Five lectures on lattices in semisimple Lie groups

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Introduction

This text is an introduction in five independant lectures to lattices in semisimple Lie groups given during the first week of the 2004 Summer School at the Fourier Institute in Grenoble. We hope that it will attract young students to this topic and convince them to read some of the many textbooks in this field quoted in the bibibliography. We illustrate five important methods of this subject: geometry, arithmetic, representations, boundaries and local fields. One for each lecture.

A lattice Γ in a real semisimple Lie group G is a discrete subgroup for which the quotient G/Γ supports a G-invariant measure of finite volume. One says that Γ is cocompact if this quotient is compact. We will suppose often that the Lie algebra $\mathfrak{g}$ is semisimple. This is the case for $\mathfrak{g} = \mathfrak{sl}(d, \mathbb{R})$ or $\mathfrak{g} = \mathfrak{so}(p, q)$. The two main sources of lattices are

- the *geometric method*: One constructs a periodic tiling of the corresponding symmetric space $X = G/K$, where K is a maximal compact subgroup of G, with one tile P either compact or of finite volume. The group of isometries of this tiling is then the required lattice. This very intuitive method, initiated by Poincaré, seems to work only in low dimension: even if one knows by theorical arguments that it does exist, the explicit description of such a tile P in any dimension is still a difficult question. The aim of the first lecture is to construct some of them for $G = \mathrm{SO}(p, 1)$, and $p \leq 9$.

- the *arithmetic method*: One think of G (or better some product of G by a compact group) as being a group of real matrices defined by polynomial equations with integral coefficients. The subgroup Γ of matrices with integral entries are then a lattice in G. This fact due to Borel and Harish-Chandra implies that G always contains a cocompact and a noncocompact lattice. The aim of the second lecture is to construct some of them for the groups $G = \mathrm{SL}(d, \mathbb{R})$ and $G = \mathrm{SO}(p, q)$.

According to theorems of Margulis and Gromov-Schoen, if $\mathfrak{g}$ is simple and not $\mathfrak{so}(p, 1)$ or $\mathfrak{su}(p, 1)$, then all lattices in G can be constructed by the arithmetic method. When $\mathfrak{g} = \mathfrak{so}(p, 1)$ or $\mathfrak{su}(p, 1)$, quite a few other methods have been developped in order to construct new lattices. Even though we will not discuss them here, let us quote:

$\star$ For $G = \mathrm{SO}(p, 1)$.
- $p = 2$. Gluing trousers (Fenchel-Nielsen); uniformization (Poincaré).
- $p = 3$. Gluing ideal tetrahedra and Dehn chirurgy (Thurston).
- All p. Hybridation of arithmetic groups (Gromov, Piatetski-Shapiro).

$\star$ For $G = \mathrm{SU}(p, 1)$.
- $p = 2$. Groups generated by pseudoreflections (Mostow); Fundamental group of algebraic surfaces (Yau, Mumford).
- $p \leq 3$. Moduli spaces of weighted points on the line; Holonomy groups of local systems (Deligne, Mostow, Thurston).
- All p. Unknown yet.

One of the main success of the theory of lattices is that it gave in a unified way many new properties of arithmetic groups. One does not use the way Γ has been constructed but just the existence of the finite invariant measure. One key tool is the theory of unitary representations and more precisely the asymptotic behavior of coefficients of vectors in unitary representations. We will explain it in the third lecture.

Another important tool are the boundaries associated to Γ. We will see in the fourth lecture how they are used in the proof of Kazhdan-Margulis normal subgroup theorem which says that: *lattices in real simple Lie groups of real rank at least* 2 *are quasisimple*, i.e. their normal subgroups are either finite or of finite index.

The general theory we described so far gives informations on arithmetic groups like $\mathrm{SL}(d, \mathbb{Z})$, $\mathrm{SO}(d, \mathbb{Z}[i])$ or $\mathrm{Sp}(d, \mathbb{Z}[\sqrt{2}])$. It can be extended to S-arithmetic groups like $\mathrm{SL}(d, \mathbb{Z}[i/N])$, $\mathrm{SO}(d, \mathbb{Z}[1/N])$ or $\mathrm{SU}(p, q, \mathbb{Z}[\sqrt{2}/N])$... The only thing one has to do is to replace the real Lie group G by a product of real and p-adic groups. The aim of the last lecture is to explain how to adapt the results of the previous lectures to that setting. For instance we will construct cocompact lattices in $\mathrm{SL}(d, \mathbb{Q}_p)$ and see that they are quasisimple for $d \geq 3$.

This text is slightly longer than the oral lecture parce qu'au tableau il est plus facile de remplacer une démonstration technique par un magnifique crobard, un principe général, un exemple insignifiant, un exercice intordable voire une grimace évocatrice. One for each lecture. Nevertheless, there are still many important classical themes in this subject which will not be discussed here, let us just quote a few: cohomological dimension and cohomology, universal extension and the congruence subgroup property, rigidity and superigidity, counting points and equirepartition, shimura varieties, quasiisometries...

Un grand merci aux auditeurs de l'Ecole d'été qui par leurs critiques m'ont permis d'améliorer ce texte.

`For an undergraduate introduction to tilings and lattices, one can read [2].`

Lecture 1: Coxeter groups

In the first lecture, we construct a few lattices in $SO(p,1)$ by the geometric method, when $p \leq 9$.

1.1 Introduction

The geometric method of construction of lattices has been initiated by Poincare in 1880. In his construction, the group G is the group $PO^+(2,1)$ of isometries of the hyperbolic plane $\mathbb{H}^2$. One begins with a polygon $P \subset \mathbb{H}^2$ and with a family of isometries which identify 2 by 2 the edges of P. When these isometries satisfy some compability conditions saying that "the first images of P give a tiling around each vertex", the Poincaré theorem says that the group Γ generated by these isometries acts properly on $\mathbb{H}^2$ with P as a fundamental domain. In particular, when P has finite volume, this group Γ is a lattice in G.

There exists a higher dimensional extension of Poincaré's theorem. One replaces $\mathbb{H}^2$ by the d-dimensional hyperbolic space $\mathbb{H}^d$, the polygon P by a polyhedron, the edges by the $(d-1)$-faces and the vertices by the $(d-2)$-faces (see [16]). In most of the explicitly known examples of such a polyhedron P, one chooses Γ to be generated by the symetries with respect to the $(d-1)$-faces of P. The aim of this lecture is to present a proof due to Vinberg of this extension of Poincaré's theorem in this case and to describe some of these explicit polyhedra for $d \leq 9$. In this case, the group Γ is a Coxeter group. As a by-product, we will obtain geometric proofs of some of the basic properties of Coxeter groups.

Even if the geometric construction may seem less efficient than the arithmetic one, it is an important tool to begin with.

1.2 Projective transformation

Let us begin by a few basic definitions and properties. Let $V := \mathbb{R}^{d+1}$, $\mathbb{S}^d = \mathbb{S}(V) := (V-0)/\mathbb{R}_+^\times$ be the projective sphere and $\mathrm{SL}^\pm(d+1,\mathbb{R})$ be the group of projective transformations of $\mathbb{S}^d$.

Definition 1.1 *A reflection σ is an element of order 2 of $\mathrm{SL}^\pm(d+1,\mathbb{R})$ which is the identity on an hyperplane. Hence, one has $\sigma = \sigma_{\alpha,v} := Id - \alpha \otimes v$ where $\alpha \in V^*$, $v \in V$ with $\alpha(v) = 2$.*
- A rotation ρ is an element of $\mathrm{SL}^\pm(d+1,\mathbb{R})$ which is the identity on a subspace of codimension 2 and is given by a matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ in a suitable supplementary basis. The real $\theta \in [0,\pi]$ is the angle of the rotation.

Let $\sigma_1 = \sigma_{\alpha_1,v_1}$, $\sigma_2 = \sigma_{\alpha_2,v_2}$ be two distinct reflections, Δ be the group they generate, $a_{12} := \alpha_1(v_2)$, $a_{21} := \alpha_2(v_1)$ and $L := \{x \in \mathbb{S}^d \ / \ \alpha_1|_x \leq 0, \alpha_2|_x \leq 0\}$. The following elementary lemma tells us when do the images $\delta(L)$ for $\delta \in \Delta$ tile a subset C of $\mathbb{S}^d$, i.e. when are the interiors $\delta(\mathring{L})$ disjoints. The set C is then the union $C := \bigcup_{\delta \in \Delta} \delta(L)$.

Lemma 1.2 *a) If $a_{12} > 0$ or $a_{21} > 0$, the $(\delta(L))_{\delta\in\Delta}$ do not tile.*
b) Suppose now $a_{12} \leq 0$ and $a_{21} \leq 0$.
b1) $a_{12}a_{21} = 0$. If both $a_{12} = a_{21} = 0$, then the product $\sigma_1\sigma_2$ is of order 2 and the $(\delta(L))_{\delta\in\Delta}$ tile $\mathbb{S}^d$. If not they do not tile.
b2) $0 < a_{12}a_{21} < 4$. The product $\sigma_1\sigma_2$ is a rotation of angle θ given by $4\cos(\theta/2)^2 = a_{12}a_{21}$. If $\theta = \pi/m$ for some integer $m \geq 3$ then $\sigma_1\sigma_2$ is of order m and the $(\delta(L))_{\delta\in\Delta}$ tile $\mathbb{S}^d$. If not they do not tile.
b3) $a_{12}a_{21} = 4$. The product $\sigma_1\sigma_2$ is unipotent and the $(\delta(L))_{\delta\in\Delta}$ tile a subset C of $\mathbb{S}^d$ whose closure is a half-sphere.
b4) $a_{12}a_{21} > 4$. The product $\sigma_1\sigma_2$ has two distinct positive eigenvalues and the $(\delta(L))_{\delta\in\Delta}$ tile a subset C of $\mathbb{S}^d$ whose closure is the intersection of two distinct half-spheres.

