CALABI-WEIL RIGIDITY

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We based these lectures on the approach developed by Raghunatan ([Rag]). The reader can also see the original papers by A. Weil (see [We1], [We2]). The text which follows is neither intended to be original nor exhaustiv. It aims at presenting in a very elementary way the theory of infinitesimal rigidity as described in [Rag].

Introduction

A trivial example The group $\mathbb{Z}$ can be viewed as a subgroup of the group of translations of the real line $\mathbb{R}$, and of infinitely many ways. More precisely let $t \in \mathbb{R}$ and let us call T_t the translation defined by

$$T_t(x) = x + t \quad \text{for} \quad x \in \mathbb{R}.$$

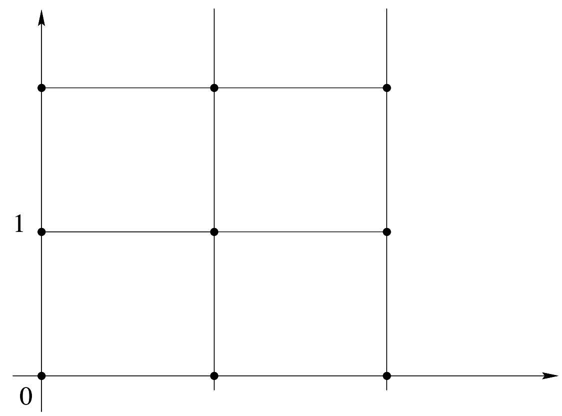
Translations are isometries of the euclidean structure on $\mathbb{R}$, so that we can define a family of morphisms

$$\begin{aligned} \rho_t : \mathbb{Z} &\hookrightarrow \mathrm{Isom}(\mathbb{E}) \\ n &\longmapsto T_t^n (x \mapsto x + nt). \end{aligned}$$

Such morphisms are called representations of $\mathbb{Z}$ as isometries of $\mathbb{R}$. We thus get a deformation of the canonical representation ρ_1.

This deformation is not trivial in the sense that there does not exist, for t close to 1, an isometry g_t of $\mathbb{R}$ such that

$$\forall n \in \mathbb{Z}\ , \quad \rho_t(n) = g_t \rho_1(n) g_t^{-1}.$$

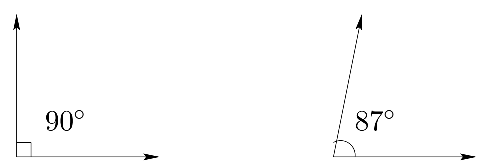


Another trivial example Again $\mathbb{Z}^2$ can be viewed as a subgroup of the group of translations of $\mathbb{R}^2$, the quotient space being a torus $\mathbb{R}^2/\mathbb{Z}^2$. The translations of $\mathbb{R}^2$ are isometries with respect to the usual euclidean structure.

The orbit of the origin is a lattice in $\mathbb{R}^2$ generated by two vectors which are image of the origin by the two translations associated to the generators $(1,0)$ and $(0,1)$ of $\mathbb{Z}^2$.

There is more flexibility here since one can play with the length of these vectors as well as with the angle between them.

The manifolds $\mathbb{R}^2/\mathbb{Z}^2$ are endowed with the metric coming from the euclidean metric of $\mathbb{R}^2$ and are thus flat riemannian manifolds. The existence of non trivial deformations corresponds to the existence of many non isometric flat tori. The following basis generated by unit length vectors gives rise to



non isometric tori.

Another remark coming from these trivial examples is that all tori are diffeomorphic but nevertheless metrically different.

The purpose of all courses on Rigidity in this school is to exhibit situations where the opposite results occur. Instead of flexibility as above we shall exhibit rigidity: rigidity of deformations (this course), situation in which diffeomorphic manifolds (and even less) are isometric and more.

1 Deformations and cohomology

A general reference for this chapter is [Br], see also [Rag].

Let Γ be a finitely generated group and G a Lie group. We call $R(\Gamma, G)$ the set of homomorphisms of Γ in G endowed with the topology of pointwise convergence. Let ρ_t, $t > 0$ be a deformation of a representation ρ_0, which we assume C^1 in t.

Let us define, for $\gamma \in \Gamma$,

$$d_{\rho_0}(t,\gamma) = \rho_t(\gamma)\rho_0(\gamma)^{-1} \quad \text{and} \quad h_{\rho_0}(\gamma) = \frac{\partial}{\partial t}\Big|_{t=0} d_{\rho_0}(t,\gamma)$$

$d_{\rho_0}(t,\gamma)$ is, for each $\gamma \in \Gamma$, a path in V.

Now, for $\gamma, \gamma' \in \Gamma$,

$$d_{\rho_0}(t,\gamma\gamma') = \rho_t(\gamma\gamma')\rho_0(\gamma\gamma')^{-1}$$

the inverse being taken in G.

$$\begin{aligned} d_{\rho_0}(t,\gamma\gamma') &= \rho_t(\gamma)\rho_t(\gamma')\rho_0(\gamma')^{-1}\rho_0(\gamma)^{-1} \\ &= \left(\rho_t(\gamma)\rho_0(\gamma)^{-1}\right)\left(\rho_0(\gamma)\rho_t(\gamma')\rho_0(\gamma')^{-1}\right) \end{aligned}$$

and

$$h_{\rho_0}(\gamma\gamma') = h_{\rho_0}(\gamma) + \rho_0(\gamma)h_{\rho_0}(\gamma')\rho_0(\gamma)^{-1} \qquad (*)$$

where the second part is $\mathrm{Ad}(\rho_0(\gamma))(h_{\rho_0}(\gamma'))$, so that h satisfies

$$h_{\rho_0}(\gamma\gamma') = h_{\rho_0}(\gamma) + (\mathrm{Ad} \circ \rho_0)(\gamma) \cdot h_{\rho_0}(\gamma').$$

A particular case of deformations, as mentioned before, consists in taking a C^1-path $g_t \in G$ with $g_0 = e$ and defining

$$u_t(\gamma) = g_t\rho_0(\gamma)g_t^{-1} \quad \text{for all} \quad \gamma \in \Gamma\ .$$

Then,

$$\begin{aligned} d_{u_0}(t,\gamma) &= g_t\rho_0(\gamma)g_t^{-1}\rho_0(\gamma)^{-1} \\ h_{u_0}(\gamma) &= X - \rho_0(\gamma)X\rho_0(\gamma)^{-1} \\ &= X - (\mathrm{Ad} \circ \rho_0)(\gamma) \cdot X \qquad (**) \end{aligned}$$

where $X = \frac{d}{dt} g_t\Big|_{t=0} \in \mathfrak{g}$, the Lie algebra of G.

We call this last type of deformations, trivial deformations. So that if there exists a non trivial deformations of the representation ρ_0, then there exists a function $h : \Gamma \to G$ satisfying $(*)$ for all $\gamma \in \Gamma$ and not of the type $(**)$, at least in spirit.

2 Cohomology of groups

See [Br] for a quite complete description of cohomology of groups.

We consider the elementary case of this theory. Let V be a vector space (an abelian group) on which Γ acts, that is, there is a representation ρ of Γ in $\mathrm{GL}(V)$. We consider the following complex

$$\cdots \longrightarrow \Lambda^k(\Gamma, V) \xrightarrow{d} \Lambda^{k+1}(\Gamma, V) \longrightarrow \cdots$$

where $C^0(\Gamma, V) = V$ and $C^k(\Gamma, V) = \Big\{ f : \underbrace{\Gamma \times \cdots \times \Gamma}_{k\text{-times}} \longrightarrow V \Big\}$ and d is given by

$$d_\rho f(\gamma_1, \ldots, \gamma_k) = \rho(\gamma_1) f(\gamma_2, \ldots, \gamma_k) + \sum_{i=2}^{k} (-1)^{i-1} f(\gamma_1, \ldots, \gamma_{i-1}\gamma_i, \ldots, \gamma_k) + (-1)^n f(\gamma_1, \ldots, \gamma_{k-1})$$

This is the non homogeneous version of group cohomology. For $p = 0, 1$, we have

i) for $v \in V = C^0(V), dv(\gamma) = \rho(\gamma)v - v$, for all $\gamma \in \Gamma$,

ii) for $f \in C^1(V)$,

$$df(\gamma_1, \gamma_2) = f(\gamma_1) + \rho(\gamma_1) f(\gamma_2) - f(\gamma_1\gamma_2)$$

for all $\gamma_1, \gamma_2 \in \Gamma$.

