Quasi-conformal geometry and Mostow rigidity

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Let $\mathbb{H}^n$ be the real hyperbolic space with $n \geq 3$. The aim of these lectures is to present the basic tools of quasi-conformal geometry of the standard $(n-1)$-sphere, and to use them to prove the two classical rigidity theorems below.

Theorem 0.1 (Mostow [M1], [M2]). *Let Γ_1, Γ_2 be cocompact lattices in $\mathrm{Isom}(\mathbb{H}^n)$. Then any abstract isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ is a conjugation by an element of $\mathrm{Isom}(\mathbb{H}^n)$.*

Theorem 0.2 (Sullivan [S] for $n = 3$, Tukia [T2] in general). *Let Γ be a finitely generated group quasi–isometric to $\mathbb{H}^n$. Then there exists Φ a cocompact lattice in $\mathrm{Isom}(\mathbb{H}^n)$ and a surjective homomorphism of groups $\Gamma \twoheadrightarrow \Phi$ with finite kernel.*

1 Quasi-conformal geometry

Let Z be the euclidean sphere S^{n-1}, of dimension $n \geq 3$. In this chapter we discuss local and global properties of quasi-conformal homeomorphisms of Z. We also establish an equality between the conformal group of Z, the Möbius group of Z, and the isometry group of $\mathbb{H}^n$.

Definition 1.1. A homeomorphism $f : Z \to Z$ is called *k-quasi-conformal* if, setting

$$H_f(x,r) := \frac{\sup\{\|f(x)-f(y)\| \ ; \ \|x-y\| \leq r\}}{\inf\{\|f(x)-f(y)\| \ ; \ \|x-y\| \geq r\}},$$

we have for all $x \in Z$

$$\overline{\lim_{r\to 0}} H_f(x,r) \leq k.$$

Example 1.2. For a linear homeomorphism f of $\mathbb{R}^{n-1}$, the number $H_f(x,r)$ is the following. Let E be the ellipse in $\mathbb{R}^{n-1}$ which is the image by f of the unit sphere centered at the origin. Denote by L and l respectively the length of the largest and of the smallest axis of E. Then $H_f(x,r) = L/l$ for every x and f is k-quasi-conformal for $k = L/l$.

This linear situation generalises easily to diffeomorphisms of the sphere : a diffeomorphism f of Z is k-quasi-conformal if and only if for every $z \in Z$ its differential is k-quasi-conformal from $T_z Z$ to $T_{f(z)} Z$.

The following theorem is due to Rademacher-Stepanov (see [V1] for a proof). It is a deep result in geometric measure theory, it establishes strong regularity properties for quasi-conformal homeomorphisms.

Theorem 1.3. *Any k-quasi-conformal homeomorphism of Z is absolutely continuous with respect to Lebesgue measure, and is differentiable almost everywhere, with k-quasi-conformal differential.*

We now turn our attention to global properties of quasi-conformal homeomorphisms. For pairwise distinct points $x, y, x', y' \in Z$ we denote their *crossratio* by:

$$[xx'yy'] = \frac{\|x-y\| \cdot \|x'-y'\|}{\|x-y'\| \cdot \|x'-y\|}.$$

Definition 1.4. (i) A homeomorphism f of Z is called *η-quasi-Möbius*, where η is an increasing homeomorphism of $[0,\infty)$, if

$$(*) \qquad \forall x, x', y, y' \in Z, \quad \left[f(x)f(x')f(y)f(y')\right] \leq \eta\left([xx'yy']\right).$$

(ii) A homeomorphism which preserves the crossratio is called a *Möbius homeomorphism*.

Note that switching $y \leftrightarrow y'$ leads to the other inequality in $(*)$ (with another function η). Inverses of quasi-Möbius homeomorphisms and compositions of quasi-Möbius homeomorphisms are quasi-Möbius as well.

It is an exercice to prove that quasi-Möbius homeomorphisms are quasi-conformal, and that Möbius homeomorphisms are conformal diffeomorphisms. The following result establishes the converse.

Theorem 1.5. *(i) Let f be a k-quasi-conformal homeomorphism of Z. Then there exists η an increasing homeomorphism of $[0;\infty)$, which only depends on n and k, such that f is a η-quasi-Möbius homeomorphism.*

(ii) In addition, if Df is conformal a.e., then f is a Möbius homeomorphism.