Proof This lemma reduces to a 2-dimensional exercise that is left to the reader. ◇

Remark The cross-ratio $[\alpha_1, \alpha_2, v_1, v_2] := \frac{\alpha_1(v_2)\alpha_2(v_1)}{\alpha_1(v_1)\alpha_2(v_2)} = \frac{a_{12}a_{21}}{4}$ is a projective invariant.

1.3 Coxeter systems

A *Coxeter system* (S, M) is the data of a finite set S and a matrix $M = (m_{s,t})_{s,t\in S}$ with diagonal coefficients $m_{s,s} = 1$ and nondiagonal coefficients $m_{s,t} = m_{t,s} \in \{2, 3, \ldots, \infty\}$. The cardinal r of S is called the *rank* of the Coxeter system. To such a Coxeter system, one associates the corresponding *Coxeter group* $W = W_S$ defined by the set of generators S and the relations $(st)^{m_{s,t}} = 1$, for all $s, t \in S$ such that $m_{s,t} \neq \infty$. For w in W, its length $\ell(w)$ is the smallest integer ℓ such that w is the product of ℓ elements of S.

A Coxeter group has a natural r-dimensional representation σ_S called the *geometric representation* which is defined in the following way. Let $(e_s)_{s\in S}$ be the canonical basis of $\mathbb{R}^S$. The *Tits form* on $\mathbb{R}^S$ is the symmetric bilinear form defined by

$$B_S(e_s, e_t) \;=\; -\cos(\tfrac{\pi}{m_{s,t}}) \;\text{ for all } s, t \in S\,.$$

According to lemma 1.2, the formulas

$$\sigma_S(s)v \;=\; v - 2B_S(e_s, v)e_s \;\forall\; s \in S,\; v \in E_S$$

define a morphism σ_S of W into the orthogonal group of the Tits form. Let P_S be the simplex in the sphere $\mathbb{S}^{r-1}$ of the dual space defined by $P_S := \{f \in \mathbb{S}^{r-1} \;/\; f(e_s) \leq 0 \;/\; \forall s \in S\}$.

As a special case of Vinberg theorem in the next sections, we will see the following theorem due to Tits:

Theorem 1.3 (Tits) *The representation σ_S is faithful with discrete image Γ_S and the translates ${}^t\gamma(P_S)$ for $\gamma \in \Gamma_S$ tile a convex set C_S of the sphere $\mathbb{S}^{r-1}$.*

Remarks - The convex C_S is called *Tits convex set.*
- For a few Coxeter groups, with $r \leq 10$, called *hyperbolic*, we will prove that the Tits form is Lorentzian of signature $(r-1,1)$ and that the group Γ_S is a lattice in the corresponding orthogonal group.

Corollary 1.4 *For all subset $S' \subset S$, the natural morphism $\rho_{S,S'} : W_{S'} \to W_S$ is injective.*

Proof of corollary 1.4 The representation $\sigma_{S'}$ is equal to the restriction of $\sigma_S \circ \rho_{S,S'}$ to the vector space $<e_s, s \in S'>$.

1.4 Groups of projective reflections

We study in this section groups generated by projective reflections fixing the faces of some convex polyhedron P of the sphere $\mathbb{S}^d$

Let $P \subset \mathbb{S}^d$ be a d-dimensional *convex polyhedron*, i.e. the image in $\mathbb{S}^d$ of a convex polyhedral cone of $\mathbb{R}^{d+1}$ with 0 omitted. A *k-face* of P is a k-dimensional convex subset of P obtained as an intersection of P with some hypersphere which does not meet the interior $\overset{\circ}{P}$. A *face* is a $(d-1)$-face and an edge is a 0-face.

Let S be the set of faces of P and for any such face s, one chooses a projective reflection $\sigma_s = Id - \alpha_s \otimes v_s$ with $\alpha_s(v_s) = 2$ which fixes s. A suitable choice of signs allows us to suppose that P is defined by the inequalities $(\alpha_s \leq 0)_{s \in S}$. Let $a_{s,t} := \alpha_s(v_t)$, for $s, t \in S$. Let Γ be the group generated by the reflections σ_s.

According to lemma 1.2, if we want the images $\gamma(P)$ to tile some subset of $\mathbb{S}^d$, the following conditions are necessary: for all faces $s, t \in S$ such that the intersection $s \cap t$ is a $(d-2)$-dimensional face of P, one has

$$a_{s,t} \leq 0 \;\; and \;\; a_{s,t} = 0 \Leftrightarrow a_{t,s} = 0 \tag{1}$$

$$a_{s,t}a_{t,s} \geq 4 \;\; or \;\; a_{s,t}a_{t,s} = 4\cos(\tfrac{\pi}{m_{s,t}}) \;\; with \;\; m_{s,t} \geq 2 \;\; \text{integer} \tag{2}$$

Conversely, the following theorem states that these conditions are also sufficient.

Let (S, M) be the Coxeter system given by these integers $m_{s,t}$ and completed by $m_{s,t} = \infty$ when either $s \cap t = \emptyset$ or $\operatorname{codim}(s \cap t) \neq 2$ or $a_{s,t}a_{t,s} \geq 4$. Note that, when the polyhedron is the simplex P_S of the previous section, then the Coxeter system is the one we started with.

Theorem 1.5 (Vinberg) *Let P be a convex polyhedron of $\mathbb{S}^d$ and, for each face s of P, let $\sigma_s = Id - \alpha_s \otimes v_s$ be a projective reflection fixing this face s. Suppose that they satisfy the conditions (1) and (2) for every s, t such that* $\operatorname{codim}(s \cap t) = 2$. *Let Γ be the group generated by these reflections σ_s. Then*
(a) The polyhedra $\gamma(P)$, for γ in Γ, tile some convex subset C of $\mathbb{S}^d$.
(b) The morphism $\sigma : W_S \to \Gamma$ given by $\sigma(s) = \sigma_s$ is an isomorphism.
(c) The group Γ is discrete.

Otherwise stated, to be sure that a convex polyhedron and its images by a group generated by projective reflections through its faces tile some part of the sphere, it is enough to check local conditions "around each 2-codimensional face".

We will still call C the *Tits convex set.* This convex set may not be open.

Remark The proof of theorem 1.3 given by Tits in [8] can be adapted to get theorem 1.5 (see [23] lemma 1). In this lecture, we will follow Vinberg's proof which is more geometric.

1.5 The universal tiling

> To prove the theorem, one introduces an abstract space X obtained by gluing copies of P indexed by the Coxeter group $W := W_S$ along their faces and proves that this space is convex.

One defines $X := W \times P/_\sim$ where the equivalence relation is defined by

$$(w, p) \sim (w', p') \Longleftrightarrow \exists s \in S \; / \; w' = ws \;\text{ and }\; p' = \sigma_s(p) \; .$$

One denote by $P^{\rm sing}$ the union of the codimension 3 faces of P and $P^{\rm reg} = P - P^{\rm sing}$, $X^{\rm sing} := W \times P^{\rm sing}/_\sim$, $X^{\rm reg} = X - X^{\rm sing}$. The Coxeter group W acts naturally on X and on $\mathbb{S}^d$. Let $\pi : X \to \mathbb{S}^d$ be the map defined by $\pi(w, p) = w\, p$.

Lemma 1.6 *a) For x in P, let $W_x \subset W$ be the subgroup generated by σ_s for $s \ni x$. Then $W_x \times P/_\sim$ is a neighborhood of x in X.*
b) The map π is W-equivariant, i.e. $\forall w \in W$, $\forall x \in X$, $\pi(w\, x) = w\, \pi(x)$.
c) For all x in $X^{\rm reg}$, there exists a neighborhood V_x of x in X such that $\pi|_{V_x}$ is an homeomorphism onto a convex subset of $\mathbb{S}^d$.

Proof a) Let P_x be an open neighborhood of x in P which does not meet the faces of P which do not contain x. Then, $W_x \times P_x/_\sim$ is open in X.
b) is easy.
c) is a consequence of a), b), lemma 1.2 and of the hypotheses (1) and (2). $\diamondsuit$

A segment on $\mathbb{S}^d$ is a 1-dimensional convex subset which is not a circle. Let us transfer this notion of segment to X:

Definition 1.7 *For every x, y in X, a segment $[x, y]$ is a compact subset of X such that the restriction of π to $[x, y]$ is an homeomorphism onto some segment of $\mathbb{S}^d$ with end points $\pi(x)$ and $\pi(y)$.*

One does not know yet wether such a segment does exist. It is what we want to show.

Let us denote by $\partial P = P - \overset{\circ}{P}$ the union of the faces of P and $\partial X := W \times \partial P/_\sim$. The following lemma is the key lemma. For each point z in $P^{\rm reg}$ one defines its multiplicity to be $m(z) = m_{s,t}$ if $z \in s \cap t$ with some $s \neq t$, otherwise $m(z) = \mathbf{1}_{\partial P}(z)$. We extend this function on $X^{\rm reg}$ by the formula $m(w\, z) = m(z)$.