A 1-cocycle with value in ρ (or in V) is a map

$$\rho : \Gamma \longrightarrow V$$

such that, for all $\gamma, \gamma' \in \Gamma$,

$$\rho(\gamma\gamma') = f(\gamma) + \rho(\gamma)\rho(\gamma') \ .$$

As usual we define

$$H^1(\Gamma, \rho) = \frac{\mathrm{Ker}\{d : C^1 \to C^2\}}{\mathrm{Im}\{d : C_0 \to C^1\}}.$$

3 Local rigidity

Let us describe the topology of $R(\Gamma, G)$. Since Γ was supposed to be finitely generated (we can even assume it to be finitely presentable), let S be a finite generating system then one can write $G = F(S)/H$ where $F(S)$ is the free group generated by S and H is the normal subgroup of F constituted by the relations. More precisely if

$$p : F(S) \longrightarrow \Gamma$$

is the natural "projection", then H =kernel(p).

Let $S = \{s_1, \ldots, s_N\}$, for $x \in H$ with $x = s_{i_1}^{\varepsilon_1} \cdots s_{i_k}^{\varepsilon_k}$, $s_{i_j} \in S$, we consider the map

$$\begin{aligned} f_x : \prod_S G = \{h = (g_{s_1}, \ldots, g_{s_N})\} &\longrightarrow G \\ h &\longmapsto g_{s_{i_1}}^{\varepsilon_1} \cdots g_{s_{i_k}}^{\varepsilon_k} \end{aligned}$$

and for $\rho \in R(\Gamma, G)$, we define $h_\rho \in \prod_S G$ by

$$h_\rho = (\rho(s_1), \ldots, \rho(s_N)) \, ,$$

we then have

Lemme 3.1. *The map $\varphi : R(\Gamma, G) \to \prod_S G$, defined by $\varphi(\rho) = h_\rho$, is a bijection between $R(\Gamma, G)$ and $\bigcap_{x \in H} f_x^{-1}(e)$. If $R(\Gamma, G)$ is endowed with the topology of pointwise convergence then φ is a homeomorphism onto its image.*

This lemma is an easy exercise left to the reader.

If Γ is finitely presented, then H is finitely generated and the above intersection can be taken to be finite (by taking x in a generating set for H). Furthermore if G is an algebraic group over a field k then $R(\Gamma, G)$ is a variety defined over k.

Finally, G acts on $R(\Gamma, G)$ by

$$(g, \rho) \longmapsto g\rho g^{-1}$$

where $g\rho g^{-1}$ is the morphism

$$\begin{aligned} \Gamma &\longrightarrow G \\ \gamma &\longmapsto g\rho(\gamma)g^{-1} \end{aligned}$$

Définition 3.2. A representation $\rho_0 \in R(\Gamma, G)$ is locally rigid if the orbit of ρ_0 under the action of G is a neighbourhood of ρ_0 in $R(\Gamma, G)$.

Remarks

i) It says that close to ρ_0 (in the topology given by a finite generating system) a representation ρ is a trivial deformation of ρ_0.

ii) In the introduction we used 1-parameter deformations which are C^1; the above definition is more general.

The key result of these lectures is the

Théorème 3.3 (A. Weil [We3]). *Let $G \subset \mathrm{GL}(n, \mathbb{C})$ an algebraic group and $\rho_0 \in R(\Gamma, G)$. If $H^1(\Gamma, \mathrm{Ad} \circ \rho_0)$ vanishes then ρ_0 is locally rigid.*

Remarks

i) This is more difficult than the sketch made in section 1. Indeed there we just showed that a C^1-deformation is tangent at the origin to a trivial deformation when $H^1(\Gamma, \mathrm{Ad} \circ \rho_0) = 0$. However since $R(\Gamma, G)$ is an algebraic variety one can get the stronger result stated above.

ii) The converse is not true. More precisely, there are examples where not all elements of $H^1(\Gamma, \mathrm{Ad} \circ \rho_0)$ give rise to a deformation. The difficulty is thus that one cannot always "integrate" an infinitesimal non trivial deformation.

We intend to prove Weil's local rigidity which is

Théorème 3.4 (A. Weil [We2]). *If G is a connected semi-simple Lie group without compact factor and $\Gamma \subset G$ is an irreducible uniform lattice. If G is not locally isomorphic to* $\mathrm{SL}_2(\mathbb{R})$*, then* $H^1(\Gamma, \mathrm{Ad} \circ i) = 0$.

Here $i : \Gamma \hookrightarrow G$ is the injection of Γ into G.

Corollaire 3.5. *In this situation Γ is locally rigid.*

Remarks

i) We shall restrict our (sketch of) proof to the case when G is a simple Lie group and thus an algebraic subgroup of some $\mathrm{GL}(n, \mathbb{C})$.

ii) The first proof of such a result is due to A. Selberg for the case $G = \mathrm{SL}(n, \mathbb{R})$, $n \geq 3$. E. Calabi in an unpublished paper has then proved a similar result for lattices in the hyperbolic space of dimension ≥ 3. The case of hermitian symmetric spaces was first established by E. Calabi and ?.Vesentini. The general case is due to A. Weil.

iii) The non-uniform case is settled by H. Garland ([Ga]) and H.Garland and M.S. Raghunatan ([G-R]). G. Margulis's results then cover all remaining cases.

iv) One should cite Y. Matushima for the use of the Bochner formula and computation of some Betti numbers.

4 Differential geometry

Let X be a connected and oriented differentiable manifold, which will be compact in the sequel and let $\Gamma = \pi_1(X)$. The group Γ acts on the universal covering $\widetilde{X}$ of X by deck transformation. We assume that Γ acts on a finite dimensional vector space V and denote by $\rho : \Gamma \to \mathrm{GL}(V)$ the corresponding representation.

This defines a vector bundle $E(\rho)$ on X, by the following standard construction: Γ acts on $\widetilde{X} \times V$ by the left action,

$$\begin{aligned} \Gamma \times (\widetilde{X} \times V) &\longrightarrow \widetilde{X} \times V \\ \gamma, (\tilde{x}, v) &\longmapsto (\gamma\tilde{x}, \rho(\gamma)v) \end{aligned}$$

and

$$E(\rho) = \widetilde{X} \times_\rho V = \widetilde{X} \times V/\Gamma \longrightarrow X \ .$$

This bundle is flat, that is to say that it has a flat connection. This means two equivalent properties:

i) There is a foliation on $E(\rho)$ transverse to the fibers of $\pi : E(\rho) \to X$. Lifted to $\widetilde{X} \times V$ the leaf of this foliation through $(\tilde{x}, v)$ is $\widetilde{X} \times \{v\}$ while the fiber of π is $\{\tilde{x}\} \times V$. This is the geometric point of view on connections.

ii) There is a flat covariant derivative. Precisely a section s of the bundle can be viewed as an ρ-equivariant map

$$\begin{aligned} \tilde{s} : \widetilde{X} &\longrightarrow (\widetilde{X} \times \widetilde{V}) \\ \tilde{x} &\longmapsto (\tilde{x}, \varphi(\tilde{x})) \end{aligned}$$

where $\varphi : \widetilde{X} \to \widetilde{V}$ satisfies

$$\forall \gamma \in \Gamma \ , \quad \varphi(\gamma\tilde{x}) = \rho(\gamma)\varphi(\tilde{x}) \ .$$

Indeed this defines a section by

$$s([\tilde{x}]) = \big[(\tilde{x}, \varphi(\tilde{x}))\big]$$

where $[\tilde{x}]$ (resp. $\big[(\tilde{x}, \varphi(\tilde{x}))\big]$ then denotes the class of $\tilde{x}$ (resp. $(\tilde{x}, \varphi(\tilde{x}))$) for the equivalence relation given by the action of Γ. One then has the commutative diagram

$$\begin{tikzcd} \widetilde{X} \times \widetilde{V} \arrow[r] \arrow[d] & E(\rho) \arrow[d, "\pi"'] \\ \widetilde{X} \arrow[r, "p"] \arrow[u, bend right, "\tilde{s}"'] & X \arrow[u, bend right, "s"'] \end{tikzcd}$$

Exercise: We showed how to get s from $\tilde{s}$. Explain how to construct $\tilde{s}$ from s.

