MINIMAL VOLUME

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1 Introduction

1.1 Definition

A very natural question in Riemannian goemetry is the following : given a C^∞ manifold M, is there a best metric or a distinguished metric on M ? One way to seek at this question is to consider some geometric functional on the space or a subspace of all riemannian metrics of M and look for an extremum of this functional. Here we are interested in the minimal volume, which was introduced by M. Gromov.

Definition 1 (Gromov [Gro], Volume and Bounded Cohomology, IHES 56, 1981). *Let M a C^∞ manifold. Consider on M all complete riemannian metrics with sectional curvature bounded in absolute value by one*

$$|K_g| \leq 1$$

The minimal volume is the infimum of the volume of those metrics

$$Minvol(M) = \inf_{|K_g|\leq 1} vol_g(M)$$

Remarks: 1) We recall that the sectional curvature assigns to each vectorial plane $P \subset T_xM$ of the tangent space a real number $K(P)$. Its geometric meaning is the following. Let $C(r)$ be the circle of radius r tangent to P, i.e. $C(r)$ is the set of all $exp_x(rv)$ where v is a unit vector o in P. Then the length of this circle is given by

$$\ell(C(r)) = 2\pi r\left(1 - \frac{K(P)}{6}r^2 + o(r^2)\right)$$

and $K(P)$ measures the defect of the perimeter to be euclidian.

2) When one scale a metric g into λg, one has

$$Vol_{\lambda g}(M) = \lambda^{n/2} vol_g(M)$$

and

$$K_{\lambda g} = \frac{1}{\lambda} K_g$$

Thus, if the sectional curvature is not idendically null, one can seek the minimal volume of M among metrics where the supremum of the absolute value of the sectional curvature is $+1$. It is equivalent to consider the infimum of

$$vol_g(M) sup|K_g|^{n/2}$$

(which is scale invariant) among complete metrics g with bounded curvature or to consider

$$\inf_{vol_g(M)=1} \sup |K_g|^{n/2}$$

Questions about this functional (see Marcel Berger [Ber], A panoramic view in Riemannian Geometry, question 266)

1. Is $Minvol(M)$ zero or positive? Try to classify manifolds for which $Minvol(M) = 0$ and those for which not.

2. If $Minvol(M) \neq 0$, is it attained by a metric? When those best metrics exists, try to classify them.

3. Compute $Minvol(M)$ for various manifolds. Can we say something about the set of values when M runs through compact manifolds? Is zero an isolated point of the set?

1.2 The 2-dimensional case

Compact surfaces

Suppose M is a nice surface, i.e. compact, oriented, without boundary. We can see K_g a function on M. Suppose $-1 \leq K_g \leq 1$. The Gauss-Bonnet formula gives

$$\begin{aligned} |\chi(M)| &= \left| \frac{1}{2\pi} \int_M K_g(x)\, dv_g(x) \right| \\ &\leq \frac{1}{2\pi} \int_M |K_g(x)|\, dv_g(x) \\ &\leq \frac{vol_g(M)}{2\pi} \end{aligned}$$

where $\chi(M)$ is the Euler characteristic. If you prefer the genus $g(M)$ of the surface, $\chi(M) = 2 - 2g(M)$. The surface of genus one, the torus, has zero minimal volume because he has flat metrics. In other cases, the minimal volume is positive and attained by metrics where $|K_g| = 1$. If the genus is 0 (the manifold is a sphere), the curvature is $+1$ and the metric is the round one. If $g(M) \geq 2$, the metric has curvature -1 thus is hyperbolic. Conevrsely any hyperbolic realizes the minimal volume. It is a classical result that the space of all hyperbolic metrics, quotiented by isometries, of a surface of genus g has dimension $6g - 6$, thus is huge. It is called the Teichmuller space. Note that the set of minimal volume is discrete set in $\mathbb{R}$: $minvol(M) \in 4\pi\mathbb{N}$ when M describes all compact surfaces.

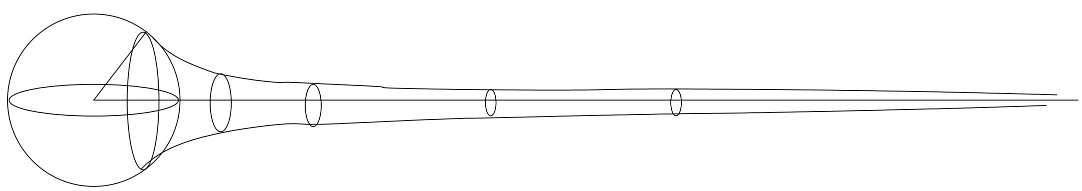
Non-compact surfaces

The Gauss-Bonnet formula holds for complete surface with bounded curvature and finite volume. Thus hyperbolic metric of finite volume are again mimimal for surfaces with $\chi(M) < 0$ and $Minvol(M) = -2\pi\chi(M)$. Moreover, the surface of infinite genus has infinite minimal volume.

For the plane $\mathbb{R}^2$, we have the following (Ch. Bavard and P. Pansu [Ba-Pa], Ann. Sci. Ec. Norm. Sup., 1986)

$$Minvol(\mathbb{R}^2) = 2\pi(1 + \sqrt{2})$$

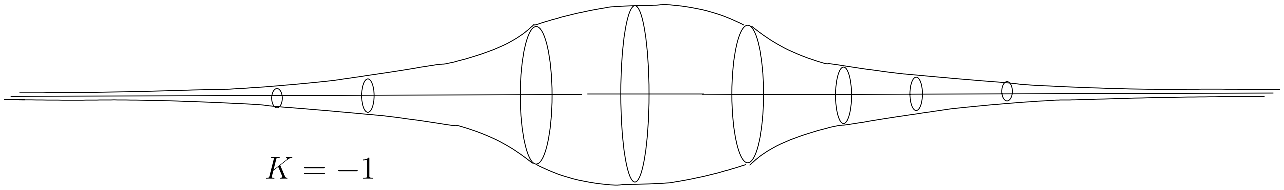
with an extremal C^1 metric obtained by pasting a spherical disk of curvature 1, which boundary has length $\pi\sqrt{2}$, with an hyperbolic cusp which boundary has same length. The metric is not C^2 through the pasting as curvature values change from $+1$ to -1.



an extremal metric on $\mathbb{R}^2$

The cylinder $S^1 \times \mathbb{R}$ has zero minimal volume.

Proof: Indeed, consider a warped product metric $g = f^2(t)d\theta^2 + dt^2$, with an hyperbolic cusp of finite volume at each end. That is $f(t) = e^{-t}$ for large $t > 0$ and $f(t) = e^t$ for large $t < 0$.



The formula for the sectional curvature is

$$K_g = -\frac{f''}{f}$$

Thus g can be choosed complete with finite volume and $-1 \leq K_g \leq 1$. Consider now any $\varepsilon > 0$ and the new metrics

$$g_\varepsilon = \varepsilon^2 f^2(t) d\theta^2 + dt^2$$

Clearly, the sectional curvature of g_ε is unchanged and $vol(g_\varepsilon)$ is arbitrary small. $\square$

We will see below large generalizations of this trick.

From now, unless specified, M will be oriented, compact with dimension ≥ 3

1.3 Manifolds with $Minvol = 0$

Here we give some examples of manifolds M with $Minvol(M) = 0$.

1. M admits a flat Riemannian metric.

2. M admits a free action of the cercle S^1. Trivial examples are the torus or the cylinder. A non trivial example is given by the Hopf's fibration $S^3 \to S^2$. Thus consider any Riemannian metric g on M. After average by the S^1 action, we can suppose the S^1 action isometric. At each point x of M, the tangent space T_xM decomposes orthogonally in a vertical part tangent to the S^1 orbit and in a horizontal part. Thus, one can write $g = g_v + g_h$. As above, consider for any $\varepsilon > 0$ the new metrics

$$g_\varepsilon = \varepsilon^2 g_v + g_h$$

   Using O'Neill formulas on the riemannian submersion $(M, g) \to (M/S^1, g_h)$ (see Besse, Einstein manifolds, chapter 9 or the Technical Chapter in Berger) one can show that sectional curvature remains bounded. The main idea is that the shrinking is one dimensional. On the other hand, the volume can be made arbitrary small.

3. A broader generalization of the S^1 action is performed by Cheeger and Gromov with the definition of *T-structure* and *F-structure*. With their words, an F-structure on a space M is a generalization of a torus action. Different tori (possibly of different dimension) act locally on finite covering spaces of subsets of M. These actions satisfy a compatibility condition, which insures that M is partitionned in different orbits. The F-structure is said to have positive dimension if all orbit have positive dimension. The definition is quite technical. Here I give the formulation given by Fukaya in his survey on Hausdorff convergence ([Fu], definition 19.1, 19.2). A *T-structure* on M is a triple $(U_i, T^{k_i}, \varphi_i)$ such that

   1) $\{U_i\}$ is an open covering of M,

   2) T^{k_i} is a k_i-dimensional torus.

