up to extraction of some subsequence, to a solution u of (1.1) such that $0 \le u \le 1$, $u(0,0) = \theta/2$, $u_y \ge 0$,

$$u(x,y) = \tilde{u}(|x|, y)$$

with $\partial_{|x|}\tilde{u}(|x|,y) \ge 0$ for all $(x,y) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Since the function u is not constant (because $f(\theta/2) \ne 0$), it follows then from Theorem 3.6 that u satisfies all the properties listed in Theorem 3.3. The fact that $\hat{x} \cdot \nabla\phi_\lambda(x) \to -\cot\alpha$ as $|x| \to +\infty$ for all $\lambda \in (0,1)$ is then a consequence of Proposition 3.9. Lastly, the positivity of $\partial_{|x|}\tilde{u}(|x|,y)$ for all $x \ne 0$ and $y \in \mathbb{R}$ follows from the strong maximum principle and Hopf lemma, because $\alpha \ne \pi/2$.

That completes the proof of Theorem 4.2. □

4.3 More existence results

Actually, we can make Theorem 4.2 more precise. In dimension $N = 2$, the solution u satisfies (1.3) and the level graphs ϕ_λ converge exponentially to the straight lines which are parallel to $\{y = \pm x \cot\alpha\}$ (see [30]). This existence result also holds for more general functions f satisfying $f'(0) < 0$, $f'(1) < 0$ and $\displaystyle\int_s^1 f > 0$ for all $0 \le s < 1$ (see [43]). As a matter of fact, the existence of solutions (c,u) of (1.1) with the asymptotic conditions (1.3) had been known in dimension $N = 2$ for angles $\alpha < \pi/2$ and close to $\pi/2$ (see [21, 32]) with a proof based on a center manifold reduction. Such solutions are examples of corner defects, under the terminology of Haragus and Scheel [32].

In dimension $N \ge 3$, the solutions u constructed in Theorem 4.2 do not satisfy (1.3). Namely,

$$\phi_\lambda(x) + |x| \cot\alpha \sim k_\lambda \ln|x| \text{ as } |x| \to +\infty,$$

with $k_\lambda \in (0,+\infty)$ (see [30]).

We also refer to Section 5 for further classification results of all solutions of (1.1) satisfying (3.2).

When f is of the bistable type (1.4) and has zero integral in $(0,1)$ ($\displaystyle\int_0^1 f = 0$), then, for any $c > 0$ and $N \ge 2$, there exists a solution $u(x,y) = \tilde{u}(|x|,y)$ of (1.1) such that $u_y > 0$ in $\mathbb{R}^N$,

$\partial_{|x|}\tilde{u}(|x|,y) > 0$ for all $x \neq 0$, $u(x,y) \to 1$ as $y \to +\infty$ uniformly in x, and $u(x,y) \to 0$ as $y \to -\infty$ locally in x. See recent results in [18]. Furthermore, in dimension $N = 2$, each level graph $\{y = \phi_\lambda(x),\ x \in \mathbb{R}\}$ is such that

$$\phi_\lambda(x) \sim -\cosh(2\sqrt{f'(1)}x) \text{ as } |x| \to +\infty,$$

for some $k \in (0,+\infty)$ which is independent of $\lambda \in (0,1)$. In dimension $N \geq 3$, one has

$$\phi_\lambda(x) \sim -c|x|^2/(2(N-2)) \text{ as } |x| \to +\infty.$$

We then see that, unlike for the solutions of Theorem 1.4, the level graphs ϕ_λ are not Lipschitz continuous anymore and the solutions u *do not* satisfy (3.3) anymore. Notice that when $c < 0$, similar results can be obtained, by changing y into $-y$ in the previous statements.

When $f(s) = u(1-u)(u-1/2)$, a well-known conjecture of De Giorgi [19] asserts that all y-monotonic solutions of (1.1) with $c = 0$ are planar at least in dimensions $n \leq 8$, namely there exist a unit vector $a \in \mathbb{R}^n$ and a function $g : \mathbb{R} \to [-1,1]$ such that $u(x,y) = g(a \cdot (x,y))$ for all (x,y) – in this conjecture, the radial symmetry in x is not assumed. This conjecture was proved recently by Savin [49] (see also [1, 3, 24]). More general nonlinearities f of the bistable type can also be considered. Thus, the recent results of [18] show that the conjecture of De Giorgi does not hold as soon as there is a transport term cu_y in the equation (1.1), with $c \neq 0$, in any dimension $n \geq 2$. Similarly, the parabolic analogue of the conjecture of De Giorgi does not hold in any dimension $n \geq 2$, namely, with a bistable nonlinearity f satisfying (1.4) and $\displaystyle\int_0^1 f = 0$, problem (1.5) admits non planar solutions of the type $v(t,x,y) = \tilde{u}(|x|, y - ct)$.

5 Further classification results

In this section, we get further classification results in dimension $N = 2$ for all solutions of (1.1) and (3.21) under the assumption that there exists a solution with asymptotic conditions (1.3). Then, we mention some additional results in dimensions $N \geq 3$.

Theorem 5.1 *Assume that $f \in C^1([0,1])$ is nonincreasing in $[0,\delta]$ and $[1-\delta,1]$ for some $\delta > 0$. Assume that there exists a, unique, solution $(c(f),U)$ of (3.9), that $c(f) > 0$ and that, for each $\alpha \in (0,\pi/2]$, there exists a solution $(c_\alpha,u_\alpha) = (c(f)/\sin\alpha, u_\alpha)$ of (1.1) and (1.3) in dimension $N = 2$.*

Let $0 \leq u \leq 1$ be a solution of (1.1) in dimension $N = 2$ and assume that (3.2) is satisfied for some Lipschitz function $\phi : \mathbb{R} \to \mathbb{R}$. Under the notations of Theorem 3.3, all level graphs ϕ_λ of u have the same Lipschitz norm denoted by $\cot\alpha$ with $\alpha \in (0,\pi/2]$. Then either u is a planar front $u(x,y) = U(\pm x\cos\alpha + y\sin\alpha + \tau)$ (for some $\tau \in \mathbb{R}$), or u is equal to u_α up to shift.

Proof. First, all conclusions of Theorems 3.3, 3.8 and Proposition 3.9 (part 2bis) apply. Namely, the function u is decreasing in any unit direction $(\tau_x,\tau_y) \in \mathbb{R}^2$ such that $\tau_y < -\cos\alpha$, where $\alpha \in (0,\pi/2]$ is such that all level graphs ϕ_λ of u have their Lipschitz norm equal to $\cot\alpha$. Furthermore, $c = c(f)/\sin\alpha$ and either u is a planar front $u(x,y) = U(\pm x\cos\alpha + y\sin\alpha + \tau)$ (for some $\tau \in \mathbb{R}$), or

$$\phi'_\lambda(x) \to \mp\cot\alpha \text{ as } x \to \pm\infty \text{ for all } \lambda \in (0,1) \tag{5.1}$$

and

$$u(x+r, y+\phi_\lambda(r)) \to U(\pm x\cos\alpha + y\sin\alpha + U^{-1}(\lambda)) \text{ as } r \to \pm\infty, \text{ in } C^2_{loc}(\mathbb{R}^2). \tag{5.2}$$

Lastly, Theorem 3.12 implies that $\alpha \le \pi/2$. In the case where u is a planar front, then we have got the desired conclusion. Therefore, one can assume that (5.1) holds. Moreover, if $\alpha = \pi/2$, then the level graphs ϕ_λ are all flat, namely the function u is planar and only depends on y; up to shift, it is then equal to $U(y)$, i.e. $u_{\pi/2}$.

One can then assume in the following that (5.1) holds with $\alpha \in (0, \pi/2)$. Because of (3.3) and the monotonicity properties recalled above (see Theorem 3.3), we get that

$$\limsup_{y+|x|\cot\alpha \to -\infty} u(x,y) = 0.$$

We shall now prove that u and u_α are equal up to shift. Notice first that u and u_α satisfy the euqation (1.1) with the same speed $c = c(f)/\sin\alpha$. From Theorem 3.13, the function u_α, solving (1.1) and (1.3), is unique up to shift, and one can assume without loss of generality that it is even in x, namely $u_\alpha(x,y) = u_\alpha(-x,y)$ for all $(x,y) \in \mathbb{R}^2$. From Theorem 3.13, one can also assume without loss of generality that $u_\alpha(x+r, y-|r|\cot\alpha) \to U(\pm x\cos\alpha + y\sin\alpha)$ in $C^2_{loc}(\mathbb{R}^2)$ as $r \to \pm\infty$. Since u_α satisfies

$$\limsup_{y+|x|\cot\alpha \to +\infty} u_\alpha(x,y) = 1,$$

Theorem 2.4 of Section 2 can then be applied to $\underline{u} = u$ and $\overline{u} = u_\alpha$, and $\phi(x) = -|x|\cot\alpha$. Therefore, there exists $\tau^* \in \mathbb{R}$ such that

$$u(x,y) \le u_\alpha(x, y+\tau^*) \ \text{ for all } (x,y) \in \mathbb{R}^2$$

and

$$\inf_{y=B-|x|\cot\alpha} u_\alpha(x,y+\tau^*) - u(x,y) = 0 \tag{5.3}$$

for all $B \in \mathbb{R}$. But

$$\left\{ \begin{array}{rcl} u_\alpha(x, B-|x|\cot\alpha+\tau^*) & \to & U((B+\tau^*)\sin\alpha) > 0 \\ u(x, B-|x|\cot\alpha) & \to & u_\pm(B) \end{array} \right. \quad \text{as } x \to \pm\infty. \tag{5.4}$$

where the limits $u_\pm(B)$ exist because u is nonincreasing in both directions $(\pm\sin\alpha, -\cos\alpha)$ and satisfy $u_\pm(B) \in [0,1)$ because $u(0,B) < 1$. From now on, let us fix $B = 0$, and define $u_\pm = u_\pm(0)$. According to the values of $u_\pm$, four cases may occur:

Case 1: $u_- = u_+ = 0$. It follows from (5.3) and (5.4) that $u_\alpha(x_0, y_0+\tau^*) = u(x_0,y_0)$ for some $(x_0,y_0) \in \mathbb{R}^2$. Since both functions $u_\alpha(\cdot, \cdot + \tau^*)$ and u are ordered and satisfy the same equation (1.1), the strong maximum principle then yields $u_\alpha(x,y+\tau^*) = u(x,y)$ for all $(x,y) \in \mathbb{R}^2$. This is impossible because of (5.4) and the assumption $u_- = u_+ = 0$.