Lemma 1.8 *Fix $w \in W$. Let $S = S_w := \{(x,y) \in \overset{\circ}{P} \times \overset{\circ}{P} \subset X \times X$ such that $\pi(x) \neq -\pi(y)$, the segment $[x, w\,y]$ exists and is contained in $X^{\mathrm{reg}}\}$. Suppose $S \neq \emptyset$. Then*
a) the sum $\sum_{z \in [x,wy]} m(z)$ is a constant $L(w)$ on S depending only on w.
b) The set S is dense in $P \times P$

The above sum counts the number of faces crossed by the segment. We will see later that this number $L(w)$ is equal to the length $\ell(w)$.

Proof Let $L(x,y,w)$ be the above sum. According to the local analysis given in lemma 1.2, when the segment $[x, w\,y]$ crosses the interior of a codimension 2 face $w'(s \cap t)$, one has $m_{s,t} < \infty$. Moreover this local analysis proves that the function $(x,y) \to L(x,y,w)$ is locally constant (this is the main point in this proof, see the remark below). Choose $L \geq 0$, such that the set $S_L := \{(x,y) \in S \;/\; L(x,y,w) = L\}$ is nonempty. One knows that S_L is open in $\overset{\circ}{P} \times \overset{\circ}{P}$. Notice that, for (x,y) in S_L, the only tiles $w'P \subset X$ crossed by the segment $[x, w\,y]$ satisfy $\ell(w') \leq L$, they are among a fixed finite set of tiles. Hence, by a compactness argument, for any (x,y) in the closure $\overline{S}_L$, the segment $[x, w\,y]$ exists and is included in the compact $\bigcup_{\ell(w') \leq L} w'(P)$. Moreover, since P^{sing} is of codimension 3, removing some subset of codimension 2 in S, one can find an open connected and dense subset S' of $\overset{\circ}{P} \times \overset{\circ}{P}$, such that $\overline{S}_L \cap S' \subset S_L$. Hence, successively, $S_L \cap S'$ is open and closed in S', S' is included in S_L, S_L is dense in $\overset{\circ}{P} \times \overset{\circ}{P}$ and $S_L = S$. ♢

The next statement is a corollary of the previous proof.

Lemma 1.9 *For every x, x' in X, there exists at least one segment $[x, x']$ relating them. Moreover, when $\pi(x) \neq -\pi(x')$, this segment is unique.*

Proof Keep the notations of the previous lemma with $x' = w\,y$.
We know that $S_w \neq \emptyset \Longrightarrow \overline{S}_w = P \times P$. This allows to prove by induction on $\ell(w)$ that $S_w \neq \emptyset$, by letting the point y move continuously through a face. The uniqueness follows from the uniqueness of the segment joining two non antipodal points on the sphere $\mathbb{S}^d$. ♢

Lemma 1.10 *The map $\pi : X \to C$ is bijective and C is convex.*

Proof Let x, x' be two points of X. According to lemma 1.9, there is a segment $[x, x']$ joining them. Hence if $\pi(x) = \pi(x')$, one must have $x = x'$. This proves that $\pi : X \to C$ is bijective. Two points of C can also be joined by a segment, hence C is convex. ♢

Proof of theorem 1.5 (a), (b) follow from lemma 1.10 and (c) follows from (a). ♢

Remark Let us point out how crucial the lemma 1.8 is. Consider the following group Γ generated by two linear transformations g_1 and g_2 of $\mathbb{R}^2$, which identify the opposite faces of a convex quadrilateral P.
- g_1 is the homothety of ratio 2,

- g_2 is a rotation whose angle α is irrational with π and
- $P := \{(x,y) \in \mathbb{R}^2 \ / \ 1 \leq x \leq 2 \ \text{ and } \ \left|\frac{y}{x}\right| \leq \tan\frac{\alpha}{2}\}$.

The successive image $\gamma(P)$ draw a kind of irrational spider web which instead of tiling an open set in $\mathbb{S}^2$ tile the universal cover of $\mathbb{R}^2 - \{0\}$. The group Γ is not discrete.

1.6 Cocompactness

The following corollary tells us when the convex set C is open.

Corollary 1.11 *With the same notations as theorem 1.5, the following are equivalent.*
(i) For every x in P, the Coxeter group W_{S_x} is finite, where $S_x := \{s \in S \ / \ x \in s\}$.
(ii) The convex C is open.
In this case, W acts properly on C with compact quotient.

To prove this corollary, we will use the following lemma.

Lemma 1.12 *a) The union of the boundaries of the tiles $\cup_{w\in W} w(\partial P)$ is the interection of C with a family of hyperspheres. This family is locally finite in $\overset{\circ}{C}$.*
b) One has $L(w) = \ell(w)$, for all w in W.
c) For every x in P, W_{S_x} is the stabilizer of x. Moreover, the union U_x of $w(P)$, for $w \in W_{S_x}$, is a neighborhood of x in C.
d) One has the equivalence: $x \in \overset{\circ}{C} \Leftrightarrow \mathrm{card}(W_{S_x}) < \infty$.
e) The group W acts properly on $\overset{\circ}{C}$.

Remark The point *b*) is related to the exchange lemma for Coxeter groups ([8] ch.IV §1).

Proof a) One just has to check that when one walks on a hypersphere containing a face and passes through a codimension 2 face then one is still on a face. But this is a consequence of the local analysis of lemma 1.2.b2.

b) $\ell(w)$ is the minimum number of faces a path from $\overset{\circ}{P}$ to $w(\overset{\circ}{P})$ has to cross. According to a), this minimum is achieved when this path is a segment. Hence $\ell(w) = L(w)$.

c) This a consequence of lemma 1.6 and 1.10.

d) If the union U_x is a neighborhood of x, by local finiteness of the tiling and by compactness of a small sphere centered at x, the index set W_{S_x} mut be finite. Conversely, if W_{S_x} is finite, the intersection of U_x with a small sphere is, by induction, simultaneously open and closed.

e) This is a consequence of c) and d) ♢

Proof of corollary 1.11 Use lemma 1.12 d) and e). ♢

Let q_0 be a quadratic form of signature $(d,1)$ and $\mathbb{H}^d \subset \mathbb{S}^d$ be the corresponding hyperbolic space: it is one of the two connected components of the set $\{x \in \mathbb{S}^d \ / \ q_0|_x < 0\}$.

Corollary 1.13 *With the same notations.*
a) If $\overset{\circ}{P} \subset \mathbb{H}^d$ *and if the symmetries* σ_s *are orthogonal for* q_0*, then* $\overset{\circ}{C} = \mathbb{H}^d$*.*
b) Moreover, if $P \subset \mathbb{H}^d$*, then* Γ *is a cocompact lattice in the orthogonal group* $O(d,1)$

In case *a)* P is called an *hyperbolic Coxeter polyhedron.*

Proof a) By contradiction, let $x_0 \in \overset{\circ}{P}$, $y \in \mathbb{H}^d - \overset{\circ}{C}$ a point minimizing the distance to x_0 and s a face of P crossed by the segment $[x_0, y]$. Then, one has $d(x_0, \sigma_s(y)) < d(x_0, y)$. Contradiction.

b) Note that $C = \mathbb{H}^d$ is open and use corollary 1.11. $\diamondsuit$

1.7 Examples

a) Consider any convex polygon in $\mathbb{H}^2$ whose angles between the edges are equal to π/m for some $m \leq 2$.

Then the group generated by the orthogonal reflections with respect to the faces is a cocompact lattice in $O(2,1)$.

b) Consider a tetrahedron in $\mathbb{H}^3$ whose group of isometry is S_4 and whose vertices are on the boundary of $\mathbb{H}^3$. The angles between the faces are $\pi/3$.

Then the group generated by the orthogonal reflections with respect to the faces is a noncocompact lattice in $O(3,1)$.

c) Consider a dodecahedron in $\mathbb{H}^3$ whose group of isometry is $\mathcal{A}_5$ such that the angles between the faces are $\pi/2$.

Then the group generated by the orthogonal reflections with respect to the faces is a cocompact lattice in $O(3,1)$.

d) Consider a convex polygon in $\mathbb{R}^2$, with sides s_1,..., s_k, $s_{k+1} = s_1$ and denote by $\sigma_i = Id - \alpha_i \otimes v_i$ the projective symmetries such that $s_i \subset Ker(\alpha_i)$ and v_i is on the intersection of the lines $Ker(\alpha_{i-1})$ and $Ker(\alpha_{i+1})$.

Then the group generated by σ_i acts cocompactly on some bounded open convex subset of $\mathbb{R}^2$ whose boundary is in general non C^2. This kind of groups has been introduced first in [14]. See [3] for more information on these examples and their higher dimensional analogs.

e) Consider the convex polyhedron $P_S \subset \mathbb{R}^{r-1}$ associated to the geometrical representation of a Coxeter group W_S given by some Coxeter system (S, M). Consider also the Tits convex set C_S tiled by the images of P_S and the Tits form B_S as in section 1.3.