   3) $\varphi_i : T^{k_i} \to Diff(U_i)$ is an effective and smooth action.

   4) When $U_i \cap U_j \neq \emptyset$, $U_i \cap U_j$ is (T^{k_i}, φ_i) and (T^{k_j}, φ_j) invariant and the two actions commute.

   Now for the *F-structure*, we have to consider finite covering $\tilde{U_i}$ of U_i and natural action of torus T^{k_i} on the cover $\tilde{U_i}$ instead of U_i. By natural we means that the orbits are well defined in U_i even if the action does not descend.

   The existence of a F-structure of positive dimension (that is, the orbit have positive dimension) on M is related to the collapsing of M. One says that M is ε-collapsed if there is a Riemannian metric g_ε with $-1 \leq K_{g_\varepsilon} \leq 1$ and injectivity radius $\leq \varepsilon$ at each point. Recall that the injectivity radius at x is the supremum of the radius $r > 0$ such that $exp_x : B(0,r) \subset T_xM \longrightarrow B(x,r) \subset M$ is a diffeomorphism. For metrics with sectional curvature between -1 and $+1$, a bound below of the volume of the unit ball $B(x,1)$ is equivalent to a bound below of the injectivity radius at x. Thus if the minimal volume is zero, the manifold is ε-collapsed for any $\varepsilon > 0$. On the contrary, a manifold can be ε-collapsed and have a not so small volume. For example, a torus $S^1 \times S^1$ where one of the circle has length ε and the other $1/\varepsilon$ is ε-collapsed but has volume $(2\pi)^2$. A fundamental result of Cheeger and Gromov ([Ch-Gr1], [Ch-Gr2]) is

$$\begin{pmatrix} \text{M has a F-structure} \\ \text{of positive dimension} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \text{M is } \varepsilon\text{-collapsed} \\ \text{for any } \varepsilon > 0 \end{pmatrix}$$

   To prove $\Rightarrow$, start with metrics g_ε defined on the U_i, shrinked in certains directions tangent to the orbit. The problem is how to patch them on $U_i \cap U_j$. If the shrinking directions are different, you have to expand the metric in directions normal to both orbits to keep curvature bounded. Maybe the volume is going to infinity. They prove a strenghtened version of the converse $\Leftarrow$. They prove the existence of a universal $\varepsilon_n > 0$ such that ε_n-collapse implies the existence a F-structure of positive dimension. For the vanishing of the minimal volume, they have the following

$$\begin{pmatrix} \text{M has a polarized F-structure} \\ \text{of positive dimension} \end{pmatrix} \Longleftrightarrow Minvol(M) = 0$$

   Roughly speaking, a F-structure is *polarized* if there is a collection of connected (non trivial) subgroups $H_i \subset T^{k_i}$ whose action is locally free and such that the H_i-orbit of

$p \in U_i \cap U_j$ either contains or is contained in the H_j-orbit of p. Cheeger and Gromov prove also that in dimension 3, if M has a F-structure of positive dimension, it has a polarized one. Thus, il the minimal volume of a three-manifold is below some ε_n, it must be zero. X. Rong ([R]) has proved this true for $n = 4$ but it remains in higher dimensions

4. Any product $M \times N$ where M is one of the above example and N is arbitrary.

2 A criterion for $Minvol(M) > 0$

Via the generalization of the Gauss-Bonnet formula, the Euler characteristic provides, in even dimension, an obstruction to the vanishing of the minimal volume:

$$Minvol(M) \geq c(n)\chi(M)$$

M. Gromov defines in [Gro] another invariant, the *simplicial volume*, as follows.

Definition 2. *The fondamental class $[M] \in H_n(M, \mathbb{R})$ can be write*

$$[M] = \sum_i a_i \sigma_i$$

where a_i are reals and σ_i are singular simplices. Then the simplicial volume is

$$||M|| = \inf \left\{ \sum_i |a_i| \ : \ [M] = \sum_i a_i \sigma_i \right\}$$

Some useful properties:

1. If $f : M \to M'$ is a continuous map of degree d, then

$$||M|| \geq d||M'||$$

   In particular, if M has a self mapping of degree $d \geq 2$ then $||M|| = 0$.

2.

$$a(n)||M||||N|| \leq ||M \times N|| \leq b(n)||M||||N||$$

   where $a(n) > 0$, $b(n) > 0$ depends only on $n = dim(M \times N)$.

3. The connected sum satisfies

$$||M\sharp N|| = ||M|| + ||N||$$

Now we state a fundamental result of M.Gromov. Recall that the Ricci curvature is a symetric bilinear form on TM, which can be defined as follows. Given $v \in T_xM$,

$$Ricci(v,v) = \sum_{i=1} n - 1K(P_{ve_i},$$

where (e_i) is an orthonormal basis of the $v^\perp \in T_xM$ and $P_{ve_i} = vect(v, e_i)$. Then,

Theorem 2.1. *If (M, g) satisfies*

$$Ric_g \geq -(n-1)g$$

then

$$vol_g(M) \geq \frac{1}{(n-1)^n n!}||M||$$

Corollary 2.2.

$$Minvol(M) \geq \frac{1}{(n-1)^n n!}||M||$$

Indeed, $K_g \geq -1$ implies $Ric_g \geq -(n-1)g$.

As a consequence, if $||M'|| > 0$ and M has a non zero degree map onto M', $f : M \to M'$, then $Minvol(M) > 0$. If $||M'|| > 0$ and M is another manifold, $Minvol(M\sharp M') > 0$.

Related to the isolation problem, we have the following

Theorem 2.3 (Gromov's isolation theorem). *There exists $\varepsilon_n > 0$ such that the following holds. If $Ric_g \geq -(n-1)g$ and $vol_g(B(p,1)) \leq \varepsilon_n$ for each $p \in M$ then*

$$||M|| = 0$$

So if the minimal volume is sufficiently small, the simplicial volume is zero. In dimension 3 (Cheeger-Gromov) and 4 (Rong [R]), small minimal volume implies zero minimal volume. The question is open in higher dimensions.

It's time to give manifolds for which $||M|| > 0$.

Theorem 2.4 (Thurston's inequality). *If $K_g \leq -1$ then*

$$||M|| \geq C(n) vol_g(M)$$

With the properties of the simplicial volume, one can produce some manifolds with non-zero simplicial volume. For example, a product of negative curvature manifolds has non-zero simplicial volume whereas it has a lot of zero curvature. But other examples are very hard to find. Only recently, J-F Lafont and B. Schmidt [LS] have shown that closed locally symetric spaces of non-compact type have non zero simplicial volume.

If the metric is hyperbolic, the Thurston's inequality is strenghtened in

Theorem 2.5 (Gromov). *If (M, g_0) is hyperbolic, then*

$$||M|| = \frac{vol_{g_0}(M)}{V_n}$$

where V_n is the volume of any ideal regular simplex of the n-hyperbolic space $\mathbb{H}^n$.

3 Best metrics

Now we turn to the question of the best metrics. The stronger result is the following

Theorem 3.1 (Besson, Courtois and Gallot, 1995 [BCG]). *Let (M, g_0) an hyperbolic manifold and g a Riemannian metric such that*

$$Ric_g \geq -(n-1)g$$

Then

$$vol_g(M) \geq vol_{g_0}(M)$$

with equality if and only if g is isometric to g_0.

In particular, the minimal volume is attained by the hyperbolic metric only. In fact, this result is corollary of the more general:

Theorem 3.2 (BCG). *let (M, g_0) a locally symmetric compact manifold of negative curvature and (N, g) another riemannian manifold. Suppose there is a map $f : N \to M$ with degree $d > 0$. Then*

$$h(g)^n vol_g(N) \geq d.h(g_0)^n vol_{g_0}(M)$$

where $h(g)$ and $h(g_0)$ are the volume entropy of the metrics g and g_0. Moreover, there is equality if and only if f is homotopic to a Riemannian covering.