Case 2: $0 < u_- < 1$ and $u_+ = 0$. Choose any real number ρ_0 and call

$$w(x,y) = u_\alpha(x+\rho_0, y+\rho_0\cot\alpha).$$

With the same arguments as previously, there exists then a real number $t = t(\rho_0)$ such that $u(x,y) \le w(x,y+t)$ for all $(x,y) \in \mathbb{R}^2$ and such that (5.3) holds, with w and t instead of u_α and τ^*. Since $u(x,y) \not\equiv w(x,y+t)$ because of the different asymptotic limits in the direction $(\sin\alpha, -\cos\alpha)$, it then follows that

$$w(x, -|x|\cot\alpha + t) - u(x, -|x|\cot\alpha) \to 0 \ \text{ as } x \to -\infty,$$

whence $U(t\sin\alpha) = u_-$. Similarly,

$$u_\alpha(x, -|x|\cot\alpha + \tau^*) - u(x, -|x|\cot\alpha) \to 0 \ \text{ as } x \to -\infty$$

yields $U(\tau^* \sin\alpha) = u_-$. As a consequence, $U(t\sin\alpha) = U(\tau^*\sin\alpha)$, i.e. $t = \tau^*$ does not depend on ρ_0 and

$$u(x,y) \le u_\alpha(x+\rho_0, y+\rho_0\cot\alpha+\tau^*) \ \text{ for all } (x,y)\in\mathbb{R}^2 \text{ and } \rho_0\in\mathbb{R}.$$

Passing to the limit as $\rho_0 \to -\infty$ implies, that

$$u(x,y) \le U(-x\cos\alpha + y\sin\alpha + \tau^*) \ \text{ for all } (x,y)\in\mathbb{R}^2.$$

Therefore, each level graph ϕ_λ of u is above a translate of the line $y = x\cot\alpha$, which is in contradiction with (5.1). Case 2 is then ruled out.

Case 3: $u_- = 0$, $0 < u_+ < 1$. The same arguments as in Case 2 lead to a contradiction.

Case 4: $0 < u_\pm < 1$. It then follows from (5.2) that

$$\sup_{x\in\mathbb{R}} |\phi_\lambda(x) + |x|\cot\alpha| < +\infty$$

for all $\lambda \in (0,1)$. Hence, the function u satisfies (1.3) and Theorem 3.13 implies then that u is equal to u_α up to shift. That completes the proof of Theorem 5.1. □

Remark 5.2 *In the case where $c(f) < 0$, the same conclusion as in Theorem 5.1 holds by changing $\alpha \in (0,\pi/2]$ into $\alpha \in [\pi/2,\pi)$.*

From the existence results mentionned in Section 4, the assumptions of Theorem 5.1 are fulfilled if f satisfies (1.2) or if f satisfies (1.4) with $\displaystyle\int_0^1 f > 0$.

In dimension $N \ge 3$, one can get a classification result of all solutions of (1.1) and (3.2) for a bistable nonlinearity under an assumption of cylindrical symmetry. Namely, if f satisfies (1.4) and $\displaystyle\int_0^1 f > 0$ (the case where the integral has a negative sign can be treated similarly), and if $0 \le u(x,y) = \tilde{u}(|x|,y) \le 1$ is a solution of (1.1) such that

$$\liminf_{y-\tilde{\phi}(|x|)\to+\infty} u(x,y) > \theta, \qquad \limsup_{y-\tilde{\phi}(|x|)\to-\infty} u(x,y) < \theta, \tag{5.5}$$

for some Lipschitz continuous function $\tilde{\phi} : \mathbb{R}_+ \to \mathbb{R}$, then $c \ge c(f)$ and, up to shift, u is equal to the solution u_α constructed in Theorem 4.2 with $\alpha = \arcsin(c(f)/c) \in (0,\pi/2]$. Here, $c(f) > 0$ denotes the unique speed solving (3.9). Furthermore, the same result holds if, instead of (5.5), one assumes that $c \ge 0$, $u \not\equiv 0$, $\inf_{\mathbb{R}^N} u < \theta$, $u_y \ge 0$ and $\partial_{|x|}\tilde{u}(|x|,y) \ge 0$ in $\mathbb{R}^N$.

6 Stability issues

This section deals with the global stability of the solutions u of problem (1.1) under asymptotic conditions of the type (3.2). We will also state more precise results in dimension $N = 2$ under the assumption of the existence of solutions satisfying (1.3).

Another way of formulating this question of the stability is to ask the question of the convergence to the travelling fronts $u(x, y+ct)$, or to some translates of them, for the solutions $v(t,x,y)$ of the Cauchy problem

$$\begin{cases} v_t = \Delta v + f(v), \quad t > 0,\ (x,y)\in\mathbb{R}^N, \\ v(0,x,y) = v_0(x,y) \text{ given},\ 0 \le v_0 \le 1 \end{cases} \tag{6.1}$$

where $v_0(x,y)$ is close, in some sense to be defined later, to a translate $u(\cdot+a,\cdot+b)$ of a solution u of (1.1).

There are many papers dealing with the stability of the travelling fronts for one-dimensional equations of the type (1.6) with various types of nonlinearities f (*see e.g.* [2], [13], [22], [35], [47], [48]), or for wrinkled travelling fronts of multidimensional equations in infinite cylinders (*see* [8], [40], [44], [45], [46]), or lastly for planar fronts in the whole space (*see* [37], [52]). However, the question of the stability of the solutions of the N-dimensional problem (1.1) under conical conditions of the type (1.3) or (3.2), is more involded. As already emphasized, the travelling fronts $u(x,y+ct)$ are special time-global solutions of (6.1) which are stationary in the frame moving downwards with speed c (or upwards with speed $|c|$ if $c\le 0$). Therefore, the question of the global stability of these travelling waves and the question of the asymptotic behaviour for large time of the solutions of the Cauchy problem (6.1) starts from the study of the global attractor of equation (6.1) under some fixed asymptotic conditions in a moving frame.

The next theorem states that, under the same type of assumptions on f as in Sections 3 to 5, the travelling waves are the only time-global solutions of (6.1) satisfying some fixed asymptotic conditions. The proof of this Liouville type result will be based on the general comparison principles proved in Section 2.

Theorem 6.1 *Assume that f is of class $C^1([0,1])$, nonincreasing in $[0,\delta]$ and in $[1-\delta,1]$, for some $\delta>0$. Let $0\le v(t,x,y)\le 1$ be a time-global solution of the equation*

$$v_t=\Delta v+f(v)\ \ \textit{for all } (x,y)\in\mathbb{R}^N \textit{ and } t\in\mathbb{R} \tag{6.2}$$

and assume that

$$\liminf_{y-\phi(x)\to+\infty,\ t\in\mathbb{R}} v(t,x,y-ct)=1,\qquad \limsup_{y-\phi(x)\to-\infty,\ t\in\mathbb{R}} v(t,x,y-ct)=0, \tag{6.3}$$

for some continuous function $\phi\ :\ \mathbb{R}^{N-1}\to\mathbb{R}$. Then there exists a solution u of (1.1) and (3.2) such that

$$v(t,x,y)=u(x,y+ct)\ \ \textit{for all } (x,y)\in\mathbb{R}^N \textit{ and } t\in\mathbb{R}.$$

Proof. Notice first that all functions u_t, u_{x_i}, $u_{x_ix_j}$ are of class $C^{0,\beta}(\mathbb{R}\times\mathbb{R}^N)$ from standard parabolic estimates, for all $\beta\in[0,1)$. Furthermore, the conditions (6.3) and the strong parabolic maximum principle imply that $0<v(t,x,y)<1$ for all $(t,x,y)\in\mathbb{R}\times\mathbb{R}^N$.

Set

$$u(t,x,y)=v(t,x,y-ct)$$

and let us prove that u does not depend on time t. The function u satisfies

$$u_t=\Delta u-cu_y+f(u)\ \ \text{for all } (x,y)\in\mathbb{R}^N \text{ and } t\in\mathbb{R}. \tag{6.4}$$

Choose now any $s\in\mathbb{R}$, and call

$$w(t,x,y)=u(t+s,x,y).$$

Theorem 2.4 can be applied to $\underline{u}=u$ and $\overline{u}=w$ and there exists then $\tau^*\in\mathbb{R}$ such that

$$u(t,x,y)\le w(t,x,y+\tau)\ \ \text{for all } (t,x,y)\in\mathbb{R}\times\mathbb{R}^N \text{ and } \tau\ge\tau^*$$

and

$$\forall\ y\in\mathbb{R},\qquad \inf_{t\in\mathbb{R},\ x\in\mathbb{R}^{N-1}} w(t,x,\phi(x)+\tau^*)-u(t,x,\phi(x)=0.$$

Notice indeed that $\tau^* > -\infty$, because $w(t,x,-\infty) = 0$ for all (t,x), while $u(t,x,y) > 0$ for all (t,x,y). Let us now prove that $\tau^* \le 0$. Assume $\tau^* > 0$. As in the proof of Theorem 2.4, there exists a sequence $(t_k, x_k) \in \mathbb{R} \times \mathbb{R}^{N-1}$ such that

$$u(t_k+s, x_k, \phi(x_k)+\tau^*) - u(t_k, x_k, \phi(x_k)) = w(t_k, x_k, \phi(x_k)+\tau^*) - u(t_k, x_k, \phi(x_k)) \to 0 \ \text{ as } k \to +\infty$$

and the functions

$$u_k(t,x,y) = u(t+t_k, x+x_k, y+\phi(x_k))$$

converge in C^1_{loc} in t and C^2_{loc} in (x,y) to a solution $0 \le U(t,x,y) \le 1$ of (6.4) such that

$$U(t,x,y) \le U(t+s, x, y+\tau^*) \ \text{ for all } (t,x,y) \in \mathbb{R} \times \mathbb{R}^N,$$

with equality at $(0,0,0)$. Furthermore,

$$\liminf_{y\to+\infty,\ t\in\mathbb{R}} U(t,0,y) = 1, \qquad \limsup_{y\to-\infty,\ t\in\mathbb{R}} U(t,0,y) = 0 \tag{6.5}$$

because of (6.3). From the strong parabolic maximum principle, one gets that $0 < U(t,x,y) < 1$ for all (t,x,y), and $U(t,x,y) = U(t+s,x,y+\tau^*)$ for all $t \le 0$ and $(x,y) \in \mathbb{R}^N$. As a consequence, $U(-ns, 0, -n\tau^*) = U(0,0,0) > 0$ for all $n \in \mathbb{N}$. But $U(-ns,0,-n\tau^*) \to 0$ as $n \to +\infty$ because of (6.5) and because τ^* is assumed to be positive. One has then reached a contradiction.

Therefore, $\tau^* \le 0$, whence

$$u(t,x,y) \le w(t,x,y) = u(t+s,x,y) \ \text{ for all } (t,x,y) \in \mathbb{R} \times \mathbb{R}^N.$$

since this last property holds for all $s \in \mathbb{R}$, one eventually concludes that u does not depend on time t, and therefore the conditions (3.2) are fulfilled, because of (6.3). That completes the proof of Theorem 6.1. □

Theorem 6.1 immediately yields the following Liouville type result for the solutions v of (6.2) which are trapped between two travelling fronts satisfying the same conditions (3.2).