To each Coxeter system (S, M), one associates its Coxeter diagram. It is a graph whose set of vertices is S and whose edges are weighted by the number $m_{s,t}$, with the convention that an edge is omitted when $m_{s,t} = 2$ and a weight is omitted when $m_{s,t} = 3$. The Coxeter system is said to be *irreducible* if the corresponding graph is connected.

The following proposition gives the list of compact (resp. finite volume) hyperbolic Coxeter simplices.

Proposition 1.14 *Let (S, M) be an irreducible Coxeter system.*
a) One has the equivalences: B_S is positive definite $\iff C_S = \mathbb{S}^{r-1} \iff \mathrm{card}(W_S) < \infty$.
In this case (S, M) is said to be elliptic.
(S, M) is said to be parabolic if B_S is positive.
b) Suppose B_S is Lorentzian then one has the equivalences:
b1) All Coxeter subsystems are elliptic $\iff W_S$ is a cocompact lattice in $O(B_S)$.
b2) All Coxeter subsystems are either elliptic or irreducible parabolic $\iff W_S$ is a lattice in $O(B_S)$.

Proof We will just prove the implications $\Rightarrow$ we need for our examples.

a) and b1) are easy consequences of theorem 1.5 and corollaries 1.11, 1.13.

b2) Use the fact that for $d \geq 2$, for any simplex S with $\overset{\circ}{S} \subset \mathbb{H}^d$, the hyperbolic volume of $\overset{\circ}{S}$ is finite. $\diamondsuit$

The list of Coxeter diagrams satisfying these properties are due to Coxeter in cases a) and b1) and to Lanner in cases b2). They can be found, for instance, in [25] p.202-208. In cases c), there are only finitely many of them with rank $r \leq 5$ in the case c1) and $r \leq 10$ in the case c2). Here are two examples.

The Coxeter diagram given by a pentagone with one edge of weight 4, gives a cocompact lattice in $0(4, 1)$.

The Coxeter diagram E_{10} (which is a segment with 9 points and a last edge starting from the third point of the segment) gives a noncocompact lattice in $0(9, 1)$.

f) The description of all compact (resp. finite volume) hyperbolic Coxeter polyhedra is known only in dimension 2 and 3. The highest dimension of known example is $d = 5$ (resp. $d = 21$) and one knows that one must have $d \leq 29$ (resp. $d \leq 995$).

Lecture 2: Arithmetic groups

The aim of this second lecture is to give explicit constructions of lattices in the real Lie groups $\mathrm{SL}(d, \mathbb{R})$ and $\mathrm{SO}(p, q)$. These examples are instances of a general arithmetic construction, due to Borel and Harish-Chandra, of lattices in any semisimple group G. In fact, according to Margulis, all "irreducible" lattices of G are obtained in this way when the real rank of G is at least 2.

2.1 Examples

Here are a few explicit examples of lattices.

Write $d = p + q$ with $p \geq q \geq 1$. For any commutative ring A, let $\mathrm{SL}(d, A) := \{g \in \mathcal{M}(d, A) \ / \ \det(A) = 1\}$.

Denote by I_d the $d \times d$ identity matrix, $I_{p,q} := \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, $J_{p,q} := \begin{pmatrix} I_p & 0 \\ 0 & -\sqrt{2}\, I_q \end{pmatrix}$ and let $\mathrm{SO}(p, q) = \{g \in \mathrm{SL}(d, \mathbb{R}) / g\, I_{p,q}\, {}^t g = I_{p,q}\}$.

Example 1 *The group* $\Gamma := \mathrm{SL}(d, \mathbb{Z})$ *is a noncocompact lattice in* $\mathrm{SL}(d, \mathbb{R})$.

Example 2 *The group* $\Gamma := \mathrm{SO}(p, q) \cap \mathrm{SL}(d, \mathbb{Z})$ *is a noncocompact lattice in* $\mathrm{SO}(p, q)$.

Example 3 Let σ be the automorphism of order 2 of $\mathbb{Q}[\sqrt{2}]$. *The group* $\Gamma := \{g \in \mathrm{SL}(d, \mathbb{Z}[\sqrt{2}]) \ / \ g\, I_{p,q}\, {}^t g^\sigma = I_{p,q}\}$ *is a noncocompact lattice in* $\mathrm{SL}(d, \mathbb{R})$.

Example 4 Let O be a subring of $\mathcal{M}(d, \mathbb{R})$ which is also a lattice in this real vector space. Suppose that $O \subset \mathrm{GL}(d, \mathbb{R}) \cup \{0\}$. We will see that such a subring does exist for every $d \geq 2$: in fact O is an "order in a central division algebra over $\mathbb{Q}$ such that $D \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathcal{M}(d, \mathbb{R})$". *The group* $\Gamma := O \cap \mathrm{SL}(d, \mathbb{R})$ *is a cocompact lattice in* $\mathrm{SL}(d, \mathbb{R})$.

Example 5 Let σ be the automorphism of order 2 of $\mathbb{Q}[\sqrt{2}]$. *The group* $\Gamma := \{(g, g^\sigma) \ / \ \ g \in \mathrm{SL}(d, \mathbb{Z}[\sqrt{2}])\}$ *is a noncocompact lattice in* $\mathrm{SL}(d, \mathbb{R}) \times \mathrm{SL}(d, \mathbb{R})$.

Example 6 *The group* $\Gamma := \mathrm{SL}(d, \mathbb{Z}[i])$ *is a noncocompact lattice in* $\mathrm{SL}(d, \mathbb{C})$.

Example 7 Let τ be the automorphism of order 2 of $\mathbb{Q}[\sqrt[4]{2}]$. *The group* $\Gamma := \{g \in \mathrm{SL}(d, \mathbb{Z}[\sqrt[4]{2}]) \ / \ g\, J_{p,q}\, {}^t g^\tau = J_{p,q}\}$ *is a cocompact lattice in* $\mathrm{SL}(d, \mathbb{R})$.

Example 8 *The group* $\Gamma := \{g \in \mathrm{SL}(d, \mathbb{Z}[\sqrt{2}]) \ / \ g\, J_{p,q}\, {}^t g = J_{p,q}\}$ *is a cocompact lattice in* $\{g \in \mathrm{SL}(d, \mathbb{R}) \ / \ g\, J_{p,q}\, {}^t g = J_{p,q}\} \simeq \mathrm{SO}(p, q)$.

The aim of this lecture is to give a complete proof for the examples 1, 4, 7 and 8, a sketch for the others, and a short survey for the general theory.

2.2 The space of lattices in $\mathbb{R}^d$

We study in this section the space X of lattices in $\mathbb{R}^d$ and the subset X_1 of lattices of covolume 1 in $\mathbb{R}^d$. As homogeneous spaces, one has $X = \mathrm{GL}(d,\mathbb{R})/\mathrm{SL}^{\pm}(d,\mathbb{Z})$ and $X_1 = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$. We will prove that X_1 has finite volume.

Proposition 2.1 (Minkowski) *The group* $\mathrm{SL}(d,\mathbb{Z})$ *is a lattice in* $\mathrm{SL}(d,\mathbb{R})$.

Let us denote by $g_{i,j}$ the entries of an element g in $G := \mathrm{GL}(d,\mathbb{R})$ and let
$K := O(d) = \{g \in G \;/\; g^{\,t}g = 1\}$,
$A := \{g \in G \;/\; g$ is diagonal with positive entries$\}$,
$A_s := \{a \in A \;/\; a_{i,i} \leq s\, a_{i+1,i+1}\ ,\ \text{for } i = 1,\ldots,d-1\ \}$, with $s \geq 1$,
$N := \{g \in G \;/\; g - 1$ is strictly upper triangular$\}$ and
$N_t := \{n \in N \;/\; |n_{i,j}| \leq t\ ,\ \text{for } 1 \leq i < j \leq d\ \}$, with $t \geq 0$.

According to the Iwasawa decomposition, the multiplication induces a diffeomorphism $K \times A \times N \simeq G$. Let $S_{s,t}$ be the *Siegel domain* $S_{s,t} := KA_sN_t$ and $\Gamma := \mathrm{SL}(d,\mathbb{Z})$.

Lemma 2.2 *For* $s \geq \frac{2}{\sqrt{3}}$, $t \geq \frac{1}{2}$, *one has* $G = S_{s,t}\Gamma$.

Proof Let g be in G and $\Lambda := g(\mathbb{Z}^d)$. One argue by induction on d. A family $(f_1,\ldots,f_d)$ of vectors of Λ is said to be *admissible* if:
- the vector f_1 is of minimal norm in $\Lambda - \{0\}$.
- the images $\dot{f}_2,\ldots,\dot{f}_d$ of $f_2,\ldots,f_d$ in the lattice $\dot{\Gamma} := \Gamma/\mathbb{Z}f_1$ of the Euclidean space $\mathbb{R}^d/\mathbb{R}f_1$ form an admissible family of $\dot{\Gamma}$.
- each f_i with $i \geq 2$ is of minimal norm among the vectors of Γ whose image in $\dot{\Gamma}$ is $\dot{f}_i$.

It is clear that Λ contains an admissible family $(f_1,\ldots,f_d)$ and that such a family is a basis of Λ. After right multiplication of g by some element of Γ, one may suppose that this family is the image of the standard basis $(e_1,\ldots,e_d)$ of $\mathbb{Z}^d$, i.e. for all $i = 1,\ldots,d$, one has $g\,e_i = f_i$.