Definition 3. *the* ***volume entropy*** *$h(g)$ of a compact riemannian manifold (Y, g) is defined as follows. Let $\tilde{Y}$ the universal covering of Y, $y \in \tilde{Y}$ and $\tilde{g}$ the lift metric. Then one defines*

$$\begin{aligned} h(g) &= \lim_{r\to\infty} \frac{1}{r} \ln(vol_{\tilde{g}}(B_{\tilde{g}}(y, r))) \\ &= \inf \left\{ c > 0, \int_{\tilde{Y}} e^{-c.\rho(y,z)}\ dvol_{\tilde{g}}(z) < \infty \right\} \end{aligned}$$

(see Manning - Ann.of Math. 110, 1979)

For example, the volume of hyperbolic balls satisfy

$$vol_{\mathbb{H}^n}(B(r)) \sim e^{(n-1)r}$$

as $r \to \infty$ thus $h(g_0) = n - 1$. For other locally symmetric spaces with negative curvature, that is the complex hyperbolic space, the quaternionic hyperbolic space or the Cayley hyperbolic space with curvatures normalized to be pinched as $-4 \leq K \leq -1$, one has $h(g_0) = (n + d - 2)$, with $d = 2, 4$ or 8, and n is the real dimension of the space. With the normalisation $-1 \leq K \leq -\frac{1}{4}$, one has for these spaces $h(g_0) = \frac{n+d-2}{2}$. To obtain 3.1 from 3.2, one applies the Bishop inequality which says: if $Ric_g \geq -(n-1)g$, then

$$vol_{\mathbb{H}^n}(B(r)) \geq vol_{\tilde{g}}(B_{\tilde{g}}(y, r))$$

thus $h(g_0) \geq h(g)$. Now the equality case in 3.1 implies also $h(g) = h(g_0)$.

Remark

1. the theorem 3.2 gives a proof of the Mostow rigidity theorem. Indeed, suppose that N and M are compact locally symmetric of negative curvature and f is an homotopy equivalence. Then inequality holds in both directions and from the equality case, f is homotopic to an isometry.

2. An interesting consequence of the inequality in 3.2 is, when g_0 is hyperbolic,

$$Minvol(N) \geq d.vol_{g_0}(M)$$

In my thesis, I prove that

Theorem 3.3 (Bes). *If (M, g_0) is hyperbolic, $f : N \to M$ a degree $d > 0$ map and*

$$Minvol(N) = d.vol_{g_0}(M)$$

then f is homotopic to a differentiable covering.

It has surprising consequences. First, the minimal volume is non additive by connected sum

Corollary 3.4 (Bes). *Let (M, g_0) hyperbolic. Then*

$$Minvol(M \sharp M) > 2vol_{g_0}(M)$$

Indeed, $M \sharp M$ cannot have an hyperbolic metric. In fact, it cannot have a metric with nonpositive curvature for then, the universal covering would be $\mathbb{R}^n$ and this contradicts $\pi_{n-1}(M \sharp M) \neq 1$.

Secondly, the minimal volume depends on the differentiable structure of the manifold. To see this, one take a differentiable manifold which is homeomorphic to an hyperbolic manifold but is not diffeomorphic. Such exotic differentiable structures had been constructed by Farell and Jones ([Fa-Jo]). Their idea is to do the connected sum of an hyperbolic manifold M, or a finite covering of it, with an exotic sphere Σ (a manifold homeomorphic to the standard sphere but not diffeomorphic). Then $M\sharp\Sigma$ is homeomorphic but not diffeomorphic to M. The existence of exotic spheres follows from the work of Kervaire and Milnor, Smale, with a dimension condition. The lower dimension is $n = 7$ where there is 28 exotic spheres. Moreover, for any $\varepsilon > 0$, Farell and Jones can construct such manifold $M\sharp\Sigma$ with pinched negative curvature metric

$$-1-\varepsilon \leq K \leq -1+\varepsilon.$$

Corollary 3.5 (Bes). *For these manifolds,*

$$Minvol(M\sharp\Sigma) > vol_{g_0}(M)$$

Jeff Boland, Chris Connell and Juan Souto ([BCS]) have extended BCG's theorem 3.2 for complete hyperbolic manifold of finite volume, with the hypothesis that the map f is proper. But there are counter-examples for the rigidity of the minimal volume in the complete case.

4 Proofs

of the inequality in 3.2 and of 3.3. For simplicity, we suppose that f has degree one. The main tools are the natural maps of BCG and the Gromov theory of convergence of riemannian manifolds. Before going inside the proof, we give some ideas.

4.1 Ideas of proofs

Natural maps

Suppose given (X, g_0) an hyperbolic manifold (or locally symmetric space of negative curvature), (Y, g) a riemannian manifold, a map $f : Y \to X$ of degree one and a constant $c > h(g)$, where $h(g)$ is the volume entropy. Then there exist a map

$$F_c : Y \to X$$

with the following properties. The map is C^1, homotopic to f and for all $y \in Y$,

$$|Jac\ F_c(y)| \leq \left(\frac{c}{h(g_0)}\right)^n \tag{1}$$

Moreover, there is equality at one point if and only if d_yF_c is an homothety of ratio $\frac{c}{h(g_0)}$. This map is called the natural map.

We explain briefly how to use it. For the inequality, one has

$$\begin{aligned}
vol_g(X) &= deg(F_c).vol_g(X) = \int_Y F_c^*(dvol_{g_0})(y) \\
&= \int_Y Jac(F_c)\ dvol_g(y) \\
&\leq \int_Y |Jac(F_c)|\ dvol_g(y) \\
&\leq \left(\frac{c}{h(g_0)}\right)^n vol_g(Y) \qquad (2)
\end{aligned}$$

Let $c \to h(g)$, this gives

$$vol_{g_0}(X) \geq \left(\frac{h(g)}{h(g_0)}\right)^n vol_g(Y).$$

If $Ric_g \geq -(n-1)g$ and g_0 is hyperbolic, recall that $h(g) \geq h(g_0)$ thus $vol_{g_0}(X) \leq vol_g(Y)$.

We give also the ideas of the proof of the equality in [BCG], which are used in [Bes]. One can normalize such that $vol_g(Y) = vol_{g_0}(X)$ and $h(g) = h(g_0)$. As $c \to h(g_0)$ one sees that $Jac(F_c)$ converge to 1 in L^1 norm. One shows that F_c has a uniform lipschitz bound for $c - h(g_0)$ small enough. Then a subsequence F_{c_k} converge to a 1-lipschitz map $F : Y \to X$. The map F is injective, differentiable almost everywhere and the derivative d_yF is isometric almost everywhere. If the map was C^1, one could conclude by the local inversion theorem. Instead, we use Federer theory of local degree to show that F is a local homeomorphism whose reciprocal is 1-lipschitz, and then F is an isometry.

If $Minvol(Y) = vol_{g_0}(X)$, we have a sequence of riemannian metrics (g_k), such that $|K(g_k)| \leq 1$ and $vol_{g_k}(Y) \to vol_{g_0}(X)$. One consider $c_k > h(g_k)$ and natural maps $F_{c_k} : Y \to X$. We have $h(g_k) \to h(g_0)$ and one can suppose $c_k \to h(g_0)$. The estimates on F_{c_k} depends of the metric g_k, thus we need a reference metric. So we consider the limit of the Riemannian manifolds (N, g_k). The relevant compactness result, due to Gromov ([Gro2]) says for any $n \in \mathbb{N}$, $D > 0$, $v > 0$, the set

$$\mathbb{M}(n, D, v) = \left\{ \begin{array}{ccc} \text{M} & | & |K| \leq 1 \\ \text{n-riemannian} & | & diam(M) \leq D \\ \text{compact manifold} & | & vol(M) \geq v \end{array} \right\}$$

is relatively compact, for the Gromov-Hausdorff or the bilipschitz topology, in the space of n-riemannian compact manifolds with metric of regularity $C^{1,\alpha}$. Gromov proved this theorem with regularity $C^{0,\alpha}$ and Peters ([Pet]) improve it to $C^{1,\alpha}$. Unfortunately, we don't have a bound on the diameter of g_k, even with the help of the natural maps, so we cant't apply directly this theorem. However,

$$||Y|| \geq deg(f)||X|| > 0,$$

thus, by the Gromov's isolation theorem 2.3, there exists for each k a point $p_k \in Y$ such that

$$vol_{g_k}(B_{g_k}(p_k, 1)) \geq \varepsilon_n \qquad (3)$$

This allows to use the pointed version of the compactness theorem. Then there exists a n-dimensional manifold Z with a complete metric g_∞ of class $C^{1,\alpha}$, and a point $p \in Z$ such that the following holds. For any $R > 0$, there exists diffeomorphisms $\varphi_{R,k} : B(p,R) \to \varphi_{R,k}(B(p,R)) \subset Y$ such that $||\varphi_{R,k}^* g_k - g_\infty||_{C^{1,\alpha}} \to 0$ on $B(p,R)$ as $k \to \infty$. Needless to say, $\varphi_{R,k}(p) = p_k$ and $\varphi_{R,k}(B(p,R)) \sim B(p_k,R)$ for k big enough relatively to R. One consider maps $F_k \circ \varphi_{R,k}$ as R and k goes to ∞. One shows that a subsequence converge to a map $F : Z \to X$. Then one shows that this map is an isometric embedding, which give a bound on the diameter of Z. One can conclude that Z is compact, diffeomorphic to Y and isometric to X.