Corollary 6.2 *Assume that f is of class $C^1([0,1])$, nonincreasing in $[0,\delta]$ and in $[1-\delta,1]$, for some $\delta > 0$. Let $0 \le u(x,y) \le 1$ be a solution of (1.1) satisfying (3.2) for some continuous function $\phi : \mathbb{R}^{N-1} \to \mathbb{R}$. If $0 \le v(t,x,y) \le 1$ is a time-global solution of (6.2) and if there exist (a,b) and $(a',b') \in \mathbb{R}^{N-1} \times \mathbb{R}$ such that*

$$u(x+a, y+ct+b) \le v(t,x,y) \le u(x+a', y+b') \ \text{ for all } (t,x,y) \in \mathbb{R} \times \mathbb{R}^N,$$

then v is a travelling front with the speed c, namely $v(t,x,y) = V(x, y+ct)$ for some solution V of (1.1) and (3.2).

Notice that, in the case where $N = 2$ and u satisfies (1.3), then the function V would itself be a translate of u, from Theorem 3.13. However, this property is not clear in general.

Another consequence of Theorem 6.1 is the characterization of all elements of the ω-limit sets of some initial conditions in dimension $N = 2$.

Corollary 6.3 *Assume that f is of class $C^1([0,1])$ and satisfies $f'(0) < 0$, $f'(1) < 0$. Let $N = 2$ and assume that there exists a solution $0 \le u(x,y) \le 1$ of (1.1) satisfying (1.3) for some $\alpha \in (0,\pi/2]$. Let $v(t,x,y)$ be a solution of the Cauchy problem (6.1) such that*

$$0 \le v_0 \le u \ \text{ in } \mathbb{R}^2 \ \text{ and } \liminf_{y+|x|\cot\alpha\to+\infty} v_0(x,y) > \theta, \tag{6.6}$$

where $\theta \in (0,1)$ and $f > 0$ in $(\theta, 1)$. Then, for every sequence $t_n \to +\infty$, there exist a subsequence $t_{n'} \to +\infty$ and $(a,b) \in \mathbb{R}^2$ such that

$$v(t_{n'} + t, x, y - ct_{n'} - ct) \to u(x+a, y+b) \ \textit{ locally uniformly in } (t,x,y) \in \mathbb{R}^3 \textit{ as } n' \to +\infty.$$

Proof. The functions $w_n(t,x,y) = v(t_n + t, x, y - ct_n - ct)$ solve

$$\partial_t w_n = \Delta w_n - c\partial_y w_n + f(w_n) \tag{6.7}$$

for $t > -t_n$. Furthermore, since u is a solution of (1.1), the parabolic maximum principle implies that

$$w_n(t,x,y) \leq u(x,y)$$

for all $(x,y) \in \mathbb{R}^2$ and for all $t \geq -t_n$. On the other hand, because of the second inequality in (6.6) and because v_0 is nonnegative, there exist $\eta \in (\theta, 1]$ and $s_0 \in \mathbb{R}$ such that

$$\forall (x,y) \in \mathbb{R}^2, \quad v_0(x,y) \geq \max(H(\pm x\cos\alpha + y\sin\alpha + s_0)) = H(|x|\cos\alpha + y\sin\alpha + s_0),$$

where $H(s) = 0$ if $s < 0$ and $H(s) = \eta$ if $s \geq 0$. Therefore,

$$\forall t \geq -t_n, \ \forall (x,y) \in \mathbb{R}^2, \quad w_n(t,x,y) \geq \max(w^+(t_n + t, x, y), w^-(t_n + t, x, y)),$$

where the functions $w^\pm$ solve equation (6.7) with initial conditions

$$w^\pm(0,x,y) = H(\pm x\cos\alpha + y\sin\alpha + s_0).$$

Consider the function w^+. Since equation (6.7) is invariant by translation and since $w^+(0,\cdot,\cdot)$ only depends on the variable $s = x\cos\alpha + y\sin\alpha$, so does $w^+(t,\cdot,\cdot)$ at any time $t \geq 0$. Therefore, $w^+(t,x,y)$ can be written as $w^+(t,x,y) = W^+(t,s)$ where W^+ solves

$$\begin{cases} W_t^+ &= W_{ss}^+ - c(f)W_s^+ + f(W^+) \\ W^+(0,s) &= H(s+s_0). \end{cases}$$

Here $c(f)$ denotes the unique speed for problem (3.9). Remember that the existence of u solving (1.1) and (1.3) with $\alpha \in (0, \pi/2]$ implies that $c = c(f)/\sin\alpha$ and $c(f) \geq 0$ (from Theorems 3.8 and 3.12). Because of the well-known stability results for the one-dimensional front U, it follows that $W^+(t,s) \to U(s + s_1)$ uniformly in $s \in \mathbb{R}$ as $t \to +\infty$, for some $s_1 \in \mathbb{R}$, where U is the solution of (3.9). By symmetry in the x-variable, it also follows that $w^-(t,x,y) \to U(s' + s_1)$ uniformly in $(x,y) \in \mathbb{R}^2$ as $t \to +\infty$, where $s' = -x\cos\alpha + y\sin\alpha$. Consequently,

$$\forall (t,x,y) \in \mathbb{R}^3, \quad \liminf_{n \to +\infty} w_n(t,x,y) \geq \max(U(\pm x\cos\alpha + y\sin\alpha + s_1)).$$

Eventually, from standard parabolic estimates, there exists a subsequence $n' \to +\infty$ such that the functions $w_{n'}$ converge locally uniformly in $\mathbb{R} \times \mathbb{R}^2$ to a classical solution $w(t,x,y)$ of $w_t = \Delta w - cw_y + f(w)$ such that

$$\max(U(\pm x\cos\alpha + y\sin\alpha + s_1)) \leq w(t,x,y) \leq u(x,y)$$

for all $(t,x,y) \in \mathbb{R}^3$.

The function $v(t,x,y) = w(t,x,y+ct)$ then satisfies (6.2) and (6.3) with $\phi(x) = -|x|\cot\alpha$. Theorem 6.1 yields that $v(t,x,y) = z(x, y+ct)$, for some solution z of (1.1) satisfying (1.3), and Theorem 3.13 implies that z is a translate of u. □

Remark 6.4 *The assumptions on f and u are especially satisfied if f is of the bistable type (1.4) and if $\displaystyle\int_0^1 f > 0$.*

Furthermore, with the same arguments, one can easily check that Corollary 6.3 holds if $\alpha \in [\pi/2, \pi)$ and if (6.6) is changed into:

$$u(x,y) \le v_0(x,y) \le 1 \quad \textit{and} \quad \limsup_{y+|x|\cot\alpha \to -\infty} v(x,y) < \theta,$$

where $\theta \in (0,1)$ and f is negative in $(0,\theta)$.

Lastly, in the case where f is of the combustion type (1.2), Corollary 6.3 still holds, because of the convergence to planar travelling fronts for the solutions of $v_t = v_{xx} + f(v)$ with step-like initial conditions of the type $H(x)$ as above.

A consequence of Corollary 6.3 is that, if v_0 satisfies (6.6), then the ω-limit set $\omega(v_0)$ is made up of travelling waves. Condition (6.6) is especially satisfied when v_0 lies between two translates of a solution u of (1.1) and (1.3). But, even under condition (6.6), the ω-limit set $\omega(v_0)$ of v_0 may well be a continuum, and one may ask for sufficient conditions for $\omega(v_0)$ to be a singleton. This is the purpose of Theorem 6.5 below.

Theorem 6.5 *Assume that f is of class $C^1([0,1])$ and satisfies: either $f'(0) < 0$ and $f'(1) < 0$, or f is of the type (1.2). Let $N = 2$ and assume that there exists a solution $0 \le u(x,y) \le 1$ of (1.1) satisfying (1.3) for some $\alpha \in (0,\pi/2) \cup (\pi/2,\pi)$. Let $v(t,x,y)$ be a solution of the Cauchy problem (6.1) with initial condition $0 \le v_0 \le 1$ such that*

$$v_0(x,y) \le u(x+a, y+b) \textit{ if } \alpha < \pi/2$$

(resp. $v_0 \ge u(x+a,y+b)$ if $\alpha > \pi/2$) in $\mathbb{R}^2$ for some $(a,b) \in \mathbb{R}^2$.

1) Assume that v_0 is uniformly continuous and

$$|v_0(x,y) - u(x,y)| \le Ce^{-\rho\sqrt{x^2+y^2}} \quad \textit{for all } (x,y) \in \mathbb{R}^2,$$

for some positive constants C and ρ. Then $v(t,x,y-ct)$ converges to u uniformly in (x,y) and exponentially in t, as $t \to +\infty$, namely

$$\|v(t,\cdot,\cdot - ct) - u\|_{L^\infty(\mathbb{R}^2)} \le C'e^{-\omega t} \quad \textit{for all } t \ge 0,$$

for some positive constants C' and ω.

2) Assume that v_0 is C^1, $|\nabla v| \in L^\infty(\mathbb{R}^2)$ and $\liminf_{y+|x|\cot\alpha \to +\infty} v_0(x,y) > \theta$ if $\alpha < \pi/2$, where $f > 0$ in $(\theta,1)$ (resp. $\limsup_{y+|x|\cot\alpha\to-\infty} v_0(x,y) < \theta$ if $\alpha > \pi/2$, where $f < 0$ in $(0,\theta)$ if $f'(0) < 0$). Also assume that

$$|(\pm\sin\alpha, -\cos\alpha)\cdot\nabla v_0(x,y)| \le Ce^{\rho(\mp x\cos\alpha + y\sin\alpha)} \quad \textit{for all } (x,y) \in \mathbb{R}^2,$$

(resp. $|(\pm\sin\alpha, -\cos\alpha)\cdot\nabla v_0(x,y)| \le Ce^{\rho(\pm x\cos\alpha - y\sin\alpha)}$), for some positive constants C and ρ. Then $v(t,\cdot,\cdot - ct)$ converges uniformly in $\mathbb{R}^2$ to a translate of u as $t \to +\infty$.

Part 1) essentially means that if v_0 is exponentially close to u at infinity, then v converges to u uniformly in time in the moving frame with speed c downwards. This condition for v_0 is especially fulfilled if $v_0 - u$ has compact support. Actually, more precise convergence results in weighted Banach spaces can be obtained. Part 2) means that the convergence phenomenon is really governed by the behaviour of the initial datum when the space variable becomes infinite along the directions $(\pm\sin\alpha, -\cos\alpha)$, which have an angle α with respect to the vector $(0,-1)$. Some recent results show that, if the initial datum v_0 has no limit in these directions, then $\omega(v_0)$ is made up of a continuum of waves.