We will give in the sequel some evidences about this theorem. We first present an essential tool for the proof of theorem 1.5.

Let A, B be disjoint continua (*i.e.* compact connected subsets of Z) not reduced to a point. The *modulus* of the pair (A, B) is defined as

$$\mathrm{Mod}(A, B) := \inf_{\rho} \left\{ \int_Z \rho^{n-1} \, dm \right\},$$

where the infimum is taken over all $\rho : Z \to \mathbb{R}_+$ which are measurable and such that $\int_\gamma \rho \geq 1$ for every rectifiable curve γ joining A to B.

Lemma 1.6. *(1) Let f be a K-quasi-conformal homeomorphism of Z. Then for every A, B as above*

$$\frac{1}{K'} \mathrm{Mod}(A, B) \leq \mathrm{Mod}(f(A), f(B)) \leq K' \, \mathrm{Mod}(A, B),$$

where K' is a function of K. In addition, if Df is conformal a.e., then f preserves moduli.

(2) Let B_1, B_2 be two closed balls in $\mathbb{R}^{n-1}$ with same center and radii $r_1 < r_2$. Then

$$\mathrm{Mod}(B_1, Z - \overset{\circ}{B}_2) = \omega_{n-2} \log \left(\frac{r_2}{r_1} \right)^{2-n},$$

where ω_{n-2} is the volume of the unit $(n-2)$-sphere.

(3) There exist increasing homeomorphisms δ_1, δ_2 of $[0; \infty)$, such that for all A, B as above, we have

$$\delta_1(\Delta(A, B)^{-1}) \leq \mathrm{Mod}(A, B) \leq \delta_2(\Delta(A, B)^{-1}),$$

where $\Delta(A, B)$ is the relative distance between A and B, i.e.

$$\Delta(A, B) = \frac{\mathrm{dist}(A, B)}{\inf\{\mathrm{diam}\, A, \mathrm{diam}\, B\}}.$$

In the sequel we abreviate this last property by saying that $\mathrm{Mod}(A, B) \approx \Delta(A, B)^{-1}$.

Sketch of proof of lemma 1.6. (1) For C^1-diffeomorphisms of Z the property follows from the formula of transformation of variables. For general quasi-conformal homeomorphisms, the same line of proof works thanks to theorem 1.3, and to another regularity property called "absolute continuity along almost all rectifiable curves" (see [V1], [Vu] for more details).

(2) Let r be the distance from x to the common center of the balls. By letting

$$\rho(x) = (\log r_2/r_1)^{n-1} r^{-1}$$

if $r_1 < r < r_2$, and $\rho(x) = 0$ if not, one obtains that the left side of expected formula is less than or equal to the right. The reverse inequality comes from Hölder inequality (see [V1]).

(3) More difficult, see for example [V1], [Vu].

□

With the above lemma we can now give the

Proof of Theorem 1.5(i). Because $[xx'yy'] = [x'xyy']^{-1}$, it is enough to prove that the crossratio $[xx'yy']$ of four distinct points of Z is small if and only if $[f(x)f(x')f(y)f(y')]$ is small, quantitatively. By lemma 1.6(3), the map f quasi-preserves the relative distances between continua. Now on the sphere Z the crossratio and the relative distances are related as follows (see [BK] lemma 2.1) : there exist functions $\delta_1, \delta_2 : \mathbb{R}^+ \to \mathbb{R}^+$, such that

i) If $[xx'yy'] \leq \delta_1(\epsilon)$, then there exist two continua C and C' of Z with $x, y \in C$, $x', y' \in C'$ and $\Delta(C, C') \geq 1/\epsilon$.

ii) If there exit two continua C, C' of Z with $x, y \in C$, $x', y' \in C'$ and $\Delta(C, C') \geq 1/\delta_2(\epsilon)$, then $[xx'yy'] < \epsilon$.

Thus we obtain that f is quasi-Möbius. □

To prove the second part of theorem 1.5 we relate the crossratio on $Z = \partial\,\mathbb{H}^n$ with the hyperbolic distance in $\mathbb{H}^n$, denoted by $d_{\mathbb{H}^n}$.