Now let U be a vector field near $x \in X$ and $\widetilde{U}$ a pulled back of U in $T\widetilde{X}$, we define $D_U s$ through the previous construction using $\widetilde{U} \cdot \varphi$, that is

$$(\widetilde{U} \cdot \varphi)(\tilde{x}) = d_{\tilde{x}}\varphi(\widetilde{U}) \ .$$

Here $p(\tilde{x}) = x$. We remark that, by construction,

$$\widetilde{U}(\gamma\tilde{x}) = d_{\tilde{x}}\gamma(\widetilde{U}(\tilde{x}))$$

then

$$\begin{aligned} d_{\gamma\tilde{x}}\varphi\big[\widetilde{U}(\gamma\tilde{x})\big] &= d_{\tilde{x}}(\varphi\circ\gamma)\big(\widetilde{U}(\tilde{x})\big) \\ &= d_{\tilde{x}}\big(\rho(\gamma)\circ\varphi\big)\big(\widetilde{U}(\tilde{x})\big) \\ &= \rho(\gamma)\big(d_{\tilde{x}}\varphi\big(\widetilde{U}(\tilde{x})\big)\big) \end{aligned}$$

since the action of $\rho(\gamma)$ on V is linear. This shows the equivariance of the derivative $\tilde{u}\cdot\varphi$ which in turn defines the value at x of a new section denoted by $(D_U s)(x)$. The operator D is a covariant derivative, the analytic version of a connection. It is clearly flat, indeed let $(x_1,\ldots,x_n)$ be a coordinate chart of X around a point $x\in X$ and let $\partial/\partial x_i$ be the canonical vector fields, then, for any section s defined in a neighbourhood of x,

$$D_{\partial/\partial x_i}D_{\partial/\partial x_j}s = D_{\partial/\partial x_j}D_{\partial/\partial x_i}s$$

which is the Schwarz lemma, asserting the flatness of D.

Exercise: Check the above formula.

For a section s, the map

$$\begin{aligned} TX &\longrightarrow E(\rho) \\ u &\longmapsto D_u s \end{aligned}$$

gives rise to a differential form on X with values in the bundle $E(\rho)$; it is thus a section of the bundle $T^*X\otimes E(\rho)\longrightarrow X$. By applying the previous construction inductively one can define k-differential forms with values in $E(\rho)$ and we denote by

$$\Lambda^k(E) = \{k\text{-forms with values in } E(\rho)\}\,.$$

The operator D allows to define a coboundary operator,

$$d^D : \Lambda^k(E)\longrightarrow\Lambda^{k+1}(E)$$

by the formula

$$\begin{aligned} d^D\omega(U_1,\ldots,U_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}D_{X_i}\omega\left(U_1,\ldots,\widehat{U}_i,\ldots,U_{k+1}\right) + \\ &\sum_{i<j}(-1)^{i+j}\omega\left([U_i,U_j],U_1,\ldots,\widehat{U}_i,\ldots,\widehat{U}_j,\ldots,U_n\right) \end{aligned}$$

where $\omega \in \Lambda^k(E)$ and U_i are vector fields on X. One can check that $(d^D)^2 = 0$ (since D is flat). It gives rise to a cohomology with values in E. We shall denote by $H^k(X, E)$ the k-th cohomology group with values in E. This is a De Rham cohomology.

Another point of view could be to work on $\widetilde{X}$ with equivariant map. More precisely one can consider differential forms on $\widetilde{X}$ with values in V (which is a modest modification of the classical notion of differential forms with values in $\mathbb{R}$ or $\mathbb{C}$) which are ρ equivariant. One builds a cohomology denoted by $H^k(\Gamma, \widetilde{X}, \rho)$. It is not difficult to see that these two cohomologies are isomorphic.

Exercise: Prove this assertion. The first step is to define the action of Γ on forms with values in V.

The main result of this section is the following

Théorème 4.1 (Eilenberg). *It $\widetilde{X}$ is contractible then $H^*(X, E)$ (and $H^*(\Gamma, \widetilde{X}, \rho)$) are naturally isomorphic to $H^*(\Gamma, \rho)$*

In particular, one has $H^1(\Gamma, \rho) \cong H^1(X, E)$. This opens the possibilities of using all the differential geometric techniques in order to prove that $H^1(\Gamma, \rho) = 0$ and thus the local rigidity of certain representations.

Remarks

i) A contractible space is a topological space such that the identity is homotopic to a constant map. The symmetric spaces of non positive curvature are contractible.

ii) The above theorem is not difficult to understand in certain easy situation. Let us assume that $\widetilde{X}$ is non positively curved (a real hyperbolic space for example) and let ω be a 1-form on $\widetilde{X}$ with values in V and ρ-equivariant. We fix an origin $x_0 \in \widetilde{X}$ and define, for $\gamma \in \Gamma$:

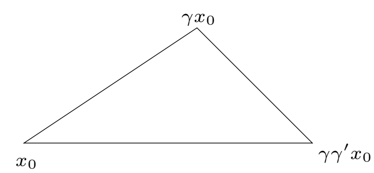
$$f(\gamma) = \int_{x_0}^{\gamma x_0} \omega = \text{ integral of } \omega \text{ on the (unique) geodesic between } x_0 \text{ and } \gamma x_0.$$

Lemme 4.2. *If ω is closed then f is a 1-cocycle.*

Proof. Since ω is closed, one has

$$f(\gamma\gamma') = \int_{x_0}^{\gamma\gamma' x_0} \omega = \int_{x_0}^{\gamma x_0} \omega + \int_{\gamma x_0}^{\gamma\gamma' x_0} \omega$$

now



$$\int_{\gamma x_0}^{\gamma\gamma' x_0} \omega = \int_{x_0}^{\gamma' x_0} \gamma^* \omega = \int_{x_0}^{\gamma' x_0} \rho(\gamma)(\omega) = \rho(\gamma) \left(\int_{x_0}^{\gamma' x_0} \omega \right)$$

by linearity of the action $\rho(\gamma)$. Here we used the equivariance of ω which reads, for $u \in T_x M$,

$$(\gamma^* \omega)_x(u) = \omega_{\gamma x}(d_x \gamma(u)) = \rho(\gamma)(\omega_x(u)) \ .$$

We thus proved

$$f(\gamma\gamma') = f(\gamma) + \rho(\gamma) f(\gamma') \Longleftrightarrow df \equiv 0 \ .$$

One can extend this construction to arbitrary forms and show that the correspondence yields an isomorphism between the two cohomologies. □

Exercise: Just do it!

5 Hodge theory

The reader can learn the Hodge theory for real-valued forms in [Wa], for example.

If X is an oriented Riemannian manifold then the scalar product (the metric) extends to a scalar product for k-forms. More precisely let ω and η be two real-valued k-forms and let $\{e_i\}_{i=1,\ldots,n}$ be an orthonormal basis of $T_x X$, then the scalar product at x is defined by

$$(\omega, \eta)(x) = \sum_{i_1 < i_2 < \cdots < i_k} \omega(e_{i_1}, \ldots, e_{i_k}) \eta(e_{i_1}, \ldots, e_{i_k}).$$

We may then talk about orthonormal basis of real-valued k-forms.