The proof of Theorem 6.5 in the case where f satisfies (1.2) is given in [28]. The proof is first based on the analysis of the linearized operator around a solution u of (1.1) with asymptotic conditions (1.3), and then on the exponential stability of the planar travelling fronts. The case where both $f'(0)$ and $f'(1)$ are negative can be treated the same way (see also [43]).

7 Interaction of KPP-type fronts in any dimension

The previous section was concerned with conical-shaped fronts in reaction-diffusion equations with combustion-type, bistable-type, or more general nonlinearities f which were nonincreasing in some neighbourhoods of 0 and 1. This section deals with another class of nonlinearities f, which are now of the Fisher or Kolmogorov-Petrovsky-Piskunov types ([23], [36]). Namely, one assumes that f is of class $C^2([0,1])$ and satisfies :

$$f(0)=f(1)=0,\ f'(0)>0,\ f'(1)<0,\ f>0 \text{ in } (0,1),\ f \text{ is concave.} \tag{7.1}$$

An example of such a function f is the quadratic nonlinearity $f(s)=s(1-s)$. Such profiles arise in models in population dynamics. As it is well-known, the equation $v_t=\Delta v+f(v)$ has, in dimension $N\geq 2$, an $N+1$-dimensional manifold of planar travelling waves, namely

$$v_{\nu,c,h}(t,z)=\varphi_c(z\cdot\nu+ct+h)$$

where ν varies in the unit sphere $\mathbb{S}^{N-1}$ of $\mathbb{R}^N$, h varies in $\mathbb{R}$ and c varies in $[c^*,+\infty[$ with

$$c^*=2\sqrt{f'(0)}>0.$$

In space dimension $N=1$, there are two 2-dimensional manifolds of travelling waves solutions:

$$v^+_{c,h}(x,t)=\varphi_c(x+ct+h) \text{ and } v^-_{c,h}(x,t)=\varphi_c(-x+ct+h)$$

([2], [13], [20], [25]). For any $c\geq c^*$, the function φ_c satisfies

$$\varphi_c''-c\varphi_c'+f(\varphi_c)=0 \ \text{ in } \mathbb{R},\quad \varphi_c(-\infty)=0 \ \text{ and } \varphi_c(+\infty)=1.$$

The function φ_c is increasing and unique up to translation. For each $c\geq c^*$, let λ_c be the positive real number defined by

$$\lambda_c=\frac{c-\sqrt{c^2-4f'(0)}}{2}=\frac{c-\sqrt{c^2-c^{*2}}}{2}>0 \quad (\lambda_c \text{ satisfies } \lambda_c^2-c\lambda_c+f'(0)=0).$$

For any $c>c^*$, we know that $\varphi_c(s)e^{-\lambda_c s}$ goes to a finite positive limit as $s\to-\infty$. Up to translation, one can then assume that

$$\forall c>c^*,\quad \varphi_c(s)\sim e^{\lambda_c s} \text{ as } s\to-\infty. \tag{7.2}$$

For the minimal speed $c^*=2\sqrt{f'(0)}$, one can assume, up to translation, that

$$\varphi_{c^*}(s)\sim |s|e^{\lambda^* s} \ \text{ as } s\to-\infty, \text{ where } \lambda^*=\lambda_{c^*}=\sqrt{f'(0)}=c^*/2.$$

Many works have been devoted to the question of the behavior for large time and the convergence to travelling waves for the solutions of the Cauchy problem for $v_t = \Delta v + f(v)$, especially in dimension 1, under a wide class of initial conditions. (*see e.g.* Bramson [13]).

The classification of all planar fronts under assumption (7.1) is thus very different from the case where f was non-increasing in some neighbourhoods of 0 and 1, for which the speed $c(f)$ for (3.9), if any, was unique. We will see in the next subsection that, under assumption (7.1), the larger set of planar fronts gives rise to an infinite-dimensional manifold of curved fronts, and especially there will be infinitely many fronts satisfying (1.1) and the asymptotic conditions (1.3). We will then prove in Section 7.2 some monotonicity properties in some cones of directions.

7.1 Existence of an infinite-dimensional manifold of curved fronts

Assume that $N \geq 2$. Let

$$B(0,c^*) = B\left(0, 2\sqrt{f'(0)}\right) = \{z \in \mathbb{R}^N,\ |z| < c^*\}$$

be the open ball of $\mathbb{R}^N$ with center 0 and radius c^*. Set $e_N = (0, \cdots, 0, 1)$. Let us define the sets

$$X = \mathbb{S}^{N-1} \times [c^*, +\infty), \quad \hat{X} = \mathbb{S}^{N-1} \times (c^*, +\infty)$$

equipped with the topology induced by the euclidean structure of $\mathbb{R}^N$. For any $c > c^* = 2\sqrt{f'(0)}$, call

$$S_c = \{(\nu, \gamma) \in \mathbb{S}^{N-1} \times [c^*, +\infty),\ ce_N \cdot \nu = \gamma\}$$

the spherical shell on the sphere of diameter $[0, ce_N]$ outside $B(0,c^*)$ (notice that in dimension 1, S_c would then reduce to the single real number c). Let now $\mathcal{M}_c$ be the set of all nonnegative and nonzero Radon-measures μ on X supported on S_c, such that the restriction μ^* of μ on the sphere $\mathbb{S}^{N-1} \times \{c^*\}$ can be written as a finite sum of Dirac masses:

$$\mu^* = \sum_{1 \leq i \leq k} m_i\ \delta_{(\nu_i, c^*)} \tag{7.3}$$

for some integer $k = k(\mu) \geq 0$, for some positive real numbers $m_i = m_i(\mu)$ and for some directions $\nu_i = \nu_i(\mu) \in \mathbb{S}^{N-1}$. For any $\mu \in \mathcal{M}_c$, call $\hat{\mu}$ the restriction of μ on the set $\hat{X}$ and $\Phi_*\hat{\mu}$ the image of $\hat{\mu}$ by the continuous, one-to-one and onto map

$$\begin{array}{rcl} \Phi\ :\ \hat{X} = \mathbb{S}^{N-1} \times (c^*, +\infty) & \longrightarrow & B(0,c^*)\backslash\{0\} \\ (\nu, c) & \longmapsto & z = 2\lambda_c \nu = \left(c - \sqrt{c^2 - c^{*2}}\right) \nu. \end{array}$$

Let $\hat{\mathcal{M}}_c$ be the set of measures $\mu \in \mathcal{M}_c$ such that $\mu^* = 0$ (*i.e.* $k(\mu) = 0$). We say that a sequence of measures $(\mu_n)_{n \in \mathbb{N}} \in \hat{\mathcal{M}}_c$ converges to a measure $\mu \in \hat{\mathcal{M}}_c$ if: 1) $\displaystyle\int_{\hat{X}} f d\hat{\mu}_n \to \int_{\hat{X}} f d\hat{\mu}$ for each continuous function f on $\hat{X}$ such that $f \equiv 0$ on $\mathbb{S}^{N-1} \times (c^*, c^* + \varepsilon)$ for some $\varepsilon > 0$, and 2) $\mu_n(\hat{X}) \to \mu(\hat{X})$.

As already underlined, there is a finite-dimensional manifold of planar travelling waves $\varphi_c(z \cdot \nu + ct + h)$ for the parabolic equation

$$v_t = \Delta v + f(v). \tag{7.4}$$

One can now wonder if there are non-planar travelling waves, namely some general time-global solutions $v(t, z)$ such that

$$\forall(t, z) \in \mathbb{R} \times \mathbb{R}^N,\ \forall\tau \in \mathbb{R}, \quad v(t + \tau, z) = v(t, z + c\nu\tau)$$

for some direction $\nu \in \mathbb{S}^{N-1}$ and some speed $c \geq 0$ (up to a change $\nu \to -\nu$, one can always assume that $c \geq 0$). Such a wave is propagating in the direction $-\nu$ with the speed c. Up to rotation of the frame, one can restrict to the case $\nu = e_N$, and the function v can then be written as

$$v(t, z) = u(z + cte_N) = u(x, y + ct),$$

where u solves the elliptic equation (1.1), namely

$$\Delta u - cu_y + f(u) = 0 \ \text{ in } \mathbb{R}^N.$$

Conversely, each such solution u gives rise to a travelling wave $u(x, y + ct)$.

The main existence result for equation (1.1) under the assumption (7.1) is the following:

Theorem 7.1 *Assume that f satisfies (7.1), and that $N \geq 2$. Let $c > c^* = 2\sqrt{f'(0)}$ be given. There exists an infinite-dimensional manifold of solutions $0 < u < 1$ of (1.1). Namely, there exists a one-to-one map*

$$\mathcal{M}_c \ni \mu \mapsto u_\mu,$$

such that each function u_μ is a solution of (1.1) ranging in $(0, 1)$, and $u_{\mu^n} \to u_\mu$ in $C^2_{loc}(\mathbb{R}^N)$ if $(\mu^n)_{n\in\mathbb{N}} \to \mu$ in $\mathcal{M}_c$. Furthermore, by construction, u_μ is the smallest solution of (1.1) such that, under the notation (7.3),

$$u_\mu(z) \geq \ \max\left(\max_{1\leq i\leq k} \varphi_{c^*}(z \cdot \nu_i + c^* \ln m_i), \int_{\hat{X}} \varphi_\gamma(z \cdot \nu + \gamma \ln \hat{M})\hat{M}^{-1} d\hat{\mu}(\nu, \gamma)\right) \tag{7.5}$$

for all $z \in \mathbb{R}^N$, where $\hat{M} = \mu(\hat{X})$ (under the convention that if $\hat{M} = 0$ then the second argument in the max *drops). Lastly, $u_{\rho\mu}(x, y) = u_\mu(x, y + c \ln \rho)$ for all $\mu \in \mathcal{M}_c$, $\rho > 0$ and $(x, y) \in \mathbb{R}^N$.*

We will only prove here the existence of u_μ and some lower and upper bounds. We will refer to [31] for the one-to-one and continuity properties, which are much more technical.

Proof. Let $c > 2\sqrt{f'(0)}$ and $\mu \in \mathcal{M}_c$ be given. Under the same notations as above, call

$$\underline{u}(z) = \max\left(\max_{1\leq i\leq k} \varphi_{c^*}(z \cdot \nu_i + c^* \ln m_i), \int_{\hat{X}} \varphi_\gamma(z \cdot \nu + \gamma \ln \hat{M})\hat{M}^{-1} d\hat{\mu}(\nu, \gamma)\right).$$

It is clear that $0 < \underline{u}(z) < 1$ for all $z \in \mathbb{R}^N$, and that $\underline{u}$ is continuous. Notice that if μ is a multiple of a Dirac mass $M\delta_{\nu,\gamma}$ for some $\gamma \geq c^*$, $\nu \in \mathbb{S}^{N-1}$ and $M > 0$, then $\underline{u}(z) = \varphi_\gamma(z \cdot \nu + \gamma \ln M)$. In the general case, $\underline{u}$ can be thought of as a superposition of travelling waves with some weights given by the measure μ.