Let us show that $g \in S_{\frac{2}{\sqrt{3}},\frac{1}{2}}$. Write $g := kan$. Since $(k^{-1}f_1,\ldots,k^{-1}f_d)$ is an admissible basis of $k^{-1}(\Lambda)$, one may suppose that $k = 1$. One has then $g = an$. Hence

$$\begin{aligned}
f_1 &= a_{1,1}e_1\\
f_2 &= a_{2,2}\,e_2 + a_{1,1}n_{1,2}\,e_1\\
&\;\;\cdots\\
f_d &= a_{d,d}\,e_d + a_{d-1,d-1}n_{d-1,d}\,e_{d-1} + \cdots + a_{1,1}n_{1,d}\,e_1.
\end{aligned}$$

By induction hypothesis, one knows that

$$\begin{aligned}
|n_{i,j}| &\leq \frac{1}{2} \quad \text{for } 2 \leq i < j \leq d \quad \text{and}\\
a_{i,i} &\leq \frac{2}{\sqrt{3}}\,a_{i+1,i+1} \quad \text{for } 2 \leq i \leq d-1\ .
\end{aligned}$$

It remains to prove these inequalities for $i = 1$.

The first inequalities are a consequence of the inequalities $\|f_j\| \leq \|f_j + p\, f_1\|$, $\forall p \in \mathbb{Z}$.

The last inequality is a consequence of the inequality $\|f_1\| \leq \|f_2\|$, because this one implies $a_{1,1}^2 \leq a_{2,2}^2 + a_{1,1}^2 n_{1,2}^2 \leq a_{2,2}^2 + \frac{1}{4} a_{1,1}^2$. ♢

Let $G' = \mathrm{SL}(d, \mathbb{R})$, $K' := K \cap G'$, $A' := A \cap G'$. One still has the Iwasawa decomposition $G' = K'A'N$. One denote $R_{s,t} = S_{s,t} \cap G'$. One still has, thanks to lemma 2.2, $G' = R_{s,t}\Gamma$.

The proposition 2.1 is now a consequence of the following lemma.

Lemma 2.3 *The volume of $R_{s,t}$ for the Haar measure is finite.*

Let us first compute the Haar measure in the Iwasawa decomposition. Let dg', dk', da' and dn be right Haar measures on the groups G', K', A' and N respectively. These are also left Haar measure, since these groups are unimodular. The modulus function of the group $A'N$ is

$$a'n \longrightarrow \rho(a'n) = \rho(a') = |\det_{\mathfrak{n}}(\mathrm{Ad}\, a')| = \prod_{i<j} \frac{a'_{i,i}}{a'_{j,j}}$$

where $\mathfrak{n}$ is the Lie algebra of N.

A left Haar measure on $A'N$ is left A'-invariant and right N-invariant, hence is, up to a multiplicative constant, equal to the product measure $da'dn$. Hence the measure $\rho(a')da'dn$ is a right Haar measure on $A'N$.

In the same way, the measures dg' and $\rho(a')dk'da'dn$ on G' are both left K-invariant and right $A'N$-invariant. They must be equal, up to a multiplicative constant. Hence

$$dg' = \rho(a')dk'da'dn\ .$$

Proof of lemma 2.3 Let $b_i := \dfrac{a'_{i,i}}{a'_{i+1,i+1}}$. The functions $b_1, \ldots, b_{d-1}$ give a coordinate system on A' for which $da' = \frac{db_1}{b_1} \cdots \frac{db_{d-1}}{b_{d-1}}$ and $\rho(a') = \prod_{1 \leq i < d} b_i^{r_i}$ with $r_i \geq 1$. Hence one has

$$\int_{R_{s,t}} dg' = \left(\int_{K'} dk'\right) \left(\prod_{1 \leq i < d} \int_0^s b_i^{r_i - 1} db_i\right) \left(\int_{N_t} dn\right)$$

which is finite because K' and N_t are compact and $r_i \geq 1$. ♢

2.3 Mahler compactness criterion

Let us prove a simple and useful criterion which tells us when some subset of the set X of lattices in $\mathbb{R}^d$ in compact.

The set X of lattices in $\mathbb{R}^d$ is a manifold as a quotient space $X := \mathrm{GL}(d, \mathbb{R})/\mathrm{SL}^{\pm}(d, \mathbb{Z})$. By definition of the quotient topology, a sequence Λ_n of lattices in $\mathbb{R}^d$ converges to some lattice Λ of $\mathbb{R}^d$ if and only if there exists a basis $(f_{n,1}, \ldots, f_{n,d})$ of Λ_n which converges to a basis $(f_1, \ldots, f_d)$ of Λ.

For any lattice Λ of $\mathbb{R}^d$, one denote $d(\Lambda)$ the volume of the torus $\mathbb{R}^d/\Lambda$. It is given by the formula $d(\Lambda) = |\det(f_1, \dots, f_d)|$ for any basis $(f_1, \dots, f_d)$ of Λ.

Lemma 2.4 (Hermite) *Any lattice Λ in $\mathbb{R}^d$ contains a nonzero vector v of norm $\|v\| \leq (\frac{4}{3})^{\frac{d-1}{4}} d(\Lambda)^{\frac{1}{d}}$.*

Proof This is a consequence of lemma 2.2, with the inequality $a_{1,1}^d \leq s^{\frac{d(d-1)}{2}} \prod a_{i,i}$. ♢

Proposition 2.5 (Mahler) *A subset $Y \subset X$ is relatively compact in X if and only if there exist constants $\alpha, \beta > 0$ such that for all $\Lambda \in Y$, one has*

$$d(\Lambda) \leq \beta \quad \text{and} \quad \inf_{v \in \Lambda - 0} \|v\| \geq \alpha \, .$$

Otherwise stated, a set of lattices is relatively compact iff their volumes are bounded and they avoid a small ball.

Proof Let us fix $s > \frac{2}{\sqrt{3}}$ and $t > \frac{1}{2}$ and denote $\Lambda_0 := \mathbb{Z}^d \in X$. Note that a subset $Y \subset X$ is relatively compact if and only if there exists a compact subset $S \subset S_{s,t}$ such that $Y \subset \{g \Lambda_0 \,/\, g \in S\}$.

$\Longrightarrow$ Let us fix $0 < r < R$ such that , for all $g = kan$ in S and all $i = 1, \dots, d$, one has $r \leq a_{i,i} \leq R$. One has then $|\det g| \leq R^d$ and $\inf_{v_0 \in \Lambda_0 - 0} \|g\, v_0\| \geq r$ because if one write $v_0 = \sum_{1 \leq i \leq \ell} m_i e_i$ with $m_\ell \neq 0$, one has

$$\|g\, v_0\| \geq |{<} k e_\ell, g\, v_0 {>}| = |{<} e_\ell, an\, v_0 {>}| \geq a_{\ell,\ell} |m_\ell| \geq r \, .$$

$\Longleftarrow$ Let $S := \{g \in S_{s,t} \,/\, g \Lambda_0 \in \overline{Y}\}$. For all $g = kan$ in S, one has, for $1 \leq i \leq d$,

$$a_{1,1} \geq \alpha \quad , \quad a_{i,i} \leq s\, a_{i+1,i+1} \quad \text{and} \quad \prod_{1 \leq j \leq d} a_{j,j} \leq \beta \, .$$

As a consequence, there exist $0 < r < R$ such that, for all $g = kan$ in S and all $i = 1, \dots, d$, one has $r \leq a_{i,i} \leq R$. Hence S is compact and $\overline{Y}$ too. ♢

The same proof can be easily adapted for the examples 5, 6. The same strategy can also be used for the examples 2, 3: thanks to the Iwasawa decomposition for G, one introduces the Siegel domain and prove that they are of finite volume and that a finite union of them surjects on G/Γ.

2.4 Algebraic groups

In this section we recall a few definitions of the theory of algebraic groups.

Let K be an algebraically closed field of characteristic 0, k be a subfield of K, $V_k \simeq k^d$ be a k-vector space, $V = K \otimes_k V_k$ and $k[V]$ be the ring of k-valued polynomials on V_k.

A *variety* $Z \subset V$ is a subset which is the set of zeros of a family of polynomials on V. Let $I(Z) \subset K[V]$ be the ideal of polynomials on V which are zero on Z. One says that Z is a *k-variety* or is defined over k if $I(Z)$ is generated by the intersection $I_k(Z) := I(Z) \cap k[V]$. Let $k[Z] := k[V]/I_k(Z)$ be the *regular functions ring* of Z. A *k-morphism* of k-varieties $\varphi : Z_1 \to Z_2$ is a map such that, for all f in $k[Z_2]$, $f \circ \varphi$ is in $k[Z_1]$.