Lemma 1.7. *For x, x', y, y' pairwise distinct points of Z and for a, a', b, b' in $\mathbb{H}^n$, we have*

$$[xx'yy'] = \lim_{a\to x, a'\to x', b\to y, b'\to y'} \exp \frac{1}{2}\Big\{ d_{\mathbb{H}^n}(a,b) + d_{\mathbb{H}^n}(a',b') - d_{\mathbb{H}^n}(a,b') - d_{\mathbb{H}^n}(a',b) \Big\}.$$

Proof. In the ball model of $\mathbb{H}^n$, let O be its center and let $a \in [Ox), b \in [Oy)$ with $d_{\mathbb{H}^n}(O,a) = d_{\mathbb{H}^n}(O,b) = t$. Let θ be the angle xOy. By standard trigonometry we get

$$\begin{aligned}\|x-y\| &= 2\sin(\theta/2) = 2\left(\frac{1-\cos\theta}{2}\right)^{1/2}\\ &= 2\left(\frac{1}{2}-\frac{\operatorname{ch}^2 t-\operatorname{ch} d(a,b)}{2\operatorname{sh}^2 t}\right)^{1/2}\\ &= 2\left(\frac{\operatorname{ch} d(a,b)}{2\operatorname{sh}^2 t}-\frac{1}{2\operatorname{sh}^2 t}\right)^{1/2}\\ &\underset{t\to\infty}{\sim} 2\exp\frac{1}{2}\left\{d(a,b)-d(O,a)-d(O,b)\right\}.\end{aligned}$$

We implement this equality in the definition of the crossratio, after cancellations we obtain the expected formula. □

Proof of theorem 1.5(ii). $*$ Recall that if f is a quasi-conformal homeomorphism of Z such that Df is conformal a.e., then f preserves moduli, (lemma 1.6(1)).

$**$ Recall that if B_1, B_2 are balls in $\mathbb{R}^{n-1}$ with same center and radii $r_1 < r_2$, then $\mathrm{Mod}(B_1, Z- \overset{\circ}{B}_2) = \omega_{n-2}(\log \frac{r_2}{r_1})^{2-n}$, (lemma 1.6(2)).

We first compute moduli $\mathrm{Mod}(C_1, C_2)$ where C_1, C_2 are disjoint closed balls in $\mathbb{R}^{n-1}$. These two balls define two disjoint totally geodesic $(n-1)$-subspaces in the upper-half space model of $\mathbb{H}^n$. Call them H_1 and H_2. Let $x_i \in H_i$ such that $d_{\mathbb{H}^n}(H_1, H_2) = d_{\mathbb{H}^n}(x_1, x_2)$. We claim that

$$\mathrm{Mod}(C_1, C_2) = \omega_{n-2}\left(d_{\mathbb{H}^n}(x_1, x_2)\right)^{2-n}.$$

Indeed let $g \in \mathrm{Isom}(\mathbb{H}^n)$ send $[x_1x_2]$ to a vertical geodesic segment. It transforms C_1 to B_1 and C_2 to $Z- \overset{\circ}{B}_2$, where B_1 and B_2 are concentric balls in $\mathbb{R}^{n-1}$, whose radii r_i satisfy $\log r_2/r_1 = d_{\mathbb{H}^n}(x_1, x_2)$. By lemma 1.7, the isometry g acts on Z as a Möbius homeomorphism, so it preserves moduli. Thus with the property $(**)$ above, we obtain the claimed result.

Now let f be a quasi-conformal homeomorphism of Z, such that Df is conformal a.e. For x, x', y, y' pairwise distinct points in $\mathbb{R}^{n-1}$, consider the balls C_1, C_2, C_3, C_4 in $\mathbb{R}^{n-1}$, of radius r, centered respectively at x, x', y, y'. By lemma 1.8 and with our claim we can express $[xx'yy']$ as the limit, when r tends to 0, of an expression which involves only $\mathrm{Mod}(C_i, C_j)$. By property $(*)$ above, Mod is f-invariant, so we get that f preserves the crossratio. The details are left to the reader. □

Here is an easy application of lemma 1.7 and theorem 1.5.