In the case where $\hat{M} = 0$, then $\underline{u}$ is the maximum of a finite number of travelling waves $\varphi_{c^*}(z \cdot \nu_i + c^* \ln m_i)$, each of them solving (1.1) because of the definition of $\mathcal{M}_c$. Therefore, $\underline{u}$ is a subsolution of (1.1). In the case where $\hat{M} > 0$, denote

$$w(z) := \int_{\hat{X}} \varphi_\gamma(z \cdot \nu + \gamma \ln \hat{M})\hat{M}^{-1} d\hat{\mu}(\nu, \gamma).$$

From standard parabolic estimates and since the function f is smooth, there exists a constant C_0 such that, if $0 \le v(t,x) \le 1$ is a time-global solution of $v_t = \Delta v + f(v)$, then $|v_t|$, $|v_{x_i}|$, $|\Delta v| \le C_0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. Any travelling wave $\varphi_\gamma(z \cdot \nu + \gamma t)$ is such a solution, whence $|\gamma \varphi'_\gamma(s)|$, $|\varphi'_\gamma(s)|$, $|\varphi''_\gamma(s)| \le C_0$ for all $\gamma \ge c^*$ and $s \in \mathbb{R}$. From Lebesgue's dominated convergence theorem, the function $w(z)$ is of class C^2 and it satisfies:

$$\begin{aligned}
\Delta w - c w_y &= \int_{\hat{X}} \left(\varphi''_\gamma(z\cdot\nu + \gamma \ln \hat{M}) - c\nu \cdot e_N \varphi'_\gamma(z\cdot\nu + \gamma\ln\hat{M})\right) \hat{M}^{-1} d\hat{\mu}(\nu,\gamma) \\
&= -\int_{\hat{X}} f(\varphi_\gamma(z\cdot\nu + \gamma\ln\hat{M}))\hat{M}^{-1} d\hat{\mu}(\nu,\gamma) \\
&\ge -f\left(\int_{\hat{X}} \varphi_\gamma(z\cdot\nu + \gamma\ln\hat{M})\hat{M}^{-1} d\hat{\mu}(\nu,\gamma)\right)
\end{aligned}$$

from the definition of $\mathcal{M}_c$ and the concavity of f on $[0,1]$. As a consequence, the function $\underline{u}$ is a sub-solution of (1.1) in the sense of distributions.

Call now
$$\overline{u}(z) = \min(1, \zeta(z)),$$
where
$$\zeta(z) = \sum_{1 \le i \le k} \varphi_{c^*}(z\cdot\nu_i + c^* \ln m_i) + \int_{\hat{X}} e^{\lambda_\gamma(z\cdot\nu + \gamma \ln \hat{M})} \hat{M}^{-1} d\hat{\mu}(\nu,\gamma).$$
One has $0 < \overline{u}(z) \le 1$. We now claim that
$$\varphi_\gamma(s) \le e^{\lambda_\gamma s} \ \text{ for all } s \in \mathbb{R} \text{ and } \gamma > c^*. \tag{7.6}$$
Indeed, the function $\eta(s) = e^{\lambda_\gamma s}$ satisfies
$$\eta'' - \gamma\eta' + f'(0)\eta = 0.$$
For each $t \in \mathbb{R}$, call $\eta^t(s) = \eta(s+t) = e^{\lambda_\gamma s + \lambda_\gamma t}$. Since φ_γ is bounded and satisfies (7.2), it follows that there exists a real number t_0 such that, for all $t \ge t_0$, $\eta^t \ge \varphi_\gamma$ in $\mathbb{R}$. Let us now define
$$\tau = \inf\ \{t \in \mathbb{R},\ \eta^t \ge \varphi_\gamma \text{ in } \mathbb{R}\}.$$
From (7.2), one gets $\tau \ge 0$ and by continuity, one has $\eta^\tau(s) \ge \varphi_\gamma(s)$ for all $s \in \mathbb{R}$. Assume now that $\tau > 0$ and consider a sequence $t^n \overset{<}{\to} \tau$ as $n \to +\infty$. There exists then a sequence of points $s_n \in \mathbb{R}$ such that $\eta^{t_n}(s_n) < \varphi_\gamma(s_n)$. Since φ_γ is bounded, the sequence (s_n) is bounded from above. Up to extraction of some subsequence, two cases may occur: $s_n \to s_\infty \in \mathbb{R}$ or $s_n \to -\infty$ as $n \to +\infty$. Assume first that $s_n \to s_\infty \in \mathbb{R}$ as $n \to +\infty$. It follows that $\eta^\tau(s_\infty) = \varphi_\gamma(s_\infty)$. Define $\omega = \eta^\tau - \varphi_\gamma$. This function ω is nonnegative and vanishes at the point s_∞. Furthermore, the function φ_γ satisfies
$$\varphi''_\gamma - \gamma\varphi'_\gamma + f'(0)\varphi_\gamma \ge \varphi''_\gamma - \gamma\varphi'_\gamma + f(\varphi_\gamma) = 0$$
since $f(s) \le f'(0)s$ for all $s \in [0,1]$. As a consequence, $\omega'' - \gamma\omega' + f'(0)\omega \le 0$. The strong maximum principle then yields that $\omega \equiv 0$. This is impossible because φ_γ is bounded, unlike η. We deduce then that $s_n \to -\infty$ as $n \to +\infty$. Now, $\varphi_\gamma(s_n) \sim e^{\lambda_\gamma s_n}$ as $s_n \to -\infty$ whereas $\varphi_\gamma(s_n) \ge \eta^{t_n}(s_n) = e^{\lambda_\gamma(s_n + t_n)}$. This is ruled out because $t_n \to \tau > 0$ as $n \to +\infty$. Eventually, we conclude that $\tau = 0$, which proves the claim (7.6).

In the region where $\zeta(z)<1$, one has, due to the definition of $\mathcal{M}_c$ and the above facts,

$$\begin{aligned}\Delta\zeta - c\zeta_y &= -\sum_{1\le i\le k} f(\varphi_{c^*}(z\cdot\nu_i + c^*\ln m_i)) + \int_{\hat X}(\lambda_\gamma^2-\gamma\lambda_\gamma)e^{\lambda_\gamma(z\cdot\nu+\gamma\ln\hat M)}\hat M^{-1}d\hat\mu(\nu,\gamma)\\ &\le -\sum_{1\le i\le k} f(\varphi_{c^*}(z\cdot\nu_i + c^*\ln m_i)) - f'(0)\int_{\hat X}\varphi_\gamma(z\cdot\nu+\gamma\ln\hat M)\hat M^{-1}d\hat\mu(\nu,\gamma)\\ &\le -\sum_{1\le i\le k} f(\varphi_{c^*}(z\cdot\nu_i + c^*\ln m_i)) - f\left(\int_{\hat X}\varphi_\gamma(z\cdot\nu+\gamma\ln\hat M)\hat M^{-1}d\hat\mu(\nu,\gamma)\right).\end{aligned}$$

But $f(\alpha_1+\cdots+\alpha_p)\le f(\alpha_1)+\cdots+f(\alpha_p)$ for all $p\in\mathbb{N}$ and $\alpha_i\in[0,1]$ such that $\alpha_1+\cdots+\alpha_p\le 1$, because of the concavity of f on $[0,1]$. Therefore, $\Delta\zeta-c\zeta_y\le -f(\zeta)$ in the region where $\zeta<1$. One then concludes that $\overline{u}$ is a super-solution of (1.1) in the sense of distributions (remember that $f(1)=0$).

From parabolic maximum principle, the solution $v(t,z)$ of $v_t=\Delta v - cv_y + f(v)$ with initial condition $v(0,z)=\underline{u}(z)$ is then nondecreasing in time and it converges as $t\to+\infty$ to a solution u_μ of (1.1) such that $\underline{u}\le u_\mu\le\overline{u}$. Furthermore, the parabolic maximum principle also implies that u_μ is the smallest solution of (1.1) above $\underline{u}$. The strong elliptic maximum principle also yields $0<u_\mu<1$ in $\mathbb{R}^N$.

It is immediate to check from the definition of $\mathcal{M}_c$ that, under obvious notations,

$$\underline{u}_{\rho\mu}(z)=\underline{u}_\mu(z+c\ln\rho\ e_N)\text{ for all }\mu\in\mathcal{M}_c,\ \rho>0\text{ and }z\in\mathbb{R}^N.$$

Therefore, $u_{\rho\mu}=u_\mu(\cdot+c\ln\rho\ e_N)$.

The fact that the map $\mu\mapsto u_\mu$ is one-to-one mainly relies on the following technical results on the asymptotic behavior of the travelling front u_μ as $t\to-\infty$: if one denotes

$$v_\mu(t,z)=u_\mu(x,y+ct)=u_\mu(z+cte_N),$$

then

$$\begin{array}{lll} v_\mu(t,-c^*t\ \nu+z)\underset{t\to-\infty}{\longrightarrow}\varphi_{c^*}(z\cdot\nu+c^*\ln m_i)\ \text{ in } C^2_{loc}(\mathbb{R}^N_z) & & \text{if } \nu=\nu_i \text{ for some } i\\ v_\mu(t,-c^*t\ \nu+z)\underset{t\to-\infty}{\longrightarrow} 0 & & \text{otherwise}\end{array}\tag{7.7}$$

and, for any sequence $t_n\to-\infty$ and any continuous function $\psi(\xi)$ with compact support on $B(0,c^*)\backslash\{0\}$,

$$\begin{aligned}&\int_{B(0,c^*)}\left(\frac{|t_n|}{4\pi}\right)^{N/2} v_\mu(t_n+t,-t_n\xi+z)\ e^{-\frac14(c^{*2}-|\xi|^2)t_n}\ \psi(\xi)\ d\xi\\ &\underset{t_n\to-\infty}{\longrightarrow}\int_{B(0,c^*)} e^{(f'(0)+\frac14|\xi|^2)(t+\ln\hat M)+\frac12 z\cdot\xi}\ \psi(\xi)\hat M^{-1}\ \Phi_*\hat\mu(d\xi)\end{aligned}\tag{7.8}$$

in C^1_{loc} in $t\in\mathbb{R}$ and C^2_{loc} in $z\in\mathbb{R}^N$, under the convention that the right-hand side is zero if $\hat M=0$. We admit this fact here, as well as the continuity of u_μ with respect to μ, and we refer to [31] for detailed proofs. □

Remark 7.2 *Under the notations of Theorem 7.1, if μ is a finite sum of Dirac masses*

$$\mu=\sum_{1\le i\le k} m_i\delta_{(\nu_i,c^*)}+\sum_{k+1\le i\le p,\ c_i<c^*} m_i\delta_{(\nu_i,c_i)}\ \in\ \mathcal{M}_c$$

for some $p \in \mathbb{N}\backslash\{0\}$ and some positive real numbers m_i, then u_μ is the smallest solution of (1.1) such that

$$u_\mu(z) \geq \underline{U}_\mu(z) := \max\left(\max_{1\leq i\leq k} \varphi_{c^*}(z\cdot\nu_i + c^*\ln m_i),\ \max_{k+1\leq i\leq p} \varphi_{c_i}(z\cdot\nu_i + c_i\ln\hat{M} + \theta_i)\right),$$

where $\hat{M} = m_{k+1} + \cdots + m_p$ and $\theta_i = \lambda_{c_i}^{-1}\ln(m_i/\hat{M})$ if $\hat{M} > 0$. In other words, the subsolution $\underline{u}$ used in the proof of Theorem 7.1 could be here replaced by $\underline{U}_\mu$ (see [31] for details). Furthermore,

$$u_\mu(z) \leq \sum_{1\leq i\leq k} \varphi_{c^*}(z\cdot\nu_i + c^*\ln m_i) + \sum_{k+1\leq i\leq p} \varphi_{c_i}(z\cdot\nu_i + c_i\ln\hat{M} + \theta_i).$$

In this case, the solution u_μ can be viewed a front obtained from the mixing of a finite number of planar fronts.