We now assume that $E(\rho) \to X$ is a metric bundle, *i.e.* the fiber above x carries a Euclidean structure g_x depending smoothly on $x \in X$.

Such a metric gives a (musical) isomorphism:

$$\flat : E(\rho)_x \longrightarrow E(\rho)_x^*$$

defined by

$$v^\flat(u) = g_x(u, v) \ .$$

This naturally extends to a pairing

$$\flat : \Lambda^k(E) \longrightarrow \Lambda^k(E^*)$$

by $(\omega \cdot u)^\flat = \omega u^\flat$ if ω is a real-valued form and $u \in E(\rho)$. On a coordinate open set on X, let $\alpha_1, \ldots, \alpha_n$ be real-valued 1-forms which are a local basis of T^*X. Then, on this set, for $\xi \in \Lambda^k(E)$, one can write

$$\xi = \sum_{i_1 < \cdots < i_k} u_{i_1 \cdots i_k} \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}$$

where $u_{i_1 \cdots i_k}$ are smooth sections of $E(\rho)$ (or $E(\rho)^*$). Now, if $\eta \in \Lambda^p(E^*)$, we define

$$\xi \wedge \eta = \sum_{\substack{i_1 < \cdots < i_k \\ j_1 < \cdots < j_\ell}} v_{j_1 \cdots j_\ell}(u_{i_1 \cdots i_k}) \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k} \wedge \alpha_{j_1} \wedge \cdots \wedge \alpha_{j_\ell} \ .$$

It is a scalar $k + \ell$-form.

The Hodge-star operator is then defined by

$$* : \Lambda^p(E) \longrightarrow \Lambda^{n-k}(E)$$

$$*\xi = \sum u_{i_1 \cdots i_k} * (\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}) \, .$$

We thus have to define the $*$-operator for real-valued forms;

$$* : \Lambda^k(X) \longrightarrow \Lambda^{n-k}(X)$$

$$\text{for} \quad \alpha \quad \text{and} \quad \beta \in \Lambda_k(X), \ \alpha \wedge *\beta = (\alpha, \beta) \, \mathrm{dvol} \, ;$$

where $(\cdot, \cdot)$ is the scalar product on k-forms on X. If $\{\alpha_i, \ldots, \alpha_\ell\}$ is a positive orthonormal basis of real valued 1-forms at $x \in X$, then

$$*(\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}) = \alpha_{j_1} \wedge \cdots \wedge \alpha_{j_l} \ , \quad k + \ell = n$$

where $(j_1, \ldots, j_\ell)$ are such that

$$\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k} \wedge \alpha_{j_1} \wedge \cdots \wedge \alpha_{j_\ell} = \alpha_1 \wedge \cdots \wedge \alpha_n \ .$$

The following properties are easily checked

i) $**\xi = (-1)^{k(n-k)}\xi$ for $\xi \in \Lambda^k(E)$

ii) for $\xi, \eta \in \Lambda^k(E)$, $\xi = \sum u_{i_1\cdots i_k}\alpha_{i_1}\wedge\cdots\wedge\alpha_{i_k}$ and $\eta = \sum v_{i_1\cdots i_k}\alpha_{i_1}\wedge\cdots\wedge\alpha_{i_k}$ where $\{\alpha_i\}_{i=1,\ldots,n}$ is a (local) orthonormal basis of real-valued 1-forms we set

$$(\xi, \eta) = \sum_{i_1<\cdots<i_k} g_x\left(u_{i_1\cdots i_k}, v_{i_1\cdots i_k}\right)$$

and we have

$$\xi \wedge (*\eta)^\flat = (\xi, \eta)\,\mathrm{dvol} \ .$$

We can also define a global scalar product

$$\langle \xi, \eta\rangle = \int_X (\eta, \xi)\,\mathrm{dvol} = \int_X \eta \wedge (*\xi)^\flat$$

when the integral makes sense, for example, when one of the forms is compactly supported.

With the help of these structures we shall construct some natural differential operators.

Définition 5.1. The codifferential $\delta^D : \Lambda^{k+1}(E) \to \Lambda^k(E)$ is the operator

$$\delta^D\xi = (-1)^{k(n-k)+k+1}\#\left(*d^D*(\xi^\flat)\right)$$

where $\# = (\flat)^{-1}$.

It is straightforward to check the following result,

Proposition 5.2. *$\langle d^D\alpha, \beta\rangle = \langle \alpha, \delta^D\beta\rangle$ for $\alpha \in \Lambda^k(E)$, $\beta \in \Lambda^{k+1}(E)$ and one of them is compactly supported.*

In other words δ is the formal adjoint of d.

Définition 5.3. We define the Hodge Laplacian acting on E-valued forms by

$$\Delta^D = d^D\delta^D + \delta^D d^D = d^D(d^D)^* + (d^D)^*(d^D)$$

The operator Δ^D preserves the degree of forms.

Proposition 5.4. *i) Δ is formally self-adjoint, that is*

$$\langle \Delta^D\xi, \eta\rangle = \langle \xi, \Delta^D\eta\rangle$$

for $\xi, \eta \in \Lambda^k(E)$ and compactly supported.

ii) With the same notations

$$\langle \Delta^D \xi, \eta \rangle = \langle d^D \xi, d^D \eta \rangle + \langle \delta^D \xi, \delta^D \eta \rangle \ .$$

Définition 5.5. A form $\xi \in \Lambda^k(E)$ is said to be harmonic when $\Delta^D \xi = 0$, which is equivalent to

$$d^D \xi = \delta^D \xi = 0 \ .$$

The main results of the Hodge theory is (see [Wa]):

Théorème 5.6. *If X is compact without boundary, every closed form $\xi \in \Lambda^k(E)$ (i.e. , $d^D \xi = 0$) is cohomologous to a unique harmonic form. Thus $H^k(X, E)$ is isomorphic to the vector space of k-harmonic E-valued forms.*

Corollaire 5.7. *Let X be compact, if there exists $c > 0$ such that, for all $\xi \in \Lambda^k(E)$,*

$$\langle \Delta \xi, \xi \rangle \geq c \|\xi\|^2 \quad \textit{then} \quad H^k(X, E) = 0 \ .$$

The goal is now to compute, in the case under consideration, d^D, δ^D and Δ^k. We shall show that

$$\Delta \xi = \overline{\Delta} \xi + Q(\xi)$$

where $\overline{\Delta}$ is a nonnegative operator and Q is an endomorphism (it does not differentiate ξ) which is positive.

6 Applications I

Let G be a semi-simple connected Lie group without compact factor and Γ a uniform lattice. Let K be maximal compact subgroup, then $\widetilde{X} = G/K$ is the symmetric space defined by G which is simply connected of non positive curvature and thus contractible by Hadamard-Cartan's theorem (for example).

We shall make the assumption that the representation

$$\rho : \Gamma \longrightarrow \mathrm{GL}(V)$$

is the restriction to Γ of a morphism, called ρ again,

$$\rho : G \longrightarrow \mathrm{GL}(V) \ .$$

Remark 6.1. This is the case in the situation of Calabi-Weil's rigidity since we have to consider

$$\rho = \mathrm{Ad} \circ i \ , \quad V = \mathfrak{g}$$

where i is the injection of Γ into G. It is the restriction of Ad.