4.2 BCG's natural maps

4.2.1 Construction

Up to homotopy, one can suppose $f : Y \to X$ smooth. Consider the universal covering $\tilde{Y}, \tilde{X}$ with the the lift metrics $\tilde{g}, \tilde{g}_0$. Given $c > h(g)$, we will define f_*-equivariant maps $\tilde{F}_c : \tilde{Y} \longrightarrow \tilde{X}$, i.e. such that

$$\tilde{F}_c(\gamma.y) = f_*(\gamma).\tilde{F}_c(y),$$

where $f_* : \Pi_1 Y \subset Isom(\tilde{Y}) \longrightarrow \Pi_1 X \subset Isom(\tilde{X})$ is the isomorphism induced by f.

The construction has three main steps. We begin with the lift $\tilde{f} : \tilde{Y} \to \tilde{X}$. Denote $\mathcal{M}(\tilde{Y})$ the space of finite positive measures on $\tilde{Y}$ and $\mathcal{M}(\partial\tilde{X})$ the space of finite positive measure on $\partial\tilde{X}$. We will use the model of the disk D^n for the hyperbolic space $\tilde{X}$, thus $\partial\tilde{X}$ will be identified with its boundary S^{n-1}. Fix some $c > h(g)$. In the first step, we assigns to each $y \in \tilde{Y}$ a measure $\nu_y^c \in \mathcal{M}(\tilde{Y})$. In the second step, this measure is push forward to a measure on $\tilde{X}$, then to a measure $\mu_y^c \in \mathcal{M}(\partial\tilde{X})$ by convolution with visual measures of $\tilde{X}$. We will recall some properties of visual measures and Buseman functions useful later. In the third step, we'll define the barycenter map from $\mathcal{M}(\partial\tilde{X})$ to $\tilde{X}$. Finally, $\tilde{F}_c(y) = bar(\mu_y^c)$. As the construction is equivariant, we have downstairs a map $F_c : Y \to X$. Here is a picture :

$$\begin{array}{ccc}
\nu_y^c \in \mathcal{M}(\tilde{Y}) & \longrightarrow & \mu_y^c \in \mathcal{M}(\partial\tilde{X}) \\
\uparrow & & \downarrow \\
y \in \tilde{Y} & \xrightarrow{\tilde{f},\tilde{F}_c} & x = bar(\mu_y^c) \in \tilde{X} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f \sim F_c} & X
\end{array}$$

step 1 For each $y \in \tilde{Y}$, one define a finite positive measure ν_y^c on $\tilde{Y}$ as follows. For each $z \in \tilde{Y}$,

$$d\nu_y^c(z) = e^{-c.\rho(y,z)} dvol_{\tilde{g}}(z)$$

step 2 This measure is pushed on a finite positive measure $\tilde{f}_*\nu_y^c$ on $\tilde{X}$ defined by

$$\tilde{f}_*\nu_y^c(U) = \nu_y^c(\tilde{f}^{-1}(U)),$$

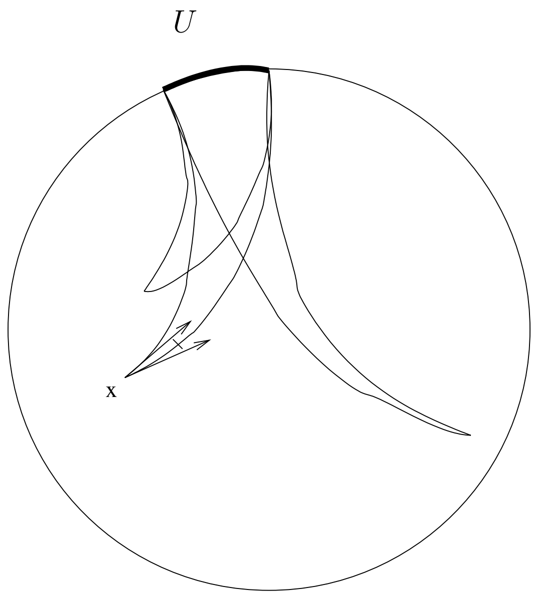
for any Borel set U in $\tilde{X}$. Then one defines a finite measure μ_y^c on $\partial\tilde{X}$, by doing a convolution with all probability visual measures P_x of $\partial\tilde{X}$:

$$\begin{aligned} \mu_y^c(U) &= \int_{\tilde{X}} P_x(U)\ d(\tilde{f}_*\nu_y^c)(x) \\ &= \int_{\tilde{Y}} P_{\tilde{f}(z)}(U)\ d\nu_y^c(z) \end{aligned}$$

Recall that the probability visual measure at x is defined as follows. For each $x \in \tilde{X}$, the unit tangent bundle $U_x\tilde{X}$ is identified with the boundary $\partial\tilde{X}$ by the map

$$v \in U_x\tilde{X} \xrightarrow{E_x} \gamma_v(\infty) \in \partial\tilde{X}$$

where $\gamma_v(t) = exp_x(tv)$. The visual probability measure P_x is the push-forward by E_x of the canonical probability measure of $U_x\tilde{X}$, i.e. $P_x(U)$ is the measure of the set of vectors $v \in U_x\tilde{X}$ such that $\gamma_v(+\infty) \in U$.



We 'll use the following facts:

Lemma 4.1. *i) P_x has no atoms on $\partial\tilde{X}$.*
ii) For each $\gamma \in Isom(\tilde{X})$,

$$P_{\gamma x} = \gamma_* P_x$$

iii) For any $x, o \in \tilde{X}$, the density of the measures satisfy

$$\frac{dP_x(\theta)}{dP_o(\theta)} = e^{-h(g_0)B(x,\theta)}$$

where $B(.,\theta)$ is the Busemann function on $\tilde{X}$ with base point o (see bellow).

We recall the definition and some properties of the Busemann functions on $\tilde{X}$.

The Busemann fonctions (see Ballman-Gromov-Schroeder [BGS] and Ballman[Bal])

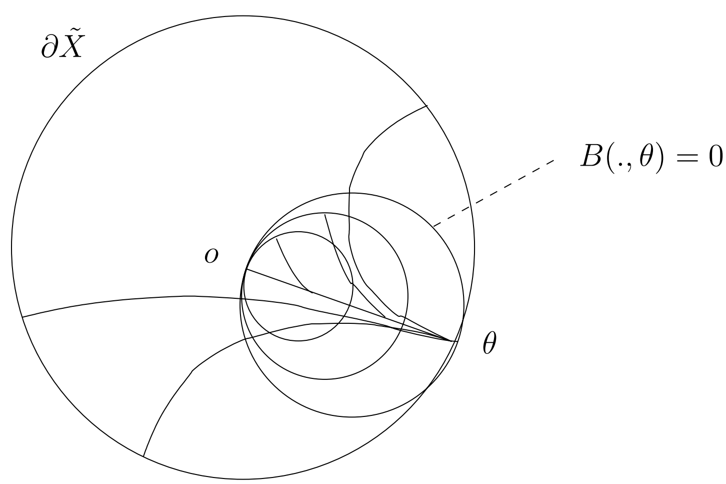
They are defined in any Hadamard space, i.e. complete simply connected riemannian manifold of nonpositive curvature. We will use some properties that holds only if the space has negative curvature.

Fix a base point $o \in \tilde{X}$ and consider a geodesic ray $c(s)$ from $c(0) = o$ to θ. For each $x \in \tilde{X}$, the function $s \to d(x, c(s)) - d(c(0), c(s))$ is monotone and bounded (use the triangle inequality) thus one define

$$B(x, \theta) = \lim_{s \to \infty} d(x, c(s)) - d(c(0), c(s))$$

By definition, $B(o, \theta) = 0$ for any θ. The following properties are relatively easy to show.

1. For each $\theta \in \partial\tilde{X}$, the Buseman function $x \mapsto B(x, \theta)$ has regularity C^1 and $\nabla B(x) = -\dot{c}(0)$ where c is a unit speed geodesic such that $c(0) = x$ and $c(+\infty) = \theta$. In fact, one can show that the regularity is C^2 (Heintze- Im Hof).
2. The Buseman function is convex on $\tilde{X}$. This follows from the convexity of the distance function in space of nonpositive curvature. Thus the sets $B_C = \{B(., \theta) \leq C\}$, which are called *horoballs*, are convex. The level sets $S_C = \{B(., \theta) = C\}$, which are called horospheres, are hypersurfaces orthogonal to all geodesic rays which ends in θ. One can see S_C as the limit, when $s \to +\infty$ of spheres $S(c(s), s + C)$.