Theorem 7.1 especially leads to the following corollary, which, as a special case, implies that the uniqueness under assumptions (1.3) is no longer true when f satisfies (7.1) (compare with Theorem 3.13). For the sake of simplicity, we will only consider the case of dimension $N = 2$, but more general statements hold in higher dimensions $N \geq 3$.

Corollary 7.3 *Assume that f satisfies (7.1), and that $N = 2$. Let $c > c^* = 2\sqrt{f'(0)}$ and $0 < \alpha_1, \alpha_2 \leq \pi/2$ be given such that*

$$c_1 := c\sin\alpha_1 \geq c^* \quad \textit{and} \quad c_2 := c\sin\alpha_2 \geq c^*.$$

Assume that α_1 and α_2 are not both equal to $\pi/2$. Call $\nu_1 = (-\cos\alpha_1, \sin\alpha_1)$, $\nu_2 = (\cos\alpha_2, \sin\alpha_2)$ and $\phi(x) = -|x|\cot\alpha_1$ for $x \leq 0$, $\phi(x) = -|x|\cot\alpha_2$ for $x \geq 0$. Let h_1 and h_2 be any real numbers. Then there exists an infinite-dimensional manifold of solutions $0 < u(x,y) < 1$ of (1.1) such that

$$\liminf_{y-\phi(x)\to+\infty} u(x,y) = 1, \qquad \limsup_{y-\phi(x)\to-\infty} u(x,y) = 0$$

and

$$\left\{\begin{array}{l} u(x-r, y-r\cot\alpha_1) \to \varphi_{c_1}(-x\cos\alpha_1 + y\sin\alpha_1 + h_1) \\ u(x+r, y-r\cot\alpha_2) \to \varphi_{c_2}(x\cos\alpha_2 + y\sin\alpha_2 + h_2) \end{array}\right. \quad \textit{in } C^2_{loc}(\mathbb{R}^2) \textit{ as } r \to +\infty.$$

In particular, if $0 < \alpha < \pi/2$ and $c\sin\alpha \geq c^*$, then there is an infinite-dimensional manifold of solutions of (1.1) satisfying the asymptotic conditions (1.3).

Therefore, equation (1.1) with a nonlinearity f satisfying (7.1) gives rise to more solutions than the same equation with combustion-type or bistable nonlinearities (1.2) or (1.4). In particular, the solutions u in Corollary 7.3 are not symmetric, up to shift, with respect to any direction, provided $c_1 \neq c_2$, namely $\alpha_1 \neq \alpha_2$.

Proof. We will actually only consider the case where $c_1 > c^*$ and $c_2 > c^*$ (the case where one of the speeds c_1 or c_2 is equal to c^* is just an adaptation of the proof below, due to the special treatment of the minimal speed c^* in the definition of $\underline{u}$ in the proof of Theorem 7.1). Under the notations of Corollary 7.3, let $M > 0$ be such that

$$e^{\lambda_{c_1}(h_1 - c_1\ln M)} + e^{\lambda_{c_2}(h_2 - c_2\ln M)} \leq 1$$

and call

$$m_1 = Me^{\lambda_{c_1}(h_1 - c_1 \ln M)} < M, \quad m_2 = Me^{\lambda_{c_2}(h_2 - c_2 \ln M)} < M.$$

Fix $\eta > 0$ such that $2\eta < \pi - \alpha_1 - \alpha_2$ (possible because α_1 and α_2 are not larger than $\pi/2$ and not both equal to $\pi/2$). Let $\tilde{\mu}$ be any nonnegative finite Radon measure on $\hat{X}$ which is concentrated on the portion of a circle defined by

$$\mathcal{C} = S_c \cap \{(r\cos\theta, r\sin\theta),\ r > 0,\ \alpha_2 + \eta \le \theta \le \pi - \alpha_1 - \eta\},$$

and such that $\tilde{\mu}(\mathcal{C}) = M - m_1 - m_2$ (≥ 0). Finally, denote

$$\mu = m_1 \delta_{(\nu_1, c_1)} + m_2 \delta_{(\nu_1, c_1)} + \tilde{\mu},$$

so that $\mu \in \mathcal{M}_c$ and $M = \mu(\hat{X})$. From the construction given in Theorem 7.1, there exists a solution $0 < u_\mu < 1$ of (1.1) such that

$$\begin{aligned} m_1 M^{-1} \varphi_{c_1}(-x\cos\alpha_1 + y\sin\alpha_1 + c_1 \ln M) + m_2 M^{-1} \varphi_{c_2}(x\cos\alpha_2 + y\sin\alpha_2 + c_2 \ln M) & \\ + \int_{\mathcal{C}} \varphi_\gamma(z\cdot\nu + \gamma\ln M) M^{-1} d\mu(\nu,\gamma) \quad \le \quad u_\mu(z) & \end{aligned}$$

and

$$\begin{aligned} u_\mu(z) \quad \le \quad & m_1 M^{-1} e^{\lambda_{c_1}(-x\cos\alpha_1 + y\sin\alpha_1 + c_1\ln M)} + m_2 M^{-1} e^{\lambda_{c_2}(x\cos\alpha_2 + y\sin\alpha_2 + c_2 \ln M)} \\ & + \int_{\mathcal{C}} e^{\lambda_\gamma(z\cdot\nu + \gamma\ln M)} M^{-1} d\mu(\nu,\gamma) \end{aligned}$$

for all $z = (x, y) \in \mathbb{R}^2$.

Let us check that this solution u_μ satisfies all properties stated in Corollary 7.3. First, the lower bound for u_μ immediately yields

$$\liminf_{y - \phi(x) \to +\infty} u_\mu(x, y) \ge \min(m_1 M^{-1}, m_2 M^{-1}) > 0.$$

Since $f > 0$ in $(0, 1)$, by using the same arguments as in the proof of Lemma 3.2, it follows then that

$$\liminf_{y - \phi(x) \to +\infty} u_\mu(x, y) = 1.$$

Take now any $A > 0$ and $x \le 0$. With the notations $\nu = (\cos\omega, \sin\omega)$ $(\alpha_2 + \eta \le \omega \le \pi - \alpha_1 - \eta)$ for the points (ν, γ) on $\mathcal{C}$, one gets from the upper bound of u_μ that

$$\begin{aligned} u_\mu(x, -|x|\cot\alpha_1 - A) \quad \le \quad & m_1 M^{-1} e^{\lambda_{c_1}(-A\sin\alpha_1 + c_1 \ln M)} \\ & + m_2 M^{-1} e^{\lambda_{c_2}(x\sin(\alpha_1 + \alpha_2)/\sin\alpha_1 - A\sin\alpha_2 + c_2 \ln M)} \\ & + \int_{\mathcal{C}} e^{\lambda_\gamma(x\sin(\alpha_1+\omega)/\sin\alpha_1 - A\sin\omega + \gamma\ln M)} M^{-1} d\mu(\nu,\gamma) \\ \le \quad & m_1 M^{-1} e^{\lambda_{c_1}(-A\sin\alpha_1 + c_1\ln M)} + m_2 M^{-1} e^{\lambda_{c_2}(-A\sin\alpha_2 + c_2\ln M)} \\ & + \int_{\mathcal{C}} e^{\lambda_\gamma(-A\sin\omega + \gamma\ln M)} M^{-1} d\mu(\nu,\gamma). \end{aligned}$$

But the qunatities $\sin\omega$, λ_γ and γ are bounded from below and above by two positive constants as (ν, γ) varies in $\mathcal{C}$. Therefore, the third term in the right-hand side of the previous inequality converges to zero as $A \to +\infty$. The same property holds for the first and second terms as well. Therefore,

$$\limsup_{y+|x|\cot\alpha_1\to-\infty,\ x\le 0} u_\mu(x,y)=0.$$

Similarly, one can prove that $\limsup_{y+|x|\cot\alpha_2\to-\infty,\ x\ge 0} u_\mu(x,y)=0$, whence

$$\limsup_{y-\phi(x)\to-\infty} u_\mu(x,y)=0.$$

Lastly, let us prove the convergence of u_μ to the fronts φ_{c_1} and φ_{c_2} in the directions $(-\sin\alpha_1,-\cos\alpha_1)$ and $(\sin\alpha_2,-\cos\alpha_2)$ respectively. Let $(r_n)_{n\in\mathbb{N}}$ be any sequence converging to $+\infty$. Up to extraction of some subsequence, the functions

$$u_n(x,y)=u_\mu(x-r_n,y-r_n\cot\alpha_1)$$

converge in $C^2_{loc}(\mathbb{R}^2)$ to a solution $0\le u(x,y)\le 1$ of (1.1) such that

$$u(x,y)\ge m_1M^{-1}\varphi_{c_1}(-x\cos\alpha_1+y\sin\alpha_1+c_1\ln M).$$

Furthermore,

$$\begin{array}{rcl} u_n(x,y) & \le & m_1M^{-1}e^{\lambda_{c_1}(-x\cos\alpha_1+y\sin\alpha_1+c_1\ln M)}\\ & & +m_2M^{-1}e^{\lambda_{c_2}(x\cos\alpha_2+y\sin\alpha_2+c_2\ln M-r_n\sin(\alpha_1+\alpha_2)/\sin\alpha_1)}\\ & & +\displaystyle\int_{\mathcal{C}}e^{\lambda_\gamma(x\cos\omega+y\sin\omega+\gamma\ln M-r_n\sin(\alpha_1+\omega)/\sin\alpha_1)}M^{-1}d\mu(\nu,\gamma).\end{array}$$