We "extend" ρ to the Lie algebra $\mathfrak{g}$. Indeed if $Y \in \mathfrak{g}$ and φ_t is the one-parameter group generated by to Y, we then set

$$v \in V \ , \quad \rho(Y)v = \frac{d}{dt}\Big|_{t=0}\rho(\varphi_t)v = d_e\rho(Y)v \ .$$

Let us consider the following diagram

$$\begin{tikzcd}
E(\rho) \arrow[d] & \widetilde{X} \times V \arrow[l] \arrow[d] & G \times V \arrow[l] \arrow[d] \\
X & \widetilde{X} \arrow[l] & G \arrow[l, "p"']
\end{tikzcd}$$

We now call p the projection of G onto $\widetilde{X}$. $E(\rho)$ is the quotient of $G \times V$ by the following action of $K \times \Gamma$

$$(k, \gamma) \cdot (g, v) = (\gamma g k, \rho(\gamma)v) \ .$$

We lift the forms from $\widetilde{X}$ to G. Let $\overline{\omega}$ be a k-form on $\widetilde{X}$ with values in V and which is ρ-equivariant, *i.e.* $\overline{\omega}$ satisfies

$$\gamma^* \overline{\omega} = \rho(\gamma) \circ \overline{\omega} \ .$$

We pulled it back to G as follows:

$$\omega_g = \rho(g)^{-1}(p^* \overline{\omega}_{p(g)}) \ .$$

The aim is to make the computations on G and use the underlying algebra. The previous formula means that if $Z_1, \ldots, Z_k$ are vector fields near g and $\overline{Z}_i = dp(Z_i)$ then

$$\omega_g(Z_1, \ldots, Z_k) = \rho(g)^{-1} \overline{\omega}_{p(g)}(\overline{Z}_1, \ldots, \overline{Z}_k) \ .$$

The space of such forms is denoted $\Lambda^k(\Gamma, G, K, \rho)$.

Proposition 6.2. *If $\omega \in \Lambda^k(\Gamma, G, K, \rho)$ then*

i) $\gamma^* \omega = \omega$, *for all* $\gamma \in \Gamma$.

ii) $R_k^*(\omega) = \rho(k)^{-1}\omega$, *for all* $k \in K$, *where* R_k *denotes the right multiplication by* $k \in K$.

iii) $i(Y)\omega = 0$ *for all* $Y \in \mathfrak{k}$.

Sketch of proof. iii) $i(Y)\omega$ is the inner product of ω by Y. With the above notations

$$i(Y)\omega(Z_1,\dots,Z_{k-1}) = \omega(Y,Z_1,\dots,Z_{k-1}) = 0$$

since $p_*(Y) = dp(Y) = 0$.

i) Let us check this case also

$$\begin{aligned}(\gamma^*\omega)_g(Z_1,\dots,Z_k) &= \omega_{\gamma g}(d\gamma(Z_1),\dots,d\gamma(Z_k)) \\ &= \rho(\gamma g)^{-1}\left(\overline{\omega}_{p(\gamma g)}(\overline{d\gamma(Z_1)},\dots,\overline{d\gamma(Z_k)}\right)\end{aligned}$$

but $p(\gamma g) = \gamma p(g)$ by definition of the action on G/K and thus $\overline{d\gamma(Z_i)} = d\gamma(\overline{Z_i})$. Then

$$\begin{aligned}(\gamma^*\omega)_g(Z_1,\dots,Z_k) &= \rho(g)^{-1}\rho(\gamma)^{-1}\left(\overline{\omega}_{\gamma p(g)}(d\gamma(\overline{Z_1}),\dots,d\gamma(\overline{Z_k})\right) \\ &= \rho(g)^{-1}\rho(\gamma)^{-1}(\gamma^*\overline{\omega}_{p(g)})(\overline{Z_1},\dots,\overline{Z_k}) \\ &= \rho(g)^{-1}\overline{\omega}_{p(g)}(\overline{Z_1},\dots,\overline{Z_k})\end{aligned}$$

ii) This is left to the reader.

Being invariant by Γ allows to view these forms as k-forms on $\Gamma\backslash G$ with values in V satisfying *ii)* and *iii)*. Now if we call $\Lambda^k(\Gamma,\widetilde{X},p)$ the ρ-equivariant k-forms on $\widetilde{X}$ with values in V, then one has the following diagram

$$\begin{array}{ccc}\Lambda^k(\Gamma,\widetilde{X},\rho) & \xrightarrow{d^D} & \Lambda^{k+1}(\Gamma,\widetilde{X},\rho) \\ \wr\| & & \wr\| \\ \Lambda^k(\Gamma,G,K,\rho) & \xrightarrow{d_\rho} & \Lambda^{k+1}(\Gamma,G,K,\rho)\end{array}$$

This gives a definition for a differential operator d_ρ. The isomorphisms $\Lambda^k(\Gamma,\widetilde{X},\rho) \simeq \Lambda^k(\Gamma,G,K,\rho)$ are easy to understand.

Proposition 6.3. *For $\omega \in \Lambda^k(\Gamma,G,K,\rho)$ and $Z_1,\dots,Z_{k+1} \in \mathfrak{g}$*

$$\begin{aligned}d_\rho\omega(Z_1,\dots,Z_{k+1}) &= \sum_{s=1}^{k}(-1)^{s+1}(Z_s+\rho(Z_s))\omega(Z_1,\dots,\widehat{Z}_s,\dots,Z_{k+1}) \\ &+ \sum_{s<t}(-1)^{s+t}\omega\left([Z_s,Z_t],Z_1,\dots,\widehat{Z}_s,\dots,\widehat{Z}_t,\dots,Z_{k+1}\right)\end{aligned}$$

Remark 6.4. $Z_s + \rho(Z_s)$ consists of a derivation by Z_s and an operator $\rho(Z_s) \in \mathrm{GL}(V)$ applied to the vector $\omega(Z_1, \ldots, \widehat{Z}_s, \ldots, Z_{k+1}) \in V$.

Sketch of proof. With the previous notations

$$(d_\rho \omega)_g = \rho(g)^{-1}(p^*(d\overline{\omega}))_g \ .$$

When one differentiates expression of the type $\rho(g)^{-1}p^*(\eta)$ one differentiates $p^*(\eta)$ but also $\rho^{-1}(g)$; In fact

$$d\omega = d(\rho^{-1}) \wedge p^*(\overline{\omega}) + \rho^{-1} d(p^*\overline{\omega})$$

where here $d\omega$ denotes the differential on G. This condensed notation means that the $\mathrm{GL}(V)$ valued 1-form $d(\rho^{-1})$ is paired with the V-valued form $p^*(\overline{\omega})$ by exterior product on the "form" part and by the action of $\mathrm{GL}(V)$ on V for the "vector" part. Thus

$$d_\rho \omega = d\omega - d(\rho^{-1}) \wedge p^*(\overline{\omega})$$

since $p^*(d\omega) = d(p^*\omega)$. It remains to compute $d_g \rho^{-1}(Y)$ for $Y \in \mathfrak{g}$. Let φ_t the one-parameter group generated by Y, then

$$\frac{d}{dt}\Big|_{t=0} \rho^{-1}(\varphi_t) = \frac{d}{dt}\Big|_{t=0} \left(\rho^{-1}(g\varphi_t)\right) \rho^{-1}(g) = -\rho(Y)\rho^{-1}(g) \ .$$

We thus have

$$d_\rho \omega = d\omega + \rho(\cdot) \wedge \omega \ .$$

Let us apply these remarks to the case of a 1-form ω. Let $Z_1, Z_2 \in \mathfrak{g}$

$$\begin{aligned} d_\rho \omega(Z_1, Z_2) &= Z_1 \cdot \omega(Z_2) - Z_2 \cdot \omega(Z_1) - \omega\left([Z_1, Z_2]\right) \\ &\quad + \rho(Z_1)\omega(Z_2) - \rho(Z_2)\omega(Z_1) \\ &= (Z_1 + \rho(Z_1))\omega(Z_2) - \left(Z_2 + \rho(Z_2)\right)\omega(Z_1) - \omega([Z_1, Z_2]) \end{aligned}$$

which is the desired formula. The other cases are left to the reader.

Remark 6.5. It is worth noticing that we never used the semi-simplicity of G. We just need at this stage G to be a connected Lie group, K to be a closed compact subgroup such that G/K is contractible and Γ a uniform lattice. These computations can be used to show that the cohomology $H^k(\Gamma, \rho)$ when G is solvable or nilpotent can be computed in terms of a cohomology of the Lie algebra $\mathfrak{g}$ of G.