A *k-group* is a k-variety $G \subset \mathrm{GL}(V) \subset \mathrm{End}(V)$ which is a group for the composition of endomorphisms. For instance, the k-groups

$$G_a := \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \ / \ x \in K\} \quad \text{and} \quad G_m := \{ \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} \ / \ y, z \in K, \ xy = 1\}$$

are called the additive and the multiplicative k-groups. One has $k[G_a] = k[x]$ and $k[G_m] = k[y, y^{-1}]$. Another example is given by $\mathrm{GL}(V)$ which can be seen as a k-group thanks to the identification

$$\mathrm{GL}(V) \simeq \{(g, \delta) \in \mathrm{End}(V) \times K \ / \ \ \delta \mathrm{det} g = 1\} \ .$$

Note that $G_k := G \cap \mathrm{GL}(d, k)$ is a subgroup of G and, more generally, for any subring A of K, $G_A := G \cap \mathrm{GL}(d, A)$ is a subgroup of G. A *k-morphism* of k-group $\varphi : G_1 \to G_2$ is a k-morphism of k-variety which is also a morphism of groups. A *k-isogeny* is a surjective k-morphism with finite kernel. A *k-character* of G is a k-morphism $\chi : G \to G_m$. A *k-cocharacter* of G is a k-morphism $\chi : G_m \to G$. A *k-representation* of G in a k-vector space W_k is a k-morphism $\rho : G \to \mathrm{GL}(W)$. The k representation is *irreducible* if 0 and W are the only invariant subspaces. It is *semisimple* if it is a direct sum of irreducible representations. A K-group G is *reductive* if all its K-representations are semisimple. A reductive K-group is *semisimple* if all its K-characters are trivial. This definition is well-suited for the goups we are dealing with since we have the following lemma.

This lemma will not be used later on. The reader may skip its proof.

Lemma 2.6 *The k-groups $\mathrm{SL}(d)$ and $\mathrm{SO}(p, q)$ are semisimple.*

Proof Say for $G = \mathrm{SL}(d, \mathbb{C})$. Since $G = [G, G]$, one has only to prove the semisimplicity of the representations of the group G in a $\mathbb{C}$-vector space V. Hence, one has to prove that any G-invariant subspace W has a G-invariant supplementary subspace. To prove this fact, we will use Weyl's unitarian trick: let $K = \mathrm{SU}(n, \mathbb{C})$ be the maximal compact subgroup of G. By averaging with respect to the Haar measure on K, one can construct a K-invariant hermitian scalar product on V. The orthogonal $W^{\perp}$ of W is then K-invariant and, since the Lie algebra of G is the complexification of the Lie algebra of K, it is also G-invariant. ♢

2.5 Arithmetic groups

We check that for a $\mathbb{Q}$-group G the subgroup $G_{\mathbb{Z}}$ does not depend, up to commensurability, on the realization of G as a group of matrices.

Lemma 2.7 *Let ρ be a $\mathbb{Q}$-representation of a $\mathbb{Q}$-group G in a $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$. Then,*
a) the group $G_{\mathbb{Z}}$ preserves some lattice $\Lambda \subset V_{\mathbb{Q}}$.
b) Any lattice $\Lambda_0 \subset V_{\mathbb{Q}}$, is preserved by some subgroup of finite index of $G_{\mathbb{Z}}$

Proof a) Choose a basis of $V_{\mathbb{Q}}$. The entries of the matrices $\rho(g) - 1$ can be expressed as polynomials with rational coefficients in the entries of the matrices $g - 1$. The constant coefficient of these polynomials is zero. Hence there is an integer $m \geq 1$ such that, if g is in the congruence subgroup $\Gamma_m := \{g \in G_{\mathbb{Z}} \ / \ g = 1 \bmod m\}$, then $\rho(g)$ has integral entries. Since Γ_m is of finite index in $G_{\mathbb{Z}}$, this group $G_{\mathbb{Z}}$ also preserves a lattice in $V_{\mathbb{Q}}$.
b) $\diamondsuit$

One easily deduces the following corollary.

Corollary 2.8 *Let $\varphi : G_1 \to G_2$ be a $\mathbb{Q}$-isomorphism of $\mathbb{Q}$-groups. Then the groups $\varphi(G_{1,\mathbb{Z}})$ and $G_{2,\mathbb{Z}}$ are commensurable.*

2.6 The embedding

The following embedding will allow us to reduce the proof of the compactness of G/Γ to Mahler's criterion

Proposition 2.9 *Let $G \subset H = \mathrm{GL}(d, \mathbb{C})$ be a $\mathbb{Q}$-group without nontrivial $\mathbb{Q}$-characters. Then the injection $G_{\mathbb{R}}/G_{\mathbb{Z}} \hookrightarrow X = H_{\mathbb{R}}/H_{\mathbb{Z}}$ is an homeomorphism onto a closed subset of X.*

We will need the following proposition.

Proposition 2.10 (Chevalley) *Let G be a k-group and $H \subset G$ be a k-subgroup. Then there exist a k-representation of G on some vector space V_k and a point $x \in \mathbb{P}(V_k)$ whose stabilizer is H, i.e. $H = \{g \in G \ / \ g\, x = x\}$.*

Proof We will need the following notations.
- $I(H) := \{P \in K[G] \ / \ P|_H = 0\}$,
- $K^m[G] := \{P \in K[G] \ / \ d^\circ P \leq m\}$ and
- $I^m(H) := I(H) \cap K^m[G]$.
Since $K[G]$ is noetherian, one can choose m such that $I^m(H)$ generates the ideal $I(H)$ of $K[G]$. The action of G in $K^m[G]$ given by $(\pi(g)P)(g') = P(g'g)$ is a k-representation. The k-representation we are looking for is the representation in the p^{th} exterior product $V := \Lambda^p(K^m[G])$, where $p := \dim I^m(H)$ and x is the line in V, $x := \Lambda^p(I^m(H))$. By construction one has the required equality $H = \{g \in G \ / \ g\, x = x\}$. $\diamondsuit$

Corollary 2.11 *Let G be a k-group and $H \subset G$ be a k-subgroup. Suppose H does not have any nontrivial k-character. Then there exist a k-representation of G on some vector space V_k and a point $v \in V_k$ whose stabilizer is H, i.e. $H = \{g \in G \ / \ g\, v = v\}$.*

Proof The action of H on the line x is trivial since all the k-characters of H are trivial. Just choose v on this line. $\diamondsuit$

Proof of proposition 2.9 We have to show that,

$$\forall g_n \in G_{\mathbb{R}},\ h \in H_{\mathbb{R}} \textit{ such that } \lim_{n\to\infty} g_n H_{\mathbb{Z}} = hH_{\mathbb{Z}},\textit{ in } H_{\mathbb{R}}/H_{\mathbb{Z}}$$
$$\exists g \in G_{\mathbb{R}},\textit{ such that } \lim_{n\to\infty} g_n G_{\mathbb{Z}} = gG_{\mathbb{Z}} \textit{ in } G_{\mathbb{R}}/G_{\mathbb{Z}}.$$

Since all $\mathbb{Q}$-characters of G are trivial, according to the corollary 2.11 (with H for G and G for H) there exists a $\mathbb{Q}$-representation of H in some $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$ and a vector $v \in V_{\mathbb{Q}}$ whose stabilizer in H is G. According to lemma 2.7, the group $H_{\mathbb{Z}}$ stabilizes some lattice Λ in $V_{\mathbb{Q}}$. One can choose Λ containing v. Hence the $H_{\mathbb{Z}}$-orbit of v is discrete in $V_{\mathbb{R}}$.

Let $h_n \in H_{\mathbb{Z}}$ such that $\lim_{n\to\infty} g_n h_n = h$. The sequence $h_n^{-1}v$ converges to $h^{-1}v$ hence is equal to $h^{-1}v$ for n large. Therefore, one can write $h_n = \gamma_n g^{-1} h$ with $g \in G_{\mathbb{R}}$, $\gamma_n \in G_{\mathbb{Z}}$. and the sequence $g_n\gamma_n$ converges to g. $\diamondsuit$

2.7 Construction of cocompact lattices

We check that the groups of examples 4 and 8 in section 2.1 are cocompact lattices in $\mathrm{SL}(d,\mathbb{R})$ and $\mathrm{SO}(p,q)$ respectively.

The following lemma can be applied directly to these examples and enlightens the strategy of the proof in the general case.

Lemma 2.12 *Let $V_{\mathbb{Q}}$ be a $\mathbb{Q}$-vector space, $G \subset \mathrm{GL}(V)$ be a $\mathbb{Q}$-subgroup without nontrivial $\mathbb{Q}$-character. Suppose there exist a G-invariant polynomial $P \in \mathbb{Q}[V]$ such that*

$$\forall v \in V_{\mathbb{Q}}\ ,\ P(v) = 0 \Longleftrightarrow v = 0\ .$$

Then the quotient $G_{\mathbb{R}}/G_{\mathbb{Z}}$ is compact.

Proof Let Λ_0 be a lattice in $V_{\mathbb{Q}}$. One can suppose that $P(\Lambda_0) \subset \mathbb{Z}$.

According to propositions 2.5 and 2.9 we only have to show that no sequence $g_n v_n$ with $g_n \in G_{\mathbb{R}}$ and $v_n \in \Lambda_0 - \{0\}$ can converge to zero.

This is a consequence of the minoration $|P(g_n v_n)| = |P(v_n)| \geq 1$. $\diamondsuit$

Corollary 2.13 *a) In example* 2.1.4*, Γ is cocompact in* $\mathrm{SL}(d,\mathbb{R})$*.*
b) In example 2.1.8*, Γ is cocompact in* $\mathrm{SO}(p,q)$*.*

Proof a) Take $V_k = D$ and $P(v) = det_D(\rho_v)$ where ρ_v is the left multiplication by v.
b) We will apply for this example Weil's recipe called "restriction of scalars". Let us denote by $\mathrm{SO}(J_{p,q},\mathbb{C})$ the special orthogonal group for the quadratic form q_0 whose matrix is $J_{p,q}$. The algebraic group

$$H := \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \in \mathrm{GL}(2d,\mathbb{C}) \ /\ a + \sqrt{2}\, b \in \mathrm{SO}(J_{p,q},\mathbb{C})\ ,\ a - \sqrt{2}\, b \in \mathrm{SO}(J_{p,q}^{\sigma},\mathbb{C}) \right\}$$

is defined over $\mathbb{Q}$, because this family of equations is σ-invariant.