Since $0<\alpha_1+\alpha_2<\pi$, the second term in the right-hand side converges to 0 as $r_n\to+\infty$, for each $(x,y)\in\mathbb{R}^2$. Similarly, one has $0<\alpha_1+\alpha_2+\eta\le\alpha_1+\omega\le\pi-\eta<\pi$ on $\mathcal{C}$, hence the third term converges to 0 as well, from Lebesgue's dominated convergence theorem. Therefore,

$$\begin{array}{c} 0<m_1M^{-1}\varphi_{c_1}(-x\cos\alpha_1+y\sin\alpha_1+c_1\ln M)\\ \le u_\mu(x,y)\le\min\left(1,m_1M^{-1}e^{\lambda_{c_1}(-x\cos\alpha_1+y\sin\alpha_1+c_1\ln M)}\right)\end{array}\tag{7.9}$$

for all $(x,y)\in\mathbb{R}^2$. Call s the variable

$$s=-x\cos\alpha_1+y\sin\alpha_1$$

and let us prove that u only depends on this variable s. First, one has $\liminf_{s\to+\infty}u(x,y)\ge m_1M^{-1}>0$ and one concludes as in the proof of Lemma 3.2 that $\liminf_{s\to+\infty}u(x,y)=1$. One also has $\liminf_{s\to-\infty}u(x,y)=0$, but one cannot apply Theorems 2.4 or 2.6, because $f'(0)>0$. However, we will adapt the proof and still use a sliding method. Fix any $r\in\mathbb{R}$ and call

$$v(x,y)=u(x+r,y+r\cot\alpha_1).$$

The function v still satisfies (7.9) and, since $\varphi_{c_1}(\xi)\sim e^{\lambda_{c_1}\xi}$ as $\xi\to-\infty$, there holds that

$$u(x,y),\ v(x,y)\ \sim\ m_1M^{-1}e^{\lambda_{c_1}(s+c_1\ln M)}\quad\text{as } s\to-\infty\tag{7.10}$$

uniformly in the variable orthogonal to s. Notice that the above formula, together with the strong elliptic maximum principle, implies that $0<u,v<1$ in $\mathbb{R}^2$. But $f'(1)<0$ and therefore the comparison Theorem 2.1 holds as $s \to +\infty$, and since $0 < u, v < 1$ have the same behavior as $s \to -\infty$, there exists $t \in \mathbb{R}$ such that

$$v^\tau(x,y) := v(x - \tau\cos\alpha_1, y + \tau\sin\alpha_1) \geq u(x,y)$$

for all $(x,y) \in \mathbb{R}^2$ and for all $\tau \geq t$. Call

$$\tau^* = \min\{t \in \mathbb{R},\ v^\tau \geq u \text{ in } \mathbb{R}^2 \text{ for all } \tau \geq t\} \ \in \ \mathbb{R}.$$

One has $v^{\tau^*} \geq u$ in $\mathbb{R}^2$. Assume now (by contradiction) that $\tau^* > 0$. From (7.10), there exists then $\varepsilon \in (0, \tau^*)$ and $A > 0$ such that

$$v^\tau \geq u \text{ in } \{s \leq -A\} \text{ for all } \tau \in [\tau^* - \varepsilon, \tau^*].$$

Furthermore, one can also assume with loss of generality that

$$v^\tau \geq 1 - \delta \text{ in } \{s \geq A\},$$

where f is decreasing in $[1-\delta, 1]$ and $\delta > 0$. If $\inf_{|s|\leq A}(v^{\tau^*} - u) > 0$, then, even if it means decreasing ε, one can assume that $v^\tau \geq u$ in $\{|s| \leq A\}$ for all $\tau \in [\tau^* - \varepsilon, \tau^*]$. Comparison Theorem 2.1 of Section 2 then implies that

$$v^\tau \geq u \text{ in } \{s \geq A\} \text{ for all } \tau \in [\tau^* - \varepsilon, \tau^*].$$

Finally, one gets a contradiction with the minimality of τ^*. Thus, there exists a sequence (x_n, y_n) in $\mathbb{R}^2$ such that

$$-A \leq s_n = -x_n\cos\alpha_1 + y_n\sin\alpha_1 \leq A \ \text{ and } \ v^{\tau^*}(x_n,y_n) - u(x_n,y_n) \to 0 \text{ as } n \to +\infty.$$

Up to extraction of some subsequence, one can assume that $s_n \to s_\infty \in [-A, A]$ and the functions $U_n(x,y) = u(x + x_n, y + x_n\cot\alpha_1)$, $V_n(x,y) = v(x + x_n, y + x_n\cot\alpha_1)$ converge in $C^2_{loc}(\mathbb{R}^2)$ to two solutions $0 \leq U(x,y),\ V(x,y) \leq 1$ of (1.1) such that

$$V^{\tau^*}(x,y) := V(x - \tau^*\cos\alpha_1, y + \tau^*\sin\alpha_1) \geq U(x,y)$$

in $\mathbb{R}^2$, with equality at the point $(0, s_\infty/\sin\alpha_1)$. The strong maximum principle yields $V^{\tau^*} \equiv U$. But U and V still satisfy (7.10) (due to the definitions of U_n and V_n), while $V^{\tau^*} \sim e^{\lambda_{c_1}\tau^*}V$ as $s \to -\infty$. One has reached a contradiction, because τ^* and λ_{c_1} are positive.

As a consequence, $\tau^* \leq 0$, whence

$$v(x,y) = u(x + r, y + r\cot\alpha_1) \geq u(x,y)$$

for all $(x,y) \in \mathbb{R}^2$. Since this is true for all $r \in \mathbb{R}$, one concludes that u depends only on the variable s, namely

$$u(x,y) = w(-x\cos\alpha_1 + y\sin\alpha_1)$$

for some function $w\ :\ \mathbb{R} \to [0,1]$. The function w satisfies

$$w'' - c\sin\alpha_1 w' + f(w) = w'' - c_1 w' + f(w) = 0 \ \text{ in } \mathbb{R}$$

and $w(-\infty) = 0$ and $w(+\infty) = 1$. From the uniqueness up to shift of such solutions w for a given speed (here, c^*), it follows that $w = \varphi_{c_1}(\cdot + s^*)$ for some $s^* \in \mathbb{R}$. But, because of the definition of m_1,

$$w(s) \sim m_1 M^{-1} e^{\lambda_{c_1}s + \lambda_{c_1}c_1\ln M} = e^{\lambda_{c_1}(s+h_1)} \sim \varphi_{c_1}(s + h_1) \ \text{ as } s \to -\infty.$$

Therefore, $s^* = h_1$ and $u(x,y) = \varphi_{c_1}(s+h_1)$. Since the limit function u does not depend on the sequence $(r_n)_{n\in\mathbb{N}}$, one concludes that

$$u(x-r, y-r\cot\alpha_1) \to \varphi_{c_1}(-x\cos\alpha_1 + y\sin\alpha_1 + h_1) \text{ in } C^2_{loc}(\mathbb{R}^2) \text{ as } r\to+\infty.$$

Similarly, one can prove that $u(x+r, y-r\cot\alpha_2) \to \varphi_{c_2}(x\cos\alpha_2 + y\sin\alpha_2 + h_2)$ in $C^2_{loc}(\mathbb{R}^2)$ as $r\to+\infty$. Because of Theorem 7.1, there exists then an infinite-dimensional manifold of solutions $0<u<1$ of (1.1) satisfying the conclusions of Corollary 7.3. □

Travelling fronts u are special solutions of (7.4) of the type $v(t,z) = u(z+ct\nu)$. The more general question of the description of the set of all time-global solutions v of $v_t = \Delta v + f(v)$ is also dealt with in [31]. There exists an infinite-dimensional manifold of solutions of this problem, given as nonlinear interactions of planar travelling fronts, in the same spirit as Theorem 7.1 above (see [31] for more precise statements). Furthermore, a partial-uniqueness result is also proved in [31].

7.2 Monotonicity and further qualitative properties

This section is devoted to the proof of the following

Theorem 7.4 *Assume that f satisfies (7.1) and let $N\ge 2$. Let $0<u<1$ be a solution (1.1) for some $c\ge 0$. Then, $c\ge c^*$. Furthermore, u is decreasing in each unit direction $\tau=(\tau_x,\tau_y)\in\mathbb{R}^N$ such that $\tau_y < -\cos(\arcsin(c^*/c))$, and for each such τ, one has $\lim_{s\to-\infty} u(a+s\tau) = 1$ and $\lim_{s\to+\infty} u(a+s\tau) = 0$ for all vector $a\in\mathbb{R}^N$. Lastly, if $c=c^*$, then $u(x,y) = \varphi_{c^*}(y+h)$ for some $h\in\mathbb{R}$.*

Theorem 7.4 can be viewed as a counterpart of the qualitative results stated in Section 3. In particular, monotonicity properties still hold under assumption (7.1), even if one cannot relate the speed c to the Lipschitz norm of the level sets of u. But the uniqueness and one-dimensional symmetry properties are still valid here for the minimal speed, as they were in Section 3 in the case where $c(f)\neq 0$.

Proof. Let us first remind the following result from [2]: if $0\le v_0\le 1$ is a continuous function which is not identically equal to 0 in $\mathbb{R}^N$, then the solution $v(t,z)$ of the Cauchy problem (7.4) with initial datum v_0 satisfies:

$$\liminf_{t\to+\infty,\ |z|\le ct} v(t,z) = 1 \tag{7.11}$$

for all $c\in[0,c^*)$. Assume by contradiction that $0<u<1$ solves (1.1) with $c\in[0,c^*)$, and call

$$v(t,z) = v(t,x,y) = u(x,y+ct).$$

The function $0<v<1$ is a time-global solution of (7.4), whence $v(t,-cte_N)\to 1$ as $t\to+\infty$, from the result recalled above. But $v(t,-cte_N) = u(0,0)$ is a fix number in $(0,1)$. One has then reached a contradiction.

Fix now any unit direction $\tau=(\tau_x,\tau_y)\in\mathbb{R}^N$ such that $\tau_y < -\cos(\arcsin(c^*/c))$, and let us prove that $\tau\cdot\nabla u < 0$ in $\mathbb{R}^N$. Call

$$w(z) = \frac{\tau\cdot\nabla u(z)}{u(z)}.$$

From standard elliptic estimates and Harnack inequality, the function $|\nabla u|/u$ is globally bounded in $\mathbb{R}^N$, and then w is globally bounded. Let us now prove that it is nonpositive. Suppose by contradiction that $\sup_{\mathbb{R}^N} w = \varepsilon > 0$. There exists a sequence $(z_n)_{n\in\mathbb{N}} \in \mathbb{R}^N$ such that $w(z_n) \to \varepsilon$ as $n \to +\infty$. Up to extraction of some subsequence, two and only two cases may occur :

Case 1: $u(z_n) \to \alpha \in (0, 1]$ as $n \to +\infty$,

Case 2: $u(z_n) \to 0$ as $n \to +\infty$.