Note that if $x \notin B_C$, then

$$B(x, \theta) = d(x, B_C) + C = d(x, S_C) + C.$$

In negative curvature, the distance function $t \to d(x(t), p)$ is strictly convex if $x(t)$ is a geodesic non colinear to the gradient of $q \mapsto d(q, p)$. The same holds for the the function $t \to d(x(t), W)$ if W is a convex set (disjoint from $x(t)$). It follows that in negative curvature $x \mapsto B(x, \theta)$ is strictly convex along geodesics which are not orthogonal to the horospheres. As a consequence, the horoballs are strictly convex. This implies that the restrictions of DdB to $\nabla B^{\perp}$, which is the second fondamental form of the horospheres, are strictly positive. In the hyperbolic case, one can compute that

$$DdB = g_0 - dB \otimes dB.$$

3. If $x \to \theta$ radialy, then $B(x, \theta) \to -\infty$. In pinched negative curvature, if $x \to \theta_0 \neq \theta$, then $B(x, \theta) \to +\infty$. Indeed, in this case, x escapes from any horoball based at θ.

Proof of lemma 4.1: Exercice. Hint: show that $\frac{dP_x(\theta)}{dP_o(\theta)}$ is constant on horospheres based at θ and then consider a dilation with axis the ray $o\theta$.

Notations: from now, we consider P_o as a fixed probability measure on $\partial\tilde{X}$ and we just note $d\theta$ its density.
We note $p_0(x, \theta) := e^{-h(g_0)B(x,\theta)}$ the density of P_x.

Some interesting facts about P_x and $p_0(x, \theta)$

1. $p_0(., \theta)$ is harmonic and p_0 is the Poisson kernel, i.e. the function

$$x \longrightarrow \int_{\partial\tilde{X}} \varphi(\theta) p_0(x, \theta)\ d\theta$$

is harmonic and coincide with φ on the boundary.

2. Let $x \in \tilde{X}$ and $\gamma \in Isom(\tilde{X})$ such that $x = \gamma^{-1}(o))$, then

$$p_0(x, \theta) = Jac(\gamma)(\theta)$$

where $Jac(\gamma)$ is the jacobian of γ acting on $\partial\tilde{X}$ by diffeomorphism.
Proof: For each Borel $U \subset \partial\tilde{X}$, $P_x(U) = P_o(\gamma U)$, thus

$$\int_U p_0(X, \theta)\ d\theta = \int_{\gamma U} d\theta = \int_U Jac(\gamma)(\theta)\ d\theta$$

3. P_x is a probability measure which converge weakly to the Dirac mass δ_{θ_0} as $x \to \theta_0$ along a geodesic.

We go back to the construction of natural maps. One can verify that μ_y^c has finite measure on $\partial\tilde{X}$, with norm

$$||\mu_y^c|| = \nu_y^c(\tilde{Y}) = ||\nu_y^c||$$

Indeed, by Fubini

$$\begin{aligned}\int_{\partial\tilde{X}}\int_{\tilde{Y}} p_0(\tilde{f}(z),\theta)e^{-c.\rho(y,z)}\ dvol_{\tilde{g}}(z)\ d\theta &= \int_{\tilde{Y}} e^{-c.\rho(y,z)}\int_{\partial\tilde{X}} p_0(\tilde{f}(z),\theta)d\theta\ dvol_{\tilde{g}}(z)\\ &= \int_{\tilde{Y}} e^{-c.\rho(y,z)}\ dvol_{\tilde{g}}(z)\end{aligned}$$

Moreover, one has

Lemma 4.2. *The map $y \to \mu_y^c$ is f_* equivariant for $f_* : \pi_1 Y \to \pi_1 X$. That is for $\gamma \in \pi_1 Y \subset Isom(\tilde{Y})$, $f_*\gamma \in Isom(\tilde{X})$ acts on $\mathcal{M}(\partial\tilde{X})$ by the push-forward action*

$$\mu_{\gamma y}^c = (f_*\gamma)_*\mu_y^c$$

Proof of the lemma 4.2: Let $U \subset \partial\tilde{X}$ a measurable set. One computes

$$\begin{aligned}\mu_{\gamma y}^c(U) &= \int_{\tilde{Y}} P_{\tilde{f}(z)}(U)e^{-c.\rho(\gamma y,z)}\ dvol_{\tilde{g}}(z)\\ &= \int_{\tilde{Y}} P_{\tilde{f}(z)}(U)e^{-c.\rho(y,\gamma^{-1}z)}\ dvol_{\tilde{g}}(z)\\ &= \int_{\tilde{Y}} P_{\tilde{f}(\gamma z')}(U))e^{-c.\rho(y,z')}\ dvol_{\tilde{g}}(z') \qquad (4)\end{aligned}$$

with the variable change $z' = \gamma^{-1}(z)$ and the fact that $Jac(\gamma)(z) = 1$ as γ acts by isometry. Now using ii) of lemma 4.1 with $\beta = f_*\gamma$,

$$\begin{aligned}(4) &= \int_{\tilde{Y}} (P_{\beta\tilde{f}(z)}(U)d\nu_y^c(z)\\ &= \int_{\tilde{Y}} P_{\tilde{f}(z)}(\beta^{-1}U)d\nu_y^c(z)\\ &= \mu_y^c(\beta^{-1}(U))\\ &= (f_*\gamma)_*\mu_y^c(U)\end{aligned}$$

by the definition of the push-forward action $\square$.

step 3 Now we take the barycenter of this measure. Recall the definition. Let $\mu \in \mathcal{M}(\partial\tilde{X})$ a finite positive measure, without atoms. Consider the function on $\tilde{X}$

$$\mathcal{B}(x) = \int_{\partial\tilde{X}} B(x,\theta)\ d\mu(\theta)$$

Lemma 4.3. *We have the two following facts:*
i) $\mathcal{B}$ is strictly convex.
ii) $\mathcal{B}(x) \to \infty$ as $x \to \theta_0 \in \partial\tilde{X}$ along a geodesic.

Proof of the lemma 4.3: i) is clear because given a geodesic $x(t)$, $B(x(t),\theta)$ is convex for all $\theta \in \partial\tilde{X}$, and stricly convex for a set of θ of full μ-measure.
ii) suppose $x \longrightarrow \theta_0$. Then $B(x,\theta) \to +\infty$ for a set of full μ-measure and $B(x,\theta) \to -\infty$ on a μ-negligeable set. But we need a more quantitative argument. We consider $x \longrightarrow \theta_0$ radially. Let $x_1 \in o\theta_0$ and $x \in x_1\theta_0$. By convexity of $B(.,\theta)$, we have for each $\theta \in \partial\tilde{X}$

$$B(x,\theta) \geq \frac{d(o,x)}{d(o,x_1)} B(x_1,\theta)$$

Note $J(x) = \{\theta \in \partial\tilde{X} : B(x,\theta) \leq 0\}$. Clearly, $\mu(J(x)) \to 0$ as $x \to \theta_0$. Let K a compact of $\partial\tilde{X}$, such that $\mu(K) > 0$ and $\theta_0 \notin K$. Suppose x_1 sufficiently close of θ_0 such that $B(x_1,\theta) \geq C > 0$ for each $\theta \in K$. Now one compute

$$\begin{aligned}
\mathcal{B}(x) &\geq \int_{J(x)} B(x,\theta)\, d\mu(\theta) + \int_K B(x,\theta)\, d\mu(\theta) \\
&\geq \frac{d(o,x)}{d(o,x_1)} \int_{J(x)} B(x_1,\theta)\, d\mu(\theta) + \frac{d(o,x)}{d(o,x_1)} \int_K B(x_1,\theta)\, d\mu(\theta) \\
&\geq \frac{d(o,x)}{d(o,x_1)} \inf_{\partial\tilde{X}}\{B(x_1,\theta)\}\mu(J(x)) + \frac{d(o,x)}{d(o,x_1)} C\mu(K) \\
&= \frac{d(o,x)}{d(o,x_1)} \left(\inf_{\partial\tilde{X}}\{B(x_1,\theta)\}\mu(J(x)) + C\mu(K) \right) \\
&\geq \frac{d(o,x)}{d(o,x_1)} \frac{C\mu(K)}{2} \longrightarrow \infty \text{ as } x \to \theta_0 \quad \Box.
\end{aligned}$$

Thus, $\mathcal{B}(x)$ has a unique minimum in $\tilde{X}$, which is called the barycenter of μ and noted $bar(\mu)$.

Lemma 4.4. *i) For any $\gamma \in Isom(\tilde{X})$, $bar(\gamma_*\mu) = \gamma(bar(\mu))$.*
ii) In particular, $bar(P_x) = x$.