7 Applications II: Semi-simple groups

In order to simplify the exposition we shall restrict ourselves to the case when G is a simple group. For the general case the reader is referred to [Rag].

We recall that we study k-forms on the compact space $\Gamma \setminus G$ with values in a finite dimensional vector space V ans satisfying the two conditions

a) $\forall k \in K$, $R_k^* \omega = \rho(k)^{-1} \omega$.

b) $\forall Y \in \mathfrak{k}$, $i(Y)\omega = 0$.

We shall need to integrate on $\Gamma \setminus G$, so we choose a bi-invariant measure μ (Haar measure) on G, which exists by uni-modularity of semi-simple groups. Now the Lie algebra of G decomposes

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$$

with the property $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ (see P. Paradan's lectures).

The forms are multiplied by $\rho(k)^{-1}$ when we right-translate them by $k \in K$. So in order to have a metric structure on V which descends to a Euclidean structure on V-valued forms on $\widetilde{X}$ we need to choose it invariant by $\rho(K)$. In fact one can do better.

Lemme 7.1. *Let ρ be a finite dimensional representation of G on a vector space V (over $\mathbb{R}$). Then there exists a Euclidean scalar product $(\cdot,\cdot)_V$ on V with respect to which*

i) $\rho(k)$ *is orthogonal,* $\forall k \in K$.

ii) $\rho(Y)$ *is symmetric,* $\forall Y \in \mathfrak{p}$

Exercise: Describe the metric structure g_x on the fiber $E(\rho)_x$ of $E(\rho) \to X$, for $x \in X$.

For $Z \in \mathfrak{g}$ we write $Z = Y + V$ with $Y \in \mathfrak{p}$ and $V \in \mathfrak{k}$, then a k-form as above is completely determined by its values on $\mathfrak{p}$ tanks to *b)*. Let us choose an orthonormal basis of $\mathfrak{p}$ denoted $\{Y_1, \ldots, Y_n\}$ where $n = \dim \widetilde{X} = \dim \mathfrak{p}$.

A form $\omega \in \Lambda^k(\Gamma, G, K, \rho)$ is completely determined by the functions,

$$\omega_{i_1,\ldots,i_k} = \omega(Y_{i_1}, \ldots, Y_{i_k}) \ ,$$

and the global scalar product of two k-forms ξ and η is given by

$$\langle \xi, \eta \rangle = \sum_{i_1 < \cdots < i_k} \int_{G/\Gamma} (\xi_{i_1 \cdots i_k}, \eta_{i_1 \cdots i_k})_V \ d\mu \ .$$

The following equalities are then consequences of the invariance of μ and Stoke's formula (precisely the divergence formula). For $Z \in \mathfrak{g}$ and f, f_1 and f_2 smooth functions on $\Gamma \setminus G$, one has

$$\int_{G/\Gamma} Z \cdot f \ d\mu = 0 \ \text{ and } \ \int_{G/\Gamma} (Z \cdot f_1) f_2 \ d\mu = - \int_{G/\Gamma} f_1 (Z \cdot f_2) \ d\mu \ .$$

These will be used to do the necessary integration by parts in order to compute the adjoint of d_ρ.

Proposition 7.2. *Let $\omega \in \Lambda^k(\Gamma, G, K, \rho)$ then, if $k > 0$,*

$$(\delta_\rho \omega)_{i_1 \cdots i_{k-1}} = - \sum_{s=1}^{n} (Y_s - \rho(Y_s)) \omega_{s i_1 \cdots i_{k-1}}$$

and $\delta_\rho \omega = 0$, if $k = 0$.

Sketch of proof.
$-Y_s$ is the adjoint of Y_s and $\rho(Y_s)$ is self-adjoint (symmetric) by our choice of $(\cdot, \cdot)_V$.

Let us look at two easy cases

1) If ω is a 1-form, $\delta_\rho \omega$ is then a V-valued function. Let η be a V-valued function, then

$$\begin{aligned} \langle \delta_\rho, \eta \rangle &= \int_{\Gamma \setminus G} \left(- \sum_{s=1}^{n} (Y_s - \rho(Y_s)) \omega(Y_s), \eta \right)_V \ d\mu \\ &= \int_{\Gamma \setminus G} \sum_{s=1}^{n} (\omega(Y_s), Y_s \eta)_V + (\omega(Y_s), \rho(Y_s) \eta)_V \ d\mu \\ &= \langle \omega, d_\rho \eta \rangle \end{aligned}$$

which is the desired formula.

2) If ω is a 2-form, $\delta_\rho\omega$ is a 1-form. Let η be another 1-form

$$\begin{aligned}
\langle \delta_\rho, \eta\rangle &= \int_{\Gamma\diagdown G} -\sum_{i=1}^{n}\sum_{s=1}^{n} ((Y_s-\rho(Y_s))\omega(Y_s,Y_i), \eta(Y_i))_V \;\; d\mu \\
&= \int_{\Gamma\diagdown G} \sum_{s<i} (\omega(Y_s,Y_i), (Y_s+\rho(Y_s))\eta(Y_i) - (Y_i+\rho(Y_i))\eta(Y_s))_V \;\; d\mu \\
&= \int_{\Gamma\diagdown G} (\omega, d_\rho\eta)_V \;\; d\mu = \langle \omega, d_\rho\eta\rangle \; .
\end{aligned}$$

In this proof we used the following fact: If $Z_i = Y_i + V_i$ where $Y_i \in \mathfrak{p}$ and $V_i \in \mathfrak{k}$, then $d_\rho\omega(Z_1,\dots,Z_k) = d_\rho\omega(Y_1,\dots,Y_k)$ for $\omega \in \Lambda^{k-1}(\Gamma, G, K, \rho)$. Indeed in the computation of $d_\rho\omega$ appears terms of the form

$$\omega\left([Y_s,Y_t], Y_1,\dots,\widehat{Y}_s,\dots,\widehat{Y}_t,\dots,Y_k\right)$$

which vanish since $[Y_s,Y_t] \in \mathfrak{k}$.

We can then compute the associated Laplacian $\Delta_\rho = d_\rho\delta_\rho + \delta_\rho d_\rho$.

Proposition 7.3. *For* $\omega \in \Lambda^k(\Gamma, G, K, \rho)$

$$\begin{aligned}
(\Delta_\rho\omega)(Y_{i_1},\dots,Y_{i_k}) = &\sum_{j=1}^{n} \left(-Y_j^2 + \rho(Y_j)^2\right)\omega_{i_1\cdots i_k} \\
&+ \sum_{j=1}^{n}\sum_{s=1}^{k} (-1)^{s+1}\left(-[Y_{i_s},Y_j] + \rho([Y_{i_s},Y_j])\right)\omega_{ji_1\cdots\hat{i}_s\cdots i_p} \; .
\end{aligned}$$

Proof. It is a straightforward computation. Let us look at the elementary case $k=0$ where $\Delta_\rho\omega = \delta_\rho d_\rho\omega$.

$$\begin{aligned}
\Delta_\rho\omega &= -\sum_{j=1}^{n} (Y_j - \rho(Y_j))\, d_\rho\omega(Y_j) \\
&= -\sum_{j=1}^{n} (Y_j - \rho(Y_j))\,(Y_j + \rho(Y_j))\omega \; .
\end{aligned}$$

Here there is only one term in the sum defining d_ρ since ω is a V-valued function on $\Gamma \diagdown G$.