The map $(a,b) \to a + \sqrt{2}\,b$ gives an isomorphism

$$H_{\mathbb{Z}} \simeq \Gamma$$

and the map $(a,b) \to (a+\sqrt{2}\,b, a-\sqrt{2}\,b)$ gives an isomorphism

$$H_{\mathbb{R}} \simeq \mathrm{SO}(J_{p,q}, \mathbb{R}) \times \mathrm{SO}(J^{\sigma}_{p,q}, \mathbb{R}) \ .$$

One applies lemma 2.12 with the natural $\mathbb{Q}$-representation in $V_{\mathbb{Q}} = \mathbb{Q}^d \times \mathbb{Q}^d$ and with $P : (u,v) \to q_0(u+\sqrt{2}\,v)\, q_0^{\sigma}(u-\sqrt{2}\,v)$. This proves that $H_{\mathbb{Z}}$ is cocompact in $H_{\mathbb{R}}$. Since $\mathrm{SO}(J^{\sigma}_{p,q}, \mathbb{R})$ is compact, Γ is a cocompact lattice in $\mathrm{SO}(J_{p,q}, \mathbb{R})$. $\diamondsuit$

Remark To convince the reader that the examples a) do exist in any dimension $d \geq 2$, we will give a construction of

a central division algebra D over $\mathbb{Q}$ such that $D \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathcal{M}(d, \mathbb{R})$

without using the well-known description of the Brauer group of $\mathbb{Q}$.

Let L be a Galois real extension of $\mathbb{Q}$ with Galois group $\mathrm{Gal}(L/\mathbb{Q}) = \mathbb{Z}/d\mathbb{Z}$ and σ be a generator of the Galois group. One can take $L = \mathbb{Q}[\eta]$ with $\eta = \sum_{1 \leq i \leq q/2d} \cos\left(2\pi g^{id}/q\right)$ where q is a prime number $q \equiv 1 \bmod 2d$ and g is a generator of the cyclic group $(\mathbb{Z}/q\mathbb{Z})^{\times}$. (for $d = 3, 4$ and 5, take $L = \mathbb{Q}[\cos\frac{2\pi}{7}]$, $L = \mathbb{Q}[\cos\frac{\pi}{17} + \cos\frac{2\pi}{17}]$ and $L = \mathbb{Q}[\cos\frac{2\pi}{11}]$).

We will construct D as a d-dimensional left L-vector space $D = L \oplus La \oplus \cdots \oplus La^{d-1}$, with the multiplication rules $\forall \ell \in L$, $a\ell a^{-1} = \sigma(\ell)$ and $a^d = p$ where p is another prime number which is inert in L (such a p does exist by Cebotarev). By construction D is an algebra with center $\mathbb{Q}$. It remains to show that every nonzero element $v = \ell_0 + \ell_1 a + \ldots + \ell_{d-1}a^{d-1} \in D$ is invertible. One may suppose that all ℓ_i are in the ring R of integers of L but that $\ell_0 \notin pR$. One computes the determinant Δ_v of the right multiplication by v as an endomorphism of the left L-vector space D. One gets

$$\Delta_v = \det \begin{pmatrix} \ell_0 & p\ell^{\sigma}_{d-1} & \cdots & p\ell_1^{\sigma^{d-1}} \\ \ell_1 & \ell_0^{\sigma} & \cdots & p\ell_2^{\sigma^{d-1}} \\ \vdots & \vdots & & \vdots \\ \ell_{d-1} & \ell^{\sigma}_{d-2} & \cdots & \ell_0^{\sigma^{d-1}} \end{pmatrix} \equiv \ell_0 \ell_0^{\sigma} \cdots \ell_0^{\sigma^{d-1}} \bmod pR \ .$$

Since p is inert, this determinant is nonzero and v is invertible. $\diamondsuit$

2.8 Godement compactness criterion

In this section, we state a general criterion for the cocompactness of an arithmetic subgroup and show how to adapt the previous arguments to prove it.

Let us first recall the definitions of *semisimple* and *unipotent* elements and some of their properties. An element g in $\mathrm{End}(V)$ is *semisimple* if it is diagonalizable over K and *unipotent* if $g-1$ is nilpotent. The following lemma is the classical Jordan decomposition.

Lemma 2.14 *Let $g \in \mathrm{GL}(V)$ and $G \subset \mathrm{GL}(V)$ be a k-group.*
i) g can be written in a unique way as $g = su = us$ with s semisimple and u unipotent.
ii) Every subspace $W \subset V$ invariant by g is also invariant by s and u.
iii) $g \in G \Longrightarrow s, u \in G$.
iv) $g \in G_k \Longrightarrow s, u \in G_k$.

Proof i) Classical.
ii) s and u can be expressed as polynomial in g.
iii) Consider the action of G on $K^m[\mathrm{End}V] := \{P \in K[\mathrm{End}V] \ / \ d^\circ P \le m\}$ given by $(\pi(g)P)(x) = P(xg)$. The subspace $I^d[G] := I[G] \cap K^d[\mathrm{End}V]$ is invariant by g. Hence it is also invariant by its semisimple and unipotent part which is nothing than $\pi(s)$ and $\pi(u)$. Hence, for all $P \in I^d[G]$, one has $P(s) = (\pi(s)P)(1) = 0$ and $P(u) = (\pi(u)P)(1) = 0$. Therefore s and u are in G.
iv) By unicity, s and u are invariant under the Galois group $\mathrm{Gal}(K/k)$. $\diamondsuit$

Lemma 2.15 *Let $\rho : G \to H$ be a k-morphism of k-groups and $g \in G$.*
a) g is semisimple $\Longrightarrow \rho(g)$ is semisimple.
b) g is unipotent $\Longrightarrow \rho(g)$ is unipotent.

Proof One can suppose that $k = K$ and G is the smallest K-group containing g. The main point then is to prove that all k-morphisms $\varphi : G_a \to G_m$ and $\psi : G_m \to G_a$ are trivial. But $y \circ \varphi$ is an invertible element of $k[x]$ hence is a constant, and $x \circ \psi$ is an element $F(y) \in k[y, y^{-1}]$ such that $F(y) = F(y^n)/n$, for all $n \ge 1$, hence is a constant. $\diamondsuit$

Note that the Lie algebra $\mathfrak{g}$ of a $\mathbb{Q}$-group G is defined over $\mathbb{Q}$, because it is invariant under $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.

Theorem 2.16 (Borel, Harish-Chandra) *Let $G \subset \mathrm{GL}(d, \mathbb{C})$ be a semisimple $\mathbb{Q}$-group and $\mathfrak{g}$ be its Lie algebra. Then the following are equivalents:*
(i) $G_{\mathbb{R}}/G_{\mathbb{Z}}$ is compact.
(ii) Every element g of $G_{\mathbb{Q}}$ is semisimple.
(iii) The only unipotent element of $G_{\mathbb{Z}}$ is 1.
(iv) The only nilpotent element of $\mathfrak{g}_{\mathbb{Q}}$ is 0.

Remark See section 2.9 for the general formulation of this theorem.

Sketch of proof of theorem 2.16 $(i) \Rightarrow (iii)$ Let $u \in G_{\mathbb{Q}}$ be a unipotent element. According to Jacobson-Morozov, there exists a Lie subgroup S of $G_{\mathbb{R}}$ containing u with Lie algebra $\mathfrak{s} \simeq \mathfrak{sl}(2, \mathbb{R})$. There exists then an element $a \in S$ such that $\lim_{n\to\infty} a^n u a^{-n} = e$. Since $G_{\mathbb{R}}/G_{\mathbb{Z}}$ is compact, one can write $a^n = k_n \gamma_n$ with k_n bounded and $\gamma_n \in G_{\mathbb{Z}}$. But then $\gamma_n u \gamma_n^{-1}$ is a sequence of elements of $G_{\mathbb{Z}} - \{e\}$ converging to e. Therefore $u = e$.
$(ii) \Leftrightarrow (iii)$ This follows from lemma 2.14.
$(iii) \Rightarrow (iv)$ The logarithm of a unipotent element of $G_{\mathbb{Q}}$ is a well-defined nilpotent element which is in $\mathfrak{g}_{\mathbb{Q}}$.