Proof of lemma 4.4: i) the barycenter $x = bar(\mu)$ is the unique solution of vectorial equation

$$\int_{\partial\tilde{X}} dB_{(x,\theta)}.u\ d\mu(\theta) = 0, \forall u \in T_x\tilde{X}$$

As γ acts on $\tilde{X}$ by isometry, $g_0(\nabla B_{(x,\theta)}, u) = g_0(D_x\gamma.\nabla B_{(x,\theta)}, D_x\gamma.u) = g_0(\nabla B_{(\gamma x,\gamma\theta)}, D_x\gamma.u)$. Thus, $\forall v \in T_{\gamma x}\tilde{X}$,

$$\begin{aligned}
0 &= \int_{\partial\tilde{X}} dB_{(\gamma x,\gamma\theta)}.v\ d\mu(\theta) \\
&= \int_{\partial\tilde{X}} dB_{(\gamma x,\alpha)}.v Jac(\gamma^{-1})(\alpha)\ d\mu(\alpha) \\
&= \int_{\partial\tilde{X}} dB_{(\gamma x,\alpha)}.v\ d(\gamma_*\mu)(\alpha)
\end{aligned}$$

thus γx is the barycenter of $\gamma_* \mu$ $\square$.
ii) By symmetry, it's clear for $x = o$. Apply i) $\square$.

Now one define $F_c(y) = bar(\mu_y^c)$ from $\tilde{Y}$ to $\tilde{X}$.

Lemma 4.5. F_c
i) is C^1*,*
ii) is equivariant under action of $\pi_1 Y$ *and* $\pi_1 X$
iii) descend in a map $F_c : Y \longrightarrow X$ *homotopic to* f*.*

Proof of lemma 4.5: i) see [BCG] proposition 2.4 and 5.4.
ii) Apply lemmas 4.2 and 4.4 i)
iii) Consider an (equivariant) homotopy between μ_y^c and $p_0(\tilde{f}(y), \theta) d\theta$ and apply lemma 4.4 ii) $\square$.

4.2.2 Jacobian and derivative estimates

To compute the jacobian of F_c, we'll use two positive definite symmetric bilinear forms of trace 1. For any $y \in \tilde{Y}$, $v \in T_{F_c(y)}\tilde{X}$,

$$h_y^c(v, v) = \int_{\partial \tilde{X}} \left(dB_{(F_c(y), \theta)}(v) \right)^2 \frac{d\mu_y^c(\theta)}{||\mu_y^c||} = g_0(H_y^c.v, v)$$

For any $y \in \tilde{Y}$, $u \in T_y \tilde{Y}$,

$$h_y'^c(u, u) = \int_{\tilde{Y}} \left(d\rho_{(y,z)}(u) \right)^2 \frac{d\nu_y^c(z)}{||\nu_y^c||} = g(H_y'^c.u, u)$$

Lemma 4.6. *For any* $y \in \tilde{Y}$*,* $u \in T_y \tilde{Y}$*,* $v \in T_{F(y)} \tilde{X}$*, one has*

$$\left| g_0((I - H_y^c) d_y F.u, v) \right| \leq c.g_0(H_y^c.v, v)^{1/2}.g(H_y'^c.u, u)^{1/2} \tag{5}$$

Proof of the lemma 4.6: From the definition of $F_c(y)$, one has for each $v \in T_{F_c(y)} \tilde{X}$,

$$0 = D_{F_c(y)} \mathcal{B}.v = \int_{\partial \tilde{X}} dB_{(F_c(y), \theta)}(v) \; d\mu_y^c(\theta) \tag{6}$$

Let V is a parallel vector field on $T\tilde{X}$, and pick $u \in T_y \tilde{Y}$. Differentiating equation (6) with $v = V_y$, one has

$$0 = \int_{\partial \tilde{X}} DdB_{(F(y),\theta)}(d_y F(u), V) d\mu_y^c(\theta) + \int_{\partial \tilde{X}} dB_{(F(y),\theta)}(v). \int_{\tilde{Y}} p(\tilde{f}(z), \theta)(-c d\rho_{(y,z)}(u)) d\nu_y^c(z) \; d\theta$$

Thus, using Cauchy-Schwarz in the second term, one has

$$\left| \int_{\partial\tilde{X}} DdB_{(F(y),\theta)}(d_yF(u),V)d\mu_y^c(\theta) \right| \leq$$

$$\int_{\partial\tilde{X}} |dB_{(F(y),\theta)}(v)| \left(\int_{\tilde{Y}} p(\tilde{f}(z),\theta)d\nu_y^c(z) \right)^{1/2} \left(\int_{\tilde{Y}} p(\tilde{f}(z),\theta)|cd\rho_{(y,z)}(u)|^2 d\nu_y^c(z) \right)^{1/2} d\theta = (E)$$

Using again Cauchy-Schwarz,

$$\begin{aligned}
(E) &\leq c \left(\int_{\partial\tilde{X}} |dB_{(F(y),\theta)}(v)|^2 \int_{\tilde{Y}} p(\tilde{f}(z),\theta)d\nu_y^c(z)d\theta \right)^{1/2} \left(\int_{\partial\tilde{X}} \int_{\tilde{Y}} p(\tilde{f}(z),\theta)|d\rho_{(y,z)}(u)|^2 d\nu_y^c(z)d\theta \right)^{1/2} \\
&= c \left(\int_{\partial\tilde{X}} |dB_{(F(y),\theta)}(v)|^2 d\mu_y^c(\theta) \right)^{1/2} \left(\int_{\tilde{Y}} |d\rho_{(y,z)}(u)|^2 \, d\nu_y^c(z) \right)^{1/2} \\
&= c||\nu_y^c||^{1/2}||\mu_y^c||^{1/2} g_0(H_y^c.v,v)^{1/2} g({H'}_y^c.u,u)^{1/2}
\end{aligned}$$

Now using $DdB = g_0 - dB \otimes dB$, the term in DdB can be computed as

$$g_0(d_yF(u),v)||\mu_y^c|| - \int_{\partial\tilde{X}} (dB_{(F_c(y),\theta)}(d_yFu)(dB_{(F_c(y),\theta)}(v) \; d\mu_y^c(\theta) \\ = g_0((I - H_y^c)d_yF(u),v))||\mu_y^c||$$

and dividing by $||\mu_y^c|| = ||\nu_y^c||$, we obtains the lemma. □

Thus d_yF_c is controled by H_y^c. Let $0 < \lambda_1^c(y) \leq ... \leq \lambda_n^c(y) < 1$ the eigenvalues of H_y^c.

Proposition 4.7. *There exists a constant $A > 0$ such that, for any $y \in Y$,*

$$|JacF_c(y)| \leq \left(\frac{c}{h(g_0)} \right)^n \left(1 - A \sum_{i=1}^n (\lambda_i^c(y) - \frac{1}{n})^2 \right) \qquad (7)$$

Thus if c is close to $h(g_0)$ and $|JacF_c(y)|$ close to 1, the eigenvalues are close to $1/n$.

Proof of the proposition 4.7: The proof follows from two lemmas:

Lemma 4.8. *At each $y \in \tilde{Y}$,*

$$|Jac(F_c)(y)| \leq \left(\frac{c}{\sqrt{n}} \right)^n \frac{det({H_y}^c)^{1/2}}{det(I - {H_y}^c)}$$

Proof of lemma 4.8: Let (u_i) an orthonormal basis of $T_{F_c(y)}\tilde{X}$ which diagonalizes ${H'}_y^c$. We can suppose d_yF_c invertible thus let $v_i' = [(I - {H_y}^c) \circ d_yF_c]^{-1}(u_i)$. The orthonormalization process of Schmidt applied to (v'_i) gives an orthonormal basis (v_i) at $T_y\tilde{Y}$. The matrix of $(I - {H_y}^c) \circ d_yF_c$ in the base (v_i) and (u_i) is triangular so

$$det(I - {H_y}^c)Jac(F_c)(y) = \prod_{i=1}^{n} g_0((I - {H_y}^c) \circ d_yF_c.v_i, u_i)$$

Thus, with (5),

$$\begin{aligned} det(I - {H_y}^c)|Jac(F_c)(y)| &\leq c^n \left(\prod_{i=1}^{n} g_0({H_y}^c v_i, v_i)\right)^{1/2} \left(\prod_{i=1}^{n} g({H'_y}^c u_i, u_i)\right)^{1/2} \\ &\leq c^n det({H_y}^c)^{1/2} \left[\frac{1}{n}\sum_{i=1}^{n} g({H'_y}^c u_i, u_i)\right]^{n/2} \end{aligned}$$

and we have the desired inequality with $tr({H'_y}^c) = 1$ □.

Lemma 4.9. *Let H a symmetric positive definite $n \times n$ matrix whose trace is equal to one then, if $n \geq 3$,*

$$\frac{det(H^{1/2})}{det(I - H)} \leq \left(\frac{n}{h(g_0)^2}\right)^{n/2} \left(1 - A\sum_{i=1}^{n}(\lambda_i - \frac{1}{n})^2\right)$$

for a constant $A(n) > 0$.

Proof of lemma 4.9: see Appendix B5 in [BCG]. This is the point where the rigidity of the natural maps fails in dimension 2. This completes the proof of the proposition □.

Thus we have obtained the inequality (1) of theorem 3.2 . Before continuing the proof of the theorem 3.3, we give useful lemmas.