Corollary 1.8. *(i) Let respectively* $\mathrm{Conf}(Z)$ *and* $\mathrm{Möb}(Z)$ *be the group of conformal diffeomorphisms and Möbius homeomorphisms of* Z*. Then*

$$\mathrm{Conf}(Z) = \mathrm{Möb}(Z) = \mathrm{Isom}(\mathbb{H}^n).$$

(ii) Let $\{f_n\}_{n\geq 1}$ *be a sequence of* k*-quasi-conformal homeomorphisms of* Z *(for the same* k*). Assume that there exist* $a, b, c \in Z$*, pairwise distinct, and fixed by each* f_n*,* $n \geq 1$*. Then, up to taking a subsequence,* $\{f_n\}_{n\geq 1}$ *converges uniformly on* Z *to a* k*-quasi-conformal homeomorphism* f_∞*.*

Proof. Part (ii) follows from theorem 1.5(i) and Ascoli theorem. The first equality in part (i) follows from theorem 1.5(ii).

Lemma 1.7 implies that $\mathrm{Isom}(\mathbb{H}^n) \subset \mathrm{Möb}(Z)$. To prove the converse it is enough to prove that a Möbius transformation of $\mathbb{R}^{n-1}$ which stabilises ∞ extends as an isometry of the upper-half space model of $\mathbb{H}^n$. This is indeed the case because such a Möbius transformation is a similarity of $\mathbb{R}^{n-1}$. $\square$

2 Quasi-isometries

This chapter will relate quasi-isometries of $\mathbb{H}^n$ with quasi-Möbius homeomorphisms of $Z = \partial\,\mathbb{H}^n$.

Definition 2.1. Let X and Y be two metric spaces. A map $f : X \to Y$ is called a *quasi-isometry* if there exists $C \geq 1$, $D \geq 0$ such that

(i) $\forall x, x' \in X$, $C^{-1} d_X(x, x') - D \leq d_Y(f(x), f(x')) \leq C d_X(x, x') + D$

(ii) $\forall y \in Y$, $\mathrm{dist}(y, f(X)) \leq D$.

Theorem 2.2 (Efremovich-Tihomirova [ET])**.** *Any quasi-isometry* f *of* $\mathbb{H}^n$ *extends to a* η*-quasi-Möbius homeomorphism of* $Z = \partial\,\mathbb{H}^n = S^{n-1}$*. Moreover,* η *depends quantitatively on the quasi-isometry constants of* f*.*

Recall that a *quasi-geodesic* in a metric space X is a map $\gamma : I \to X$, defined on an interval I, such that there exists $C \geq 1$, $D \geq 0$, with the following property

$$\forall t, t' \in I, \quad C^{-1}|t - t'| - D \leq d_X(\gamma(t), \gamma(t')) \leq C|t - t'| + D.$$

The proof of theorem 2.2 relies on the following lemma.

Lemma 2.3 (Morse lemma)**.** *Any quasi-geodesic* γ *in* $\mathbb{H}^n$ *lies within bounded distance from a true geodesic of* $\mathbb{H}^n$*. Morever the distance depends quantitatively on the quasi-geodesic constants of* γ*.*

We refer to [K] for two different proofs of the above lemma. One of them is an application of asymptotic cone technics. We now indicate how Morse lemma implies theorem 2.2.

Proof of Theorem 2.2. Let O be an origin in $\mathbb{H}^n$. In order to extend the quasi-isometry f to a map $\partial f : Z \to Z$, consider $x \in Z$ and the geodesic ray $[Ox)$. Its image by f is a quasi-geodesic ray. By Morse lemma it lies within bounded distance from a geodesic ray $[f(O)y)$, with $y \in Z$. Define $\partial f(x) = y$. It is easy to see that ∂f is bijective.

We now prove that ∂f is quasi-Möbius. We claim that there exists a constant C such that for every x, x', y, y' pairwise distinct points in Z, we have

$$d_{\mathbb{H}^n}\big((xy'),(x'y)\big) - C \le \max\big\{0, \log[xx'yy']\big\} \le d_{\mathbb{H}^n}\big((xy'),(x'y)\big) + C$$

To this end recall that by lemma 1.7,
$(*)$

$$\log[xx'yy'] = \lim_{a\to x, a'\to x', b\to y, b'\to y'} \frac{1}{2}\big\{d(a,b) + d(a',b') - d(a,b') - d(a',b)\big\}.$$

First assume both $d\left((xy'),(x'y)\right)$ and $d\left((xy),(x'y')\right)$ are smaller than 1. Then one can find a point Ω in $\mathbb{H}^n$ whose distance from each of the four geodesics $(xy'), (x'y), (xy), (x'y)$ is smaller than an universal constant. With the formula $(*)$ we get that $\log[xx'yy']$ is bounded by an universal constant; the claim follows.