Let us first deal with Case 1. The function w satisfies

$$\Delta w + 2\frac{\nabla u}{u} \cdot \nabla w - cw_y + \left(f'(u) - \frac{f(u)}{u}\right) w = 0 \ \text{ in } \mathbb{R}^N.$$

Let us set

$$u_n(z) = u(z + z_n) \ \text{ and } \ w_n(z) = w(z + z_n).$$

From standard elliptic estimates, the functions u_n converge (up to extraction of some subsequence) in $C^2_{loc}(\mathbb{R}^N)$ to a solution $0 \le u_\infty \le 1$ of (1.1). Furthermore,

$$u_\infty(0, 0) = \alpha \in (0, 1]$$

by assumption. Therefore, the function u_∞ is positive everywhere because of the strong maximum principle, and the globally bounded sequences of functions $\nabla u_n/u_n$, $f'(u_n)$ and $f(u_n)/u_n$ converge to the globally bounded functions $\nabla u_\infty/u_\infty$, $f'(u_\infty)$ and $f(u_\infty)/u_\infty$, respectively. Similarly, the globally bounded functions w_n converge in $C^2_{loc}(\mathbb{R}^N)$ (up to extraction of some subsequence) to a globally bounded function w_∞, which is equal to

$$w_\infty = \frac{\tau \cdot \nabla u_\infty}{u_\infty}$$

because of the positivity of u_∞. Moreover, $w_\infty \le \varepsilon$ in $\mathbb{R}^N$, $w_\infty(0) = \varepsilon$ and

$$\Delta w_\infty + 2\frac{\nabla u_\infty}{u_\infty} \cdot \nabla w_\infty - c(w_\infty)_y + \left(f'(u_\infty) - \frac{f(u_\infty)}{u_\infty}\right) w_\infty = 0 \ \text{ in } \mathbb{R}^N.$$

From the profile of f in (7.1), it follows that $f'(s) - f(s)/s \le 0$ for all $s \in (0, 1]$. Therefore, the strong elliptic maximum principle implies that $w_\infty \equiv \varepsilon$, namely $\tau \cdot \nabla u_\infty \equiv \varepsilon u_\infty$. Since u_∞ and ε are positive, one gets especially that $\tau \cdot \nabla u_\infty > 0$ in $\mathbb{R}^N$. As a consequence, $u_\infty < 1$ (otherwise $u_\infty \equiv 1$ in $\mathbb{R}^N$ from the strong maximum principle, whence $\tau \cdot \nabla u_\infty \equiv 0$). Call

$$\rho = c\tau_y\tau - ce_N.$$

One has

$$|\rho|^2 = c^2(1 - \tau_y^2) < c^2(1 - \cos^2(\arcsin(c^*/c))) = (c^*)^2.$$

The function

$$0 < V(t, z) = u_\infty(z + cte_N) < 1$$

satisfies (7.4) and then

$$\zeta(t) := V(t, \rho t) = V(t, ct\tau_y\tau - cte_N) \to 1 \text{ as } t \to +\infty,$$

because of (7.11), while $\zeta(t) = u_\infty(ct\tau_y\tau)$ ranges in $(0, 1)$ and is decreasing because $\tau \cdot \nabla u_\infty > 0$ and $c\tau_y < 0$. Case 1 is then ruled out.

Let us now deal with Case 2. Under the same notations as above, one has $v_t + \rho \cdot \nabla v = c\tau_y\tau \cdot \nabla v$ and

$$(v_t + \rho \cdot \nabla v)/v \ge c\tau_y\varepsilon \text{ in } \mathbb{R}^N$$

(observe that $c\tau_y \le 0$) and

$$(v_t(0,z_n)+\rho\cdot\nabla v(0,z_n))/v(0,z_n) \to c\tau_y\varepsilon \text{ as } n\to+\infty.$$

Denote

$$w_n(t,z)=\frac{v(t,z+\rho t+z_n)}{v(0,z_n)}\ e^{\frac{1}{2}\rho\cdot z},\quad \text{for all } (t,z)\in\mathbb{R}\times\mathbb{R}^N.$$

Since the fields v_t/v and $\nabla v/v$ are globally bounded, there exists a constant C such that $w_n(t,z)\le e^{C(|t|+|z|)}$ for all $(t,z)\in\mathbb{R}\times\mathbb{R}^N$ and all $n\in\mathbb{N}$. In particular, the sequence (w_n) is locally bounded and the functions $(t,z)\mapsto v(t,z+z_n,)$ approach 0 locally in $\mathbb{R}\times\mathbb{R}^N$ because $u(z_n)=v(0,z_n)\to 0$. On the other hand, each function w_n satisfies

$$(w_n)_t=\Delta w_n+\left(\frac{f(v(t,z+\rho t+z_n))}{v(t,z+\rho t+z_n)}-\frac{1}{4}|\rho|^2\right)\ w_n,\quad (t,z)\in\mathbb{R}\times\mathbb{R}^N.$$

From standard parabolic estimates, the functions w_n converge in $C^1_{loc}(\mathbb{R}_t)$ and $C^2_{loc}(\mathbb{R}^N_z)$ (up to extraction of some subsequence), to a nonnegative and locally bounded function w_∞. The function w_∞ solves

$$(w_\infty)_t=\Delta w_\infty+(f'(0)-\frac{1}{4}|\rho|^2)\ w_\infty\ \text{ in } \mathbb{R}\times\mathbb{R}^N \tag{7.12}$$

and it satisfies

$$\forall t\in\mathbb{R},\ \forall z\in\mathbb{R}^N,\quad w_\infty(t,z)\le e^{C(|t|+|z|)}. \tag{7.13}$$

Due to the definition of w_n and to the choice of (z_n), one has

$$(w_n)_t(0,0)=\frac{v_t(0,z_n)+\rho\cdot\nabla v(0,z_n)}{v(0,z_n)}\to c\tau_y\varepsilon\ \text{ as } n\to+\infty.$$

Hence,

$$(w_\infty)_t(0,0)=c\tau_y\varepsilon<0. \tag{7.14}$$

Choose now any point $(t,z)\in\mathbb{R}\times\mathbb{R}^N$. Because of (7.12) and (7.13), $w_\infty(t,z)$ can be written as

$$w_\infty(t,z)=e^{(f'(0)-\frac{1}{4}|\rho|^2)(t+k)}\int_{\mathbb{R}^N}p(t+k,z-\xi)\ w_\infty(-k,\xi)\ d\xi$$

for all $k>|t|$, where $p(s,\xi)=(4\pi s)^{-N/2}e^{-\frac{|\xi|^2}{4s}}$ for any $s>0$ and $\xi\in\mathbb{R}^N$. As a consequence,

$$(w_\infty)_t(t,z)=e^{(f'(0)-\frac{1}{4}|\rho|^2)(t+k)}\int_{\mathbb{R}^N}p_t(t+k,z-\xi)\ w_\infty(-k,\xi)\ d\xi\ \ +\ (f'(0)-\frac{1}{4}|\rho|^2)\ w_\infty(t,z).$$

Notice that $p_s(s,\xi)\ge-\frac{N}{2s}p(s,\xi)$ for all $s>0$ and $\xi\in\mathbb{R}^N$. Since w_∞ is nonnegative, it follows that

$$(w_\infty)_t(t,z)\ge\left(f'(0)-\frac{1}{4}|\rho|^2-\frac{N}{2(t+k)}\right)\ w_\infty(t,z).$$

Passing to the limit $k\to+\infty$ in the above formula leads to

$$(w_\infty)_t(t,z)\ge(f'(0)-\frac{1}{4}|\rho|^2)\ w_\infty(t,z).$$

Since $|\rho|<c^*=2\sqrt{f'(0)}$ and $w_\infty\ge0$, one gets $(w_\infty)_t(t,z)\ge0$ for all $(t,z)\in\mathbb{R}\times\mathbb{R}^N$. That is in contradiction with (7.14). Therefore, Case 2 is ruled out too.

Thus $\tau \cdot \nabla u \le 0$ in $\mathbb{R}^N$. Hence, the function $\zeta(t) = u(ct\tau_y\tau)$ ranges in $(0,1)$ and is nondecreasing (remember that $c\tau_y < 0$), while $\zeta(+\infty) = 1$ as already underlined. Therefore, $\tau \cdot \nabla u \not\equiv 0$ and the strong maximum principle yields $\tau \cdot \nabla u < 0$ in $\mathbb{R}^N$.

Observe also that the same arguments as above imply that $u(a + s\tau) \to 1$ (resp. $\to 0$) as $s \to -\infty$ (resp. $s \to +\infty$) for all $a \in \mathbb{R}^N$ and $\tau \in \mathbb{S}^{N-1}$ such that $\tau_y < -\cos(\arcsin(c^*/c))$. Notice indeed that

$$\limsup_{|z|\le c|t|,\ t\to-\infty} V(t,z) = 0 \quad \text{for any } c \in [0, c^*)$$

and for any time-global solution $0 < V(t,z) < 1$ of (7.4), because of (7.11).

Let us now consider the case where $c = c^*$. From the previous arguments and by continuity, the function u is then nonincreasing in any direction $\tau \in \mathbb{R}^N$ such that $\tau_y \le 0$. It is then both nondecreasing and nonincreasing in any direction τ such that $\tau_y = 0$. Therefore, $v = v(y)$ depends on y only and it solves $v'' - c^*v' + f(v) = 0$. Furthermore, the previous results imply that $v(-\infty) = 0$ and $v(+\infty) = 1$. One concludes that v is a translate of $\varphi_{c^*}(y)$. That completes the proof of Theorem 7.4. □

Remark 7.5 *Property (7.11) implies that* $\limsup_{t\to-\infty,\ |z|\le c|t|} V(t,z) = 0$ *for any* $c \in [0, c^*)$ *and for any time-global solution* $0 < V(t,z) < 1$ *of (7.4). If* $0 < u < 1$ *is a solution of (1.1) such that* $v(t,z) = u(z + cte_N)$ *satisfies*

$$\limsup_{t\to-\infty,\ |z|\le(c^*+\varepsilon)|t|} v(t,z) = 0$$

for some $\varepsilon > 0$*, then* u *is of the type given in Theorem 7.1. This almost-uniqueness result holds in the framework of general time-global solutions of (7.4) and it is proved in [31].*

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