Lemma 4.10. *Suppose that $|Jac\ F_c(y)| = \left(\frac{c}{h(g_0)}\right)^n$. Then d_yF_c is an homothety of ratio $\frac{c}{h(g_0)}$.*

We have $H_y^c = \frac{Id}{n}$. From lemma 4.6, we deduce that

$$||d_yF_c.u||^2 \leq n\left(\frac{c}{n-1}\right)^2 g({H'_y}^c u, u)$$

thus with $tr({H'_y}^c) = 1$ we find

$$tr(F_c^* g_0)(y) \leq n\left(\frac{c}{n-1}\right)^2 \tag{8}$$

Now

$$\begin{aligned} \left(\frac{c}{h(g_0)}\right)^{2n} &= det(F^*g_0) \leq \left[\frac{1}{n}tr(F^*g_0)\right]^n \\ &= \left(\frac{c}{n-1}\right)^{2n} \end{aligned}$$

thus $det(F^*g_0) = \left[\frac{1}{n}tr(F^*g_0)\right]^n$ and we must have ${F_c}^*g_0(y) = \left(\frac{c}{h(g_0)}\right)^2 g_y$ □.

The same arguments shows that

Lemma 4.11. *For $\varepsilon > 0$, there exists $\alpha(\varepsilon) > 0$, going to zero with as 0 such that if*

$$\left(\frac{c}{h(g_0)}\right)^n - |Jac(F_c)(y)| \leq \varepsilon$$

then for any $u \in T_y\tilde{Y}$,

$$(1-\alpha(\varepsilon))\frac{c}{h(g_0)}||u|| \leq ||d_yF_c.u|| \leq (1+\alpha(\varepsilon))\frac{c}{h(g_0)}||u||$$

We shall call $Y_{c,\varepsilon} = \left\{y \in Y, \left(\frac{c}{h(g_0)}\right)^n - |Jac(F_c)(y)| \leq \varepsilon\right\}$. Of course, it depends also of the metric g. Now we continue the proof of the theorem 3.3. Note (Y, g_k, y_k) the sequence of pointed riemannian manifolds and suppose that y_k satisfies the inequality 3

Lemma 4.12. *If $vol_{g_k}(Y) \to vol_{g_0}(X)$ and $c_k \to h(g_0)$, then*

$$vol_{g_k}(Y) - vol_{g_k}(Y_{c_k,\varepsilon}) \to 0$$

Proof of lemma 4.12: Suppose $\varepsilon > 0$ small and let $\varepsilon_k = \left(\frac{c_k}{h(g_0)}\right)^n - 1$. Suppose k big enough such that

$$|JacF_{c_k}(y)| \leq 1 - \frac{\varepsilon}{2}$$

on $Y - Y_{c_k,\varepsilon}$. We have

$$\begin{aligned} vol_{g_0}(X) &\leq \int_Y |JacF_{c_k}(y)|\, dvol_{g_k}(y) && (9)\\ &= \int_{Y_{c_k,\varepsilon}} |JacF_c(y)|\, dvol_{g_k}(y) + \int_{Y-Y_{c_k,\varepsilon}} |JacF_{c_k}(y)|\, dvol_{g_k}(y) && (10)\\ &\leq (1+\varepsilon_k)vol_g(Y_{c_k,\varepsilon}) + (1-\frac{\varepsilon}{2})vol_{g_k}(Y - Y_{c_k,\varepsilon}) && (11)\\ &= vol_{g_k}(Y) + \varepsilon_k vol_{g_k}(Y_{c_k,\varepsilon}) - \frac{\varepsilon}{2}vol_{g_k}(Y - Y_{c_k,\varepsilon}) && (12) \end{aligned}$$

Thus,

$$\begin{aligned} vol_{g_k}(Y - Y_{c_k,\varepsilon}) &\leq \frac{2}{\varepsilon}\left(vol_{g_k}(Y) - vol_{g_0}(X) + \varepsilon_k vol_{g_k}(Y_\alpha)\right) && (13)\\ &\longrightarrow 0 && (14) \end{aligned}$$

as $k \to \infty$ □.

If we make $\varepsilon_k \to 0$, $c_k \to h(g_0)$ for appropriate sequences, those results says that on sets of arbitrary large relatively volume, $d_yF_{c_k}$ tends to be isometric. We need a litle more. Suppose that c_k is close to $h(g_0)$, say $\frac{c_k}{h(g_0)} \leq 1,001$. We have the following.

Lemma 4.13. *There exists $r(n) > 0$, $\varepsilon(n) > 0$ such that if $y_0 \in Y_{c,\varepsilon(n)}$, $\varepsilon \leq \varepsilon(n)$ then for each $y \in B_g(y_0, r(n))$, one has*

$$||d_y F_c|| \leq 2\sqrt{n}$$

Proof of lemma 4.13: Name $g = g_k$, $c = c_k$. The equation (5) allows us to controll $\|d_y F_c\|$ with $\lambda_n^c(y)$, the maximal eigenvalue of H_y^c. Indeed, let u a normal vector in $T_y\tilde{Y}$, $v = d_y F_c.u$, (5) gives

$$(1 - \lambda_n^c(y))\,|g_0(d_y F_c.u, d_y F_c.u)| \leq c.\,(\lambda_n^c(y).g_0(d_y F_c.u, d_y F_c.u))^{1/2} \tag{15}$$

thus

$$\|d_y F_c.u\|_{g_0} \leq \frac{c\sqrt{\lambda_n^c(y)}}{1 - \lambda_n^c(y)} \tag{16}$$

Fix some $\eta(n) > 0$ such that if $\lambda_n^c \leq \frac{1}{n} + \eta$, the quotient above is lower $\leq 2\sqrt{n}$. We will show that $\lambda_n^c \leq \frac{1}{n} + \eta$ on some $B_g(y_0, r(n))$. From proposition 4.7 , we know that for $\varepsilon(n)$ is sufficiently small, $\lambda_n^c(y_0) \leq \frac{1}{n} + \frac{\eta}{2}$. We want to controll λ_n^c along small rays from y_0. Recall that H_y^c is defined by

$$h_y^c(u,v) = \int_{\partial\tilde{X}} dB_{(F_c(y),\theta)}(u) dB_{(F_c(y),\theta)}(v)\ d\sigma_y^c(\theta) = g_0(H_y^c.u, v)$$

Let u, v two orthonormal vectors at $F_c(y_0)$, and U,V parallel extensions in a neighbourhood of $F_c(y_0)$. We compute the derivative of $h_y^c(U,V)$ in a direction $w \in T_yY$. Note $d\sigma_y^c(\theta) = \frac{d\mu_y^c}{||\mu_y^c||}$.

$$w.h_y^c(U,V) = \int_{\partial\tilde{X}} DdB_{(F(y),\theta)}(d_yF(w), U) dB_{(F_c(y),\theta)}(V) d\sigma_y^c(\theta) +$$
$$\int_{\partial\tilde{X}} dB_{(F_c(y),\theta)}(U) DdB_{(F(y),\theta)}(d_yF(w), V) d\sigma_y^c(\theta) + \int_{\partial\tilde{X}} dB_{(F_c(y),\theta)}(U) dB_{(F_c(y),\theta)}(V) w.d\sigma_y^c(\theta)$$

Thus $|w.h_y^c(U,V)| \leq 2\|D_y F_c.w\|_{g_0} + \left|\int_{\partial\tilde{X}} w.d\sigma_y^c(\theta)\right|$ because $\|DdB\| \leq 1$ and $\|dB\| \leq 1$. Recall that

$$d\sigma_y^c(\theta) = \frac{d\mu_y^c}{\mu_y^c(\partial\tilde{X})} = \frac{\int_{\tilde{Y}} p(\tilde{f}(z),\theta) e^{-c\rho(y,z)}\ dV_{\tilde{g}}(z)\ d\theta}{\int_{\tilde{Y}} e^{-c\rho(y,z)}\ dV_{\tilde{g}}(z)}$$

Thus,

$$w.d\sigma_y^c(\theta) = \frac{\int_{\tilde{Y}} p(\tilde{f}(z),\theta)(-c.d\rho_{(y,z)}(w)) e^{-c\rho(y,z)}\ dV_{\tilde{g}}(z)\ d\theta}{\mu_y^c(\partial\tilde{X})} -$$
$$\frac{d\mu_y^c}{\mu_y^c(\partial\tilde{X})^2}.\int_{\tilde{Y}} (-c.d\rho_{(y,z)}(w)) e^{-c\rho(y,z)}\ dV_{\tilde{g}}(z) \tag{17}$$

As $|c.d\rho_{(y,z)}(w)| \leq c.\|w\|_g$, we have

$$\left|\int_{\partial\tilde{X}} w.d\sigma_y^c(\theta)\right| \leq \int_{\partial\tilde{X}} 2c.\|w\|_g d\sigma_y^c(\theta) = 2c.\|w\|_g \tag{18}$$