Assume now that $d\left((xy'),(x'y)\right) \ge 1$. Let $A \in (xy')$ and $B \in (x'y)$ such that $d(A,B) = d\left((xy'),(x'y)\right)$. Consider

$$T = (xy') \cup (x'y) \cup [AB],$$

and equip it with the length metric induced by the hyperbolic one. By standard trigonometry in $\mathbb{H}^n$ and because $d(A,B) \ge 1$, there exists a universal constant C_0 such that distances in T differ from the hyperbolic ones by an additive factor which is less than C_0. For distances in T, the right side of $(*)$ is precisely equal to $d\left((xy'),(x'y)\right)$. So using $(*)$ we get that $\log[xx'yy']$ is equal to $d\left((xy'),(x'y)\right)$ up to $4C_0$; the claim follows.

Finally assume that $d\left((xy'),(x'y)\right) \le 1$ and that $d\left((xy),(x'y')\right) \ge 1$. Switching y and y' and applying the previous case we get that $\log[xx'y'y]$ is equal to $d\left((xy),(x'y')\right)$ up to $4C_0$. In particular it is larger than $-4C_0$. Thus we obtain

$$\log[xx'yy'] = -\log[xx'y'y] \le 4C_0,$$

which implies our claim.

So in every case we have proved the claim. By combining it with Morse lemma, one obtains immediatly that ∂f is quasi-Möbius. Finally observe that a map which qusi-preserves the crossratio is automatically continuous. $\square$

3 Mostow rigidity (proof)

The theorem is stated in the introduction. Start with $\varphi : \Gamma_1 \cong \Gamma_2$ of the statement.

We construct first a quasi-isometry $F : \mathbb{H}^n \to \mathbb{H}^n$ out of φ, as follows. Choose an origin O in $\mathbb{H}^n$ whose stabiliser in Γ_1 is trivial. Define F on the Γ_1-orbit of O by $F(g \cdot O) = \varphi(g) \cdot O$. Extend arbitrarily F to all of $\mathbb{H}^n$ as a quasi-isometry.

Applying theorem 2.2, the quasi-isometry F extends to $f = \partial F$ as a quasi-Möbius homeomorphism of $Z = \partial \mathbb{H}^n$. Note that f is φ-equivariant, namely the restriction of F to $\Gamma_1 \cdot O$ is.

We want to prove that f is a Möbius homeomorphism of Z. This will imply the theorem because $\text{Möb}(Z) = \text{Isom}(\mathbb{H}^n)$ by corollary 1.8. To this end consider the bundle E which is the projectivisation of the tangent bundle of Z. Its elements are the lines in $\mathbb{R}^n$ which are tangent to Z. Because E is homogeneous under $\text{Isom}(\mathbb{H}^n)$, we write it as $E = G/H$ with $G = \text{Isom}(\mathbb{H}^n)$, and H is the stabiliser in G of a fixed element in E. Observe that H is non-compact.

For τ a non zero tangent vector to Z, denote by $[\tau]$ the line generated by τ. Then define $h : E \to \mathbb{R}$ as follows : for non-zero $\tau \in T_z Z$, let

$$h([\tau]) = \frac{\|D_z f(\tau)\|}{\|\tau\| \cdot \|D_z f\|},$$

(recall that f is differentiable a.e., thanks to theorem 1.3). Because f is φ-equivariant, and because the groups Γ_i act conformally on Z, one can check that h is Γ_1-invariant. Now, here is a general theorem, due to Moore (see [Z] for a proof) :

Theorem 3.1. *Let G be a non-compact, connected simple Lie group, with finite center. Let $H < G$ be a closed non-compact subgroup of G. Let $\Gamma < G$ be a lattice. Then Γ acts ergodically on G/H,* i.e. *any measurable Γ-invariant function on G/H is constant a.e.*

We get that h is constant a.e.; this implies that Df is conformal a.e., so f is a Möbius homeomorphism by theorem 1.5(ii).