Let us recall that $\rho(Y_j)$ is a matrix which is constant, *i.e.* it does not depend on $g \in G$, thus

$$Y_j\cdot \rho(Y_j)\omega = \rho(Y_j)(Y_j\cdot \omega)$$

and

$$\Delta_\rho \omega = -\sum_{j=1}^{n} Y_j^2 \cdot \omega + \sum_{j=1}^{n} \rho(Y_j)^2 \omega \ .$$

In the other degrees, expressions of the form

$$(Y_j + \rho(Y_j))(Y_s - \rho(Y_s))$$

appear and yield the commutators in the above formula. □

Now, Δ_ρ can be decomposed in the sum of two operators

$$\Delta_\rho = \Delta_D + H_\rho$$

where

$$(\Delta_D \omega)_{i_1 \cdots i_p} = \sum_{j=1}^{n} -Y_j^2 \cdot \omega_{i_1 \cdots i_k} - \sum_{j=1}^{n} \sum_{s=1}^{k} (-1)^{s+1} [Y_{i_s}, Y_j] \omega_{j i_1 \cdots \hat{i}_s \cdots i_k}$$

and

$(H_p \omega)_{i_1 \cdots i_p}$ = same formula with derivatives by Y replaced by the operator $\rho(Y)$ for $Y \in \mathfrak{p}$.

Even better, there exist two operators D_ρ and T_ρ such that

$$\Delta_D = D_\rho^* D_\rho + D_\rho D_\rho^* \ \text{ and } \ H_\rho = T_\rho^* T_\rho + T_\rho T_\rho^*$$

where D_ρ^* is the adjoint of the differential operator D_ρ, *i.e.*

$$\langle D_\rho \xi, \eta \rangle = \langle \xi, D_\rho^* \eta \rangle \ \text{ for } \xi \in \Lambda^k \text{ and } \eta \in \Lambda^{k+1}$$

and T_ρ^* is the pointwise adjoint of T_ρ, *i.e.*

$$(T_\rho \xi, \eta)_V = (\xi, T_\rho^* \eta)_V \ .$$

More precisely

$$\begin{aligned}
(D_\rho \xi)_{i_1 \cdots i_{k+1}} &= \sum_{s=1}^{k+1} (-1)^{s+1} Y_{i_s} \cdot \xi_{i_1 \cdots \hat{i}_s \cdots i_{k+1}} \\
(D_\rho^* \xi)_{i_1 \cdots i_{k-1}} &= \sum_{s=1}^{n} (-Y_s) \xi_{s i_1 \cdots i_{k-1}} \\
(T_\rho \xi)_{i_1 \cdots i_{k+1}} &= \sum_{s=1}^{k+1} (-1)^{s+1} \rho(Y_{i_s}) \xi_{i_1 \cdots \hat{i}_s \cdots i_{k+1}} \\
(T_\rho^* \xi)_{i_1 \cdots i_{k-1}} &= \sum_{s=1}^{n} \rho(Y_s) \xi_{s i_1 \cdots i_{k-1}}.
\end{aligned}$$

These remarks lead to

$$\langle \Delta_p \omega, \omega \rangle = \langle \Delta_D \omega, \omega \rangle + \langle H_\rho \omega, \omega \rangle$$

for $\omega \in \Lambda^k(\Gamma, G, K, \rho)$, and both operators Δ_D and H_ρ are non negative. Let us define

$$\overline{Q}_p^k(\omega) = \langle H_\rho \omega, \omega \rangle$$

which is a quadratic form on $\Lambda^k(\Gamma, G, K, \rho)$. We aim at showing that it is positive and bounded below.

It is worth noticing that

Exercise: Show that H_ρ does not depend on the choice of the basis of $\mathfrak{p}$.

The k-forms ω of $\Lambda^k(\Gamma, G, K, \pi)$ are horizontal with respect to the fibration $G \to \widetilde{X}$ thanks to the property *b)* above.

Let us point out the following two facts

i) H_ρ is a constant operator. Indeed it does not depend on $g \in G$ since it involves expression of the type $\rho(Y)$ for $Y \in \mathfrak{p}$.

ii) The above remark says that a k-form $\omega \in \Lambda^k(\Gamma, G, K, \rho)$ can be viewed as a k-linear map, called ω again

$$\omega : \Lambda^k \mathfrak{p} \longrightarrow V$$

so we can consider ω as being an element of $\mathrm{Hom}(\Lambda^k \mathfrak{p}, V)$.

Let us now define

$$Q_\rho^1(\omega) = (H_\rho^1(\omega), \omega)$$

for $\omega \in \mathrm{Hom}(\mathfrak{p}, V)$.

Proposition 7.4. *If Q_ρ^1 is positive definite on $\mathrm{Hom}(\mathfrak{p}, V)$ then there exists $c > 0$ such that for all $\xi \in \Lambda^1(\Gamma, G, K, \rho)$ one has $\langle \Delta_\rho \xi, \xi \rangle \geq c \|\xi\|^2$.*

Corollaire 7.5. *$H^1(\Gamma, \rho) = 0$.*

We now specify the situation to the case under consideration; precisely, $\rho = \mathrm{Ad}$ and $V = \mathfrak{g}$. The canonical metric on $\mathfrak{g}$ satisfies the properties required, namely

i) $\mathrm{Ad}(k)$ is orthogonal, $\forall k \in K$.

ii) $\mathrm{ad}(Y)$ is symmetric for all $Y \in \mathfrak{p}$.

Notice that the extension of Ad to the Lie algebra is ad.

We thus have to study Q^1_{Ad} on the space

$$\mathrm{Hom}(\mathfrak{p},\mathfrak{g}) \simeq \mathrm{Hom}(\mathfrak{p},\mathfrak{k}) \oplus \mathrm{Hom}(\mathfrak{p},\mathfrak{g}) \ .$$

If $\omega \in \mathrm{Hom}(\mathfrak{p},\mathfrak{g})$, then we compute

$$Q^1_{\mathrm{Ad}}(\omega) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \mathrm{ad}(Y_j)^2 \omega(Y_i) + \sum_{j=1}^{n} \mathrm{ad}([Y_i, Y_j])\omega(Y_j), \omega(T_i) \right)_{\mathfrak{g}} \ .$$

Now the subspaces $\mathfrak{p}$ and $\mathfrak{k}$ of $\mathfrak{g}$ are $\mathrm{ad}(\mathfrak{k})$-invariant; therefore if we write $\omega = \omega_{\mathfrak{p}} + \omega_{\mathfrak{k}}$, $\omega_{\mathfrak{p}} \in \mathrm{Hom}(\mathfrak{p},\mathfrak{p})$ and $\omega_{\mathfrak{k}} \in \mathrm{Hom}(\mathfrak{p},\mathfrak{k})$, they satisfy the same conditions, namely

i) $\forall X \in \mathfrak{k}$, $0 = \omega(X) = i_X\omega = \omega_{\mathfrak{p}}(X) + \omega_{\mathfrak{k}}(X)$ implies that $\omega_{\mathfrak{p}}(X) = 0$ and $\omega_{\mathfrak{k}}(X) = 0$.

ii) $R_k^*\omega = \underbrace{\mathrm{ad}(k^{-1})\omega_{\mathfrak{p}}}_{\in\mathfrak{p}} + \underbrace{\mathrm{ad}(k^{-1})\omega_{\mathfrak{k}}}_{\in\mathfrak{k}}$ for all $k \in K$.

Furthermore, let us recall that

$$Q^1_{\mathrm{Ad}}(\omega) = \left(H^1_{\mathrm{Ad}}\omega, \omega\right) .$$

Lemme 7.6. *H^1_{Ad} leaves $\mathrm{Hom}(\mathfrak{p},\mathfrak{p})$ and $\mathrm{Hom}(\mathfrak{p},\mathfrak{k})$ stable and moreover these two sub-spaces are orthogonal for the canonical scalar product on $\mathrm{Hom}(\mathfrak{p},\mathfrak{g})$.*

Proof.

i) Let $\omega \in \mathrm{Hom}(\mathfrak{p},\mathfrak{p})$, then

$$H^1_{\mathrm{Ad}}\omega(Y_i) = \sum_{j=1}^{n} \mathrm{ad}(Y_j)^2\omega(Y_i) + \sum_{j=1}^{n} \mathrm{ad}([Y_i, Y_j])\omega(Y_j)$$

is in $\mathfrak{p}$ for all Y_i.

ii) Similarly $H^1_{\mathrm{Ad}}\omega \in \mathrm{Hom}(\mathfrak{p},\mathfrak{k})$ when $\omega \in \mathrm{Hom}(\mathfrak{p},\mathfrak{k})$.

iii) If $\zeta \in \mathrm{Hom}(\mathfrak{p},\mathfrak{p})$ and $\beta \in \mathrm{Hom}(\mathfrak{p},\mathfrak{k})$ then

$$(\alpha, \beta) = \sum_{i=1}^{n} \big(\alpha(Y_i), \beta(Y_j)\big)_{\mathfrak{g}} = 0$$

since $\mathfrak{p}$ and $\mathfrak{k}$ are orthogonal.