Thus, $\left|w.h^c_y(U,V)\right| \leq 2\|D_yF_c.w\|_{g_0} + 2c.\|w\|_g$. Now suppose w is normal and use (16):

$$\left|w.h^c_y(U,V)\right| \leq 2c\left(\frac{\sqrt{\lambda^c_n(y)}}{1-\lambda^c_n(y)}+1\right) \tag{19}$$

Suppose there exists $y \in Y$ such that $\lambda^c_n(y) \geq \frac{1}{n}+\eta$. Take a point y such that $\lambda^c_n(y) = \frac{1}{n}+\eta$ and $r = d(y_0, y) > 0$ is minimal. Let γ a unit speed geodesic from y_0 to y. Let $U(t)$ a parallel vector field along $F_c(\gamma)$ such that $U(r)$ is an unit eigenvector for $\lambda^c_n(\gamma(r))$. Then with (19),

$$\begin{aligned}
|\lambda^c_n(\gamma(r)) - \lambda^c_n(\gamma(0))| &\leq \left|h^c_{\gamma(r)}(U(r),U(r)) - h^c_{\gamma(0)}(U(0),U(0))\right| && (20)\\
&\leq 2c.\int_0^r \frac{\sqrt{\lambda^c_n(\gamma(t))}}{1-\lambda^c_n(\gamma(t))}+1\ dt && (21)\\
&\leq 2cr.\left(\frac{\sqrt{\frac{1}{n}+\eta}}{1-(\frac{1}{n}+\eta)}+1\right) && (22)
\end{aligned}$$

Thus

$$\frac{\eta}{2} \leq 2cr.\left(\frac{\sqrt{\frac{1}{n}+\eta}}{1-(\frac{1}{n}+\eta)}+1\right)$$

and we have a uniform bound below for $r(n)$.

We deduces

Lemma 4.14. *For any $R > 0$, for any $k \geq k(R)$, the inequality*

$$||d_yF_{c_k}|| \leq 2\sqrt{n} \tag{23}$$

holds on $B(y_k, R)$.

Proof of lemma 4.14: by 4.13, (23) holds on the $r(n)$-neighborhood of $Y_{c_k,\varepsilon}$. On the other hand, by lemma 4.12 $vol_{g_k}(Y) - vol_{g_k}(Y_{c_k,\varepsilon}) \to 0$ as $k \to +\infty$. Thus if we have a uniform lower bound for the volume of the $r(n)$-balls of $B(y_k, R)$, the $r(n)$-neighborhood of $Y_{c_k,\varepsilon}$ covers $B(y_k, R)$ for k large enough. Recall that $vol_{g_k}(B(y_k, R))$ is uniformly bounded below by (3). By Bishop-Gromov 's theorem [Gro], for any $y \in B(y_k, R)$,

$$vol_{g_k}(B(y, r(n)) \geq vol_{g_k}(B(y, 2R)\frac{V_{-1}(r(n))}{V_{-1}(2R)} \geq > vol_{g_k}(B(y_k, R)\frac{V_{-1}(r(n))}{V_{-1}(2R)}$$

is uniformly bounded below. □.

Considering $R \to \infty$ and the maps $F_{c_k} \circ \varphi_{R,k} : B(p, R) \subset Z \to X$, one obtain uniform convergence on compact sets to a map

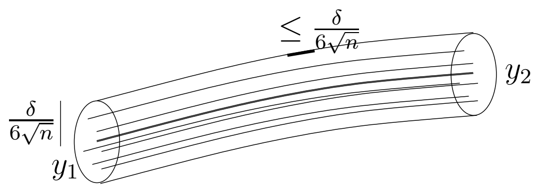
$$F : (Z, g_\infty) \to (X, g_0)$$

Proposition 4.15. *F is 1-lipschitz, injective and $Jac(F) = 1$ almost everywhere.*

Proof: By construction, this map is $2\sqrt{n}$-lipschitz. Consider $y_1, y_2 \in B(p_k, R)$ and $\delta > 0$. We claim that for k big enough,

$$d_{g_0}(F_{c_k}(y_1), F_{c_k}(y_2)) \leq (1+\delta) d_{g_k}(y_1, y_2) + \delta$$

Consider a tube of g_k-geodesics of radius $\frac{\delta}{6\sqrt{n}}$ from y_1 to y_2.



We have a uniform lower bound for the volume of this tube. As $vol_{g_k}(Y - Y_{c_k,\varepsilon_k}) \to 0$, for k big enough, there are geodesics γ whose intersection with $Y - Y_{c_k,\varepsilon_k}$ has length $\leq \frac{\delta}{6\sqrt{n}}$. Hence $F_{c_k}(\gamma)$ has length $\leq \frac{\delta}{3} + (1+\delta)\ell(\gamma)$. Thus F is 1-lipschitz. By Rademacher theorem, F is differentiable almost everywhere and $|Jac(F)| \leq 1$. To show that $Jac(F) = 1$ almost everywhere, we argue as follows.

Lemma 4.16. *For any measurable $B \subset Z$, one has*

$$vol_{g_0}(F(B)) = B$$

Proof: Suppose B compact and let $B_k = \varphi_{k,R}(B) \subset B(p_k, R) \subset Y$. We use the fact that it is almost true for F_{c_k}. We know that $|Jac(F_{c_k})|$ is close to one on sets Y_{c_k,ε_k} of arbitrarly large relative volume. Moreover, we have that the set of points in X which have exactly one antecedent by F_{c_k} has also arbitrarly large relative volume in X. More precisely, for $x \in X$, let $N(x, c_k)$ the number (possibly infinite) of $y \in Y$ such that $F_{c_k}(y) = x$. Using the aira formula (Morgan, Geometric Measure Theory, 3.7)

$$\int_Y |Jac(F_{c_k})| dvol_{g_k} = \int_X N(x, c_k)\ dvol_{g_0}(x),$$

one shows that

$$\int_{N(x,c_k)>1} N(x, c_k)\ dvol_{g_0}(x) \to 0$$

as $k \to \infty$. Thus one can show that $Jac(F_{c_k})$ is close to 1 on sets of relatively large volume and

$$\lim_{k\to\infty} vol_{g_0}(F_{c_k}(B)) = \lim_{k\to\infty} vol_{g_k}(B_k) = vol_{g_\infty}(B)$$

and the proof of the lemma is easily finished. $\Box$.

To see that F is an injective map, consider $z_1 \neq z_2 \in Z$ and suppose $x = F(z_1) = F(z_2)$. Thus $F(B(z_1, r) \cup B(z_2, r)) \subset B(x, r)$. As $r \to 0$, the volume of the balls approches $c(n)r^n$, the euclidian one (recall that g_∞ has regularity $C^{1,\alpha}$). By the previous result, $vol_{g_0}(F(B(z_1, r) \cup B(z_2, r)) = vol_{g_\infty}(B(z_1, r)) + vol_{g_\infty}(B(z_2, r)) \sim 2c(n)r^n \leq vol_{g_0}(B(x, r)) \sim c(n)r^n$ and we have a contradiction for r small. □.

Proposition 4.17. *F is an isometric embedding.*

Proof: We have that F injective and $d_z F$ is almost everywhere an isometry. If F was C^1, we could conclude by the local inversion theorem. The main step is to show that F is a local homeomorphism. We use Federer theory of local degree for lipschitz map. Let $x = F(z)$ and $B = B_{g_\infty}(z, r)$. We note F_B the restriction of F to B. The local degree is defined by

$$deg(F_B)(x') = \sum_{F_B(z')=x'} signJac(F)(z')$$

for almost all $x' \in X$ and $deg(F_B)$ is constant on each connected component of $X - F(\partial B)$ ([Fed], corollary 4.1.26). As F is injective, $x \notin F(\partial B)$. As $F(\partial B)$ is compact, the connected component of $X - F(\partial B)$ which contains in x has a set of point where $deg(F_B) = 1$ of non-zero volume. Thus $deg(F_B) = 1$ on this component and F is surjective. Thus F is an homeomorphism from a neighbourhood of z to a neighbourhood of x. Now we show that F is locally isometric. As Z is a complete manifold of same dimension than X, it will imply that F is an isometric imbedding. It suffices to show that F^{-1} is locally lipschitz. For then F^{-1} is differentiable almost everywhere and the derivative is an isometry. To show that F^{-1} is locally 2-lipschitz, one can argue by contradiction. If not, there are points z,z' arbitrarly close such that $d_{g_0}(F(z), F(z')) \leq 2d_{g_\infty}(z, z')$. Considering balls centered in z, z' and with radius half the distance $d(z, z')$, one gets a contradiction in showing that the overlapping of the images decreases sufficiently the volume.□

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