4 Sullivan-Tukia's theorem (proof)

The theorem is stated in the introduction. Consider the isometric action of Γ on itself by left translations. Because Γ is quasi-isometric to $\mathbb{H}^n$, each element of Γ induces a quasi-isometry of $\mathbb{H}^n$, which is unique up to bounded distance, and with uniform quasi-isometry constants. Thus by theorem 2.2, we get a Γ-action on $Z = \partial \mathbb{H}^n$ by K-quasi-conformal homeomorphisms (with K uniform). The kernel of this action is finite, we still denote by Γ its quotient by the kernel.

Definition 4.1. A measurable field of ellipses on Z is a measurable map which assigns to a.e. $z \in Z$ an ellipse centered at 0 in $T_z Z$.

We are only concerned with non degenerate ellipses, up to homothety, and centered at 0. The space of those ellipses in $\mathbb{R}^{n-1}$ is the symmetric space

$$X := \mathrm{SL}_{n-1}(\mathbb{R}) / \mathrm{SO}(n-1).$$

Any quasi-conformal homeomorphism f of Z acts on the left on the space of measurable fields of ellipses, as follows : if $\xi = \{\xi_z\}_{z \in Z}$ is a measurable field of ellipses, then we set

$$(f_* \xi)_z := D_{f^{-1}(z)} f(\xi_{f^{-1}(z)}).$$

Thus we get a Γ-action on the set of measurable fields of ellipses.

Lemma 4.2. *There exists a measurable field of ellipses $\{\xi_z\}_{z \in Z}$ which is Γ-invariant.*

Proof. For every $z \in Z$, let

$$E_z = \left\{ D_{\gamma^{-1}(z)} \gamma(S_{\gamma^{-1} z});\ \gamma \in \Gamma \right\},$$

where S_x is the unit sphere in $T_x Z$. By choosing a measurable trivialisation of the orthonormal frame bundle of Z, each set E_z identifies with a subset of the symmetric space X defined above. In addition we have for $\gamma \in \Gamma$, and $z \in Z$

$$E_{\gamma(z)} = D_z \gamma(E_z),$$

where $D_z \gamma$ acts on X by isometry (indeed $\mathrm{SL}_{n-1}(\mathbb{R})$ does). The eccentricity of ellipses in E_z is bounded by K, the quasi-conformal constant of the Γ-action on Z. Thus E_z is a bounded subset in X. A bounded set A in a complete, simply connected, non-positively curved, riemannian manifold, has a well-defined barycenter, namely the center of the unique smallest ball containing A. Define ξ_z to be the barycenter of E_z. The field $\{\xi_z\}_{z\in Z}$ posseses the expected properties.

□

Let $\xi = \{\xi_z\}_{z\in Z}$ be a Γ-invariant measurable field of ellipses. Our goal is now to find a quasi-conformal homeomorphism h of Z such that $h_*\xi = \mathcal{S}$, where $\mathcal{S}$ is the field of round spheres. This will imply that $h\Gamma h^{-1}$ stabilizes $\mathcal{S}$, hence we will get

$$h\Gamma h^{-1} < \mathrm{Conf}(Z) = \mathrm{Isom}(\mathbb{H}^n).$$

For $n = 3$, existence of h follows from Ahlfors-Bers theorem (see [A]). When $n \geq 4$, Ahlfors-Bers theorem is not valid, instead Tukia has proposed the following argument.

The field ξ is measurable, so it is *approximately continuous* a.e., *i.e.* for a.e. $z \in Z$, and for every $\varepsilon > 0$, we have

$$\lim_{r\to 0}\ m\big\{x \in B(z,r);\ d_X(\xi_z,\xi_x) < \varepsilon\big\}/m(B(z,r)) = 1,$$

where m denotes the Lebesgue measure on Z.

In the upper half-space model of $\mathbb{H}^n$, let 0 be the origin and let e_n be the point whose euclidean coordinates are $(0,...,0,1)$. Up to conjugating Γ by a affine map, we may assume that ξ is approximately continuous at 0, and that ξ_0 is a round sphere.

Let $\{g_k\}_{k\geq 1}$ be a sequence in Γ, such that $g_k\cdot e_n \underset{k\to\infty}{\to} 0$ and such that the distances $d_{\mathbb{H}^n}(g_k\cdot e_n, [0\infty))$ are uniformly bounded (existence comes from the fact that Γ and $\mathbb{H}^n$ are quasi-isometric).