Consequently it suffices to study Q^1_{Ad} on each space.

Remark 7.7. We shall leave aside the case $\mathrm{Hom}(\mathfrak{p},\mathfrak{k})$ which does not contain any information. Furthermore Q^1_{Ad} is positive on this space without any restriction on the group (see [Rag]). Let us consider $\mathrm{Hom}(\mathfrak{p},\mathfrak{p})$. Any ω in this space can be decomposed in

$$\omega = \omega_S + \omega_A$$

where ω_S is the symmetric part of the endomorphism ω and ω_A the skew-symmetric part.

Lemme 7.8. *This decomposition is again stable by H^1_{Ad} and the two subspaces are orthogonal.*

It is thus sufficient to study the two cases separately. Again we shall leave aside the case when ω is skew-symmetric since Q^1_{Ad} is positive on this space without any restriction on G (see [Rag]).

Let $\omega \in \mathrm{Hom}(\mathfrak{p},\mathfrak{p})$ be a symmetric endomorphism. We want to show that if $(H^1_{\mathrm{Ad}}\omega,\omega) = 0$ then $\omega = 0$. Let us recall that

$$H^1_{\mathrm{Ad}} = T^1_{\mathrm{Ad}}(T^1_{\mathrm{Ad}})^* + (T^1_{\mathrm{Ad}})^* T^1_{\mathrm{Ad}} \ .$$

Thus if $(H^1_{\mathrm{Ad}}\omega,\omega) = 0$ then $T^1_{\mathrm{Ad}}\omega = 0$. Here

$$T^1_{\mathrm{Ad}}\omega(Y_i,Y_j) = \mathrm{ad}(Y_i)\omega(Y_j) - \mathrm{ad}(Y_j)\omega(Y_i) \ .$$

Since ω is symmetric, one can choose the basis to be constituted of eigenvectors for ω, *i.e.*

$$\omega(Y_i) = \lambda_i Y_i \ .$$

Now $T^1_{\mathrm{Ad}}\omega = 0$, implies that for all i and j

$$\lambda_j[Y_i,Y_j] = \lambda_i[Y_j,Y_i]$$

showing that, if $[Y_i,Y_j] \neq 0$, then $\lambda_i = -\lambda_j$. We shall use the following lemma

Lemme 7.9. *Let $\mathfrak{g}$ be a non compact simple Lie algebra and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition. Let $\{Y_1,\dots,Y_n\}$ be a basis of $\mathfrak{p}$. Then for all $0 \le r \le n$ there exists $r_1,\dots,r_k$, with $1 \le r_i \le n$ such that $r_1 = 1$, $r_k = r$ and $[Y_{r_i},Y_{r_{i+1}}] \neq 0$ for all $1 \le i \le k-1$.*

It says that one can "reach" any Y_r by a sequence of Y_i's such that two successive Y_i's do not commute. Here the simplicity of G is essential.

This lemma implies that, for all $1 \leq i \leq n$,

$$\lambda_i = \pm\lambda_1.$$

Let us study these eigenvalues and the corresponding eigenspaces. We let

$$E = \{Y \in \mathfrak{p} \mid \omega(Y) = \lambda_1 Y\}$$
$$F = \{Y \in \mathfrak{p} \mid \omega(Y) = -\lambda_1 Y\}$$

Then, since ω is symmetric

$$\mathfrak{p} = E \overset{\perp}{\bigoplus} F\ .$$

E is not reduced to $\{0\}$ since it contains at least Y_1, and F is not trivial either thanks to the above lemma.

Now for $Y, Y' \in E$ (resp. F) one has $[Y, Y'] = 0$ otherwise the eigenvalues would have different signs.

If $Z \in \mathfrak{k}$, by symmetry of $\mathrm{ad}(Y)$ we get that for all $Y, Y' \in E$ (resp. F)

$$\langle [Z, Y], Y' \rangle = -\langle Z, [Y, Y'] \rangle = 0$$

thus $\mathrm{ad}(Z)(E) \subset E^{\perp} = F$ (resp. $\mathrm{ad}(Z)(F) \subset E$). The endomorphism $\mathrm{ad}(Z)\,\mathrm{ad}(Z')$ then preserves E and F for $Z, Z' \in \mathfrak{k}$. We conclude that

$$\mathrm{ad}([Z, Z'])(E) \subset \begin{cases} F & \text{since} [Z, Z'] \in \mathfrak{k} \\ E & \text{because of the above remark.} \end{cases}$$

which leads to

$$\mathrm{ad}([Z, Z'])(E) = \mathrm{ad}([Z, Z'])(F) = 0 \Longrightarrow \mathrm{ad}([Z, Z'])_{|\mathfrak{p}} = 0$$

that is the action of $\mathrm{ad}(\mathfrak{k})$ on $\mathfrak{p}$ is abelian. On the other hand since G is simple, this action is irreducible and faithful. Thus $\mathfrak{k}$ is abelian by the faithfulness and $\dim \mathfrak{p} = 1$ or 2 by the irreducibility. Moreover this action is skew-symmetric (*i.e.* $\mathrm{ad}(Z)$) is skew-symmetric for all $Z \in \mathfrak{k}$). Thus $\dim \mathfrak{p} = 2$ that is, in a suitable basis, for all $Z \in \mathfrak{k}$

$$\mathrm{ad}(Z) = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\ ,\quad a \in \mathbb{R}\ ,\quad a > 0\ .$$

The vector space $\{\mathrm{ad}(Z) \mid Z \in \mathfrak{k}\}$ is then 1-dimensional and again by faithfulness, $\dim \mathfrak{k} = 1$. The Lie algebra $\mathfrak{g}$ is described as follows: Let Z be a generator of $\mathfrak{k}$ and Y_1 and Y_2 a basis of $\mathfrak{p}$ in which $\mathrm{ad}(Z)$ is written as above, then one has

$$[X, Y_1] = -aY_2 \ , \quad [X, Y_2] = aY_1 \quad \text{and} \quad [Y_1, Y_2] = bX \ .$$

Here $b \neq 0$ can be chosen positive ($\mathfrak{p}$ generates $\mathfrak{g}$ as a Lie algebra). Now by setting $X' = \frac{1}{a}X$ and $Y_i' = \frac{1}{\sqrt{ab}}Y_i$ we get the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$!

Thus for $\omega \neq 0$, $T^1_{\mathrm{Ad}}(\omega) = 0$ if and only if $\mathfrak{g} = s\ell(2, \mathbb{R})$ and ω is a symmetric endomorphism with zero-trace (it has opposite eigenvalues). It is not difficult to check that indeed in the case of $s\ell(2, \mathbb{R})$ such a ω gives $Q^1_{\mathrm{ad}}(\omega) = 0$. The dimension of the kernel of H^1_{Ad} is 1 in this case.

8 Extensions of this technique

This technique can be extended in various ways:

1. One can consider the non uniform case, then the Hodge theory works if one knows the existence of L^2-harmonic forms. This relies on the study of the structure of the cusps. Here one more case has to be excluded, this is the case when $G = \mathrm{SL}_2(\mathbb{C})$, see [Ga].

2. This has been applied to the study of the rigidity of some hyperbolic metrics with conical singularities on a 3-manifold (see [K-H]). Here the problem reduces to the study of the Hodge theory with boundary.

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