Let $\{\lambda_k\}_{k\geq 1}$ be a sequence of positive numbers such that the distances $d_{\mathbb{H}^n}\big(\lambda_k g_k\cdot e_n, e_n\big)$ are uniformly bounded. The maps $\lambda_k g_k$, $k \geq 1$, are quasi-isometries of $\mathbb{H}^n$ with uniformly bounded quasi-isometry constants, and which almost stabilise e_n. Thus, by Ascoli theorem we get, up to a subsequence, that $\{\lambda_k g_k\}_{k\geq 1}$ converge uniformly on Z to a K-quasi-conformal homeomorphism h. (Note that corollary 1.8(ii) gives another way of establishing this convergence). It follows that

$$(*)\qquad h_*\xi = \lim_{k\to\infty}(\lambda_k g_k)_*\xi = \lim_{k\to\infty}(\lambda_k)_*\xi.$$

In addition, because ξ is approximately continuous at 0, up to a subsequence, the sequence $\{(\lambda_k)_*\xi\}_{k\geq 1}$ tends a.e. to the constant field equal to ξ_0 (namely convergence in measure implies convergence a.e. of a subsequence). Finally we obtain that $h_*\xi = \mathcal{S}$, which implies that $h\Gamma h^{-1}$ is contained in $\mathrm{Isom}(\mathbb{H}^n)$.

Remark : The first equality in ($*$) is not at all obvious. Indeed one doesn't know anything about convergence of the differentials of the $\lambda_k g_k$. At this stage one needs a more delicate argument based on approximations of $h_*\xi$ by $(\lambda_k g_k)_*\xi$ on subsets with complementary measure arbitrary close to 0. We refer to Tukia's paper [T2] for details.

It remains to prove that $h\Gamma h^{-1}$ is a cocompact lattice of $\mathrm{Isom}\,\mathbb{H}^n$. By reusing the quasi-isometry between Γ and $\mathbb{H}^n$, one can see that $h\Gamma h^{-1}$ acts properly discontinuously and cocompactly on $\mathbb{H}^n$. So it is a cocompact lattice in $\mathrm{Isom}(\mathbb{H}^n)$.

Notes : Quasi-Möbius homeomorphisms have been defined first by Väisälä [V2]. Equivalence between quasi-conformal and quasi-Möbius homeomorphisms (theorem 1.5(i)) is due to Gehring [G1] for $\mathbb{R}^2$, and to Gehring-Väisälä for $\mathbb{R}^n$ (see [V1]). Note that this result is false for general domains in $\mathbb{R}^n$ (see [V2]). The statement (ii) in theorem 1.5 is also true for domains in $\mathbb{R}^n$ with $n > 2$, see [G2] and [R]. This is a generalisation of Liouville theorem which requires the mappings to be sufficiently smooth (C^3 is enough). The fact that moduli depend only on the relative position of the continua (lemma 1.6(3)), was known to Grötzsch and Teichmüller for $\mathbb{R}^2$. For $\mathbb{R}^n$ it was first observed by Loewner [Lo]. For general domains in $\mathbb{R}^n$ it is false.

Morse lemma was first stated and used by Mostow in [M2]. Theorem 2.2 and its proof generalises to Gromov-hyperbolic spaces. Tukia [T1] has proved the converse of theorem 2.2, namely quasi-Möbius homeomorphisms of Z extend to quasi-isometries of $\mathbb{H}^n$. Again this phenomenon generalizes to most of the Gromov-hyperbolic spaces (see [Pau], [BHK]).

The proof of Mostow theorem we gave is taken from [GP]. In [K], M. Kapovich gives a more elementary proof which does not make use of Moore ergodic theorem. For the proof of Sullivan-Tukia theorem, we have followed rather closely Tukia's paper [T2].

Mostow theorem is the first rigidity result based on connections between hyperbolic geometry and quasi-conformal geometry. This circle of ideas is still an active domain of research, see [GP], [BP] for surveys of further developments. Recently J. Heinonen and P. Koskela [HK] have extended the euclidean theory of quasi-conformal homeomorphisms to a much larger class of metric spaces, called *Loewner spaces*. In [C], J. Cheeger has developped a differential calculus on Loewner spaces. These new ideas seem promising.

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