$(iv) \Rightarrow (i)$ The group $\mathrm{Aut}(\mathfrak{g})$ is a $\mathbb{Q}$-group and the adjoint map $Ad : G \to \mathrm{Aut}(\mathfrak{g})$ is a $\mathbb{Q}$-isogeny, i.e. it is a surjective $\mathbb{Q}$-morphism with finite kernel. Thanks to the following lemma, one can suppose that $G = \mathrm{Aut}(\mathfrak{g})$. We can then apply lemma 2.12 with P the G-invariant polynomial on $\mathfrak{g}$ given by $P(X) = (tr\, X)^2 + (tr\, X^2)^2 + \cdots + (tr\, X^d)^2$ where $d = \dim \mathfrak{g}$, since one has the equivalence: $P(X) = 0 \Longleftrightarrow X$ is nilpotent. $\diamondsuit$

In this proof, we have used the following lemma

Lemma 2.17 *Let $\varphi : G \to H$ be a $\mathbb{Q}$-isogeny between two semisimple $\mathbb{Q}$-group. Then $\varphi(G_{\mathbb{Z}})$ and $H_{\mathbb{Z}}$ are commensurable.*

Remark One must be aware that, even though φ is surjective, $\varphi(G_{\mathbb{Q}})$ and $H_{\mathbb{Q}}$ are not commensurable. Take for instance $G = \mathrm{SL}(2)$ and $H = \mathrm{PGL}(2)$, and look at the elements of $H_{\mathbb{Q}}$ given by $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

Proof One may suppose that $H = G/C$ where C is the center of G. Let $G \subset \mathrm{End}(V)$ be this $\mathbb{Q}$-group, $A = \mathrm{End}_C(V)$ be the commuting of C in $\mathrm{End}(V)$, $A_{\mathbb{Z}} = A \cap \mathrm{End}(V_{\mathbb{Z}})$, $\Gamma := G_{\mathbb{Z}} = \{g \in G \;/\; gA_{\mathbb{Z}} = A_{\mathbb{Z}}\}$ and $\Delta := \{g \in G \;/\; gA_{\mathbb{Z}}g^{-1} = A_{\mathbb{Z}}\}$. Using the fact that a bijective $\mathbb{Q}$-morphism is a $\mathbb{Q}$-isomorphism, one only has to show that Δ/Γ is finite.

According to propositions 2.5 and 2.9, we only have to show that no sequence $d_n a_n$ with $d_n \in \Delta$ and $a_n \in A_{\mathbb{Z}} - \{0\}$ can converge to zero. Since the semisimple associative algebra A is the direct sum of its minimal bilateral ideals B, one may suppose that a_n is in some $B_{\mathbb{Z}} - \{0\}$. Let b_i be a basis of B. Since $\det_B d_n = 1$, according to Minkowski lemma 2.4, one can find a constant C_0 and nonzero elements $c_n \in B_{\mathbb{Z}}$ such that $\|c_n d_n^{-1}\| \le C_0$. Since the elements $a_n b_i c_n$ are in $B_{\mathbb{Z}}$, the elements $d_n a_n b_i c_n d_n^{-1}$ are also in $B_{\mathbb{Z}}$ and converge to zero. Hence, successively, for $n \gg 0$, $a_n b_i c_n = 0$, $a_n B c_n B = 0$, $a_n B = 0$ and $a_n = 0$. Contradiction. $\diamondsuit$

Corollary 2.18 *In example* 2.1.7, Γ *is cocompact in* $\mathrm{SL}(d, \mathbb{R})$.

Proof The proof is similar to corollary 2.13 using “restriction of scalar”.
The algebraic group

$$G = \left\{ \begin{pmatrix} a & \sqrt{2}\, b \\ b & a \end{pmatrix} \in \mathrm{GL}(2d, \mathbb{C}) / (a + \sqrt[4]{2}\, b)\, J_{p,q}\, ({}^t a - \sqrt[4]{2}\, {}^t b) = J_{p,q} \;,\; \det(a + \sqrt[4]{2}\, b) = 1 \right\}$$

is defined over $k_0 = \mathbb{Q}[\sqrt{2}]$, because this family of equations is τ-invariant. The “image” of G by the Galois involution σ of k_0 is the algebraic group

$$G^\sigma = \left\{ \begin{pmatrix} a & -\sqrt{2}\, b \\ b & a \end{pmatrix} \in \mathrm{GL}(2d, \mathbb{C}) / (a + i\sqrt[4]{2}\, b) J^\sigma_{p,q} ({}^t a - i\sqrt[4]{2}\, {}^t b) = J^\sigma_{p,q} \;,\; \det(a + i\sqrt[4]{2}\, b) = 1 \right\}$$

which is also defined over $k_0 = \mathbb{Q}[\sqrt{2}]$.

Using the diagonal embedding $x \to (x, x^\sigma)$ of $\mathbb{Q}[\sqrt{2}]$ in $\mathbb{R} \times \mathbb{R}$, one constructs a semisimple $\mathbb{Q}$-group

$$H := \left\{ \begin{pmatrix} c & 2\,d \\ d & c \end{pmatrix} \in \mathrm{GL}(4d,\mathbb{C}) \;/\; c+\sqrt{2}\,d \in G \;,\; c-\sqrt{2}\,d \in G^{\sigma} \right\}$$

The maps $(c,d) \to c+\sqrt{2}\,d$ and $(a,b) \to a+\sqrt[4]{2}\,b$ give isomorphisms

$$H_{\mathbb{Z}} \simeq G_{\mathbb{Z}[\sqrt{2}]} \simeq \Gamma$$

and the map $(c,d) \to (c+\sqrt{2}\,d, c-\sqrt{2}\,d)$ gives an isomorphism

$$H_{\mathbb{R}} \simeq G_{\mathbb{R}} \times (G^{\sigma})_{\mathbb{R}} \simeq \mathrm{SL}(d,\mathbb{R}) \times \mathrm{SU}(d,\mathbb{R}) \;.$$

Since $\sqrt[4]{2} \in \mathbb{R}$, the group $G_{\mathbb{R}}$ is isomorphic to $\mathrm{SL}(d,\mathbb{R})$. Since the hermitian form h_0 on $\mathbb{C}^d$ whose matrix is $J^{\sigma}_{p,q}$ is positive definite, the group $(G^{\sigma})_{\mathbb{R}}$ is compact. Two apply theorem 2.16, one uses lemma 2.15 and notice that $H_{\mathbb{Q}}$ does not contain unipotent element, since its image by $(c,d) \to c-\sqrt{2}\,d$ lives in the compact group $(G^{\sigma})_{\mathbb{R}}$. This proves that $H_{\mathbb{Z}}$ is cocompact in $H_{\mathbb{R}}$. Since $(G^{\sigma})_{\mathbb{R}}$ is compact, Γ is a cocompact lattice in $\mathrm{SL}(d,\mathbb{R})$. $\diamondsuit$

2.9 A general overview

Let us now describe, without proof, the general theory that these examples illustrate. Roughly speaking this theory says that for $d \geq 3$ and $q \geq 2$ all the lattices of $\mathrm{SL}(d,\mathbb{R})$ and $\mathrm{SO}(p,q)$ are constructed in a similar way.

More precisely. Let $H \subset \mathrm{GL}(d,\mathbb{C})$ be a $\mathbb{Q}$-group. Then one has the equivalences:

$$\begin{aligned} \mathrm{vol}(H_{\mathbb{R}}/H_{\mathbb{Z}}) < \infty &\iff H \text{ has no nontrivial } \mathbb{Q}-\text{character.} \\ H_{\mathbb{R}}/H_{\mathbb{Z}} \text{ is compact} &\iff H \text{ has no nontrivial } \mathbb{Q}-\text{cocharacter.} \end{aligned}$$

One says that H is $\mathbb{Q}$-*anisotropic* when it does not have nontrivial cocharacter i.e. when it does not contain $\mathbb{Q}$-subgroups $\mathbb{Q}$-isomorphic to G_m.

These facts due to Borel and Harish-Chandra are the main motivations of Borel's book [4] and are illustred by the examples 1 to 4.

There is a very important construction of lattices which is simultaneously an extension and a by-product of the previous construction: let $L \subset \mathrm{GL}(d,\mathbb{C})$ be a semisimple algebraic group defined over a number field k, $\mathcal{O}$ be the ring of integers of k and $\sigma_1, .., \sigma_{r_1}$ be the real embeddings of k and $\sigma_{r_1+1}, .., \sigma_{r_1+r_2}$ the complex embeddings up to complex conjugation. Recall that the image of the diagonal map $\sigma : \mathcal{O} \to \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ is a lattice in this real vector space. Then, the diagonal image of the group $L_{\mathcal{O}} := L \cap \mathrm{SL}(d,\mathcal{O})$ in the product $L^{\sigma_1}_{\mathbb{R}} \times \cdots \times L^{\sigma_{r_1}}_{\mathbb{R}} \times L^{\sigma_{r_1+1}}_{\mathbb{C}} \times \cdots \times L^{\sigma_{r_1+r_2}}_{\mathbb{C}}$ is a lattice. According to Weil's trick called "restriction of scalars" this construction with $L_{\mathcal{O}}$ can be seen as a special case of the previous construction with $H_{\mathbb{Z}}$ for some suitable algebraic group H defined over $\mathbb{Q}$ for which $H_{\mathbb{Q}} \simeq L_k$ (this is illustred by the examples 5 and 6).

Suppose that $r_1 + r_2 > 2$ and that, in this product, all the factors are compact except one, then, as a consequence $L_{\mathcal{O}}$ is a cocompact lattice in the noncompact factor (this is illustred by the examples 7 and 8).

These examples are the main motivation for the following

Definition 2.19 *A subgroup Γ of a real linear semisimple Lie group G is said to be arithmetic, if there exists an algebraic group H defined over $\mathbb{Q}$ and a surjective morphism $\pi : H_{\mathbb{R}} \to G$ with compact kernel such that the groups Γ and $\pi(H_{\mathbb{Z}})$ are commensurable, (i.e. the intersection is of finite index in both of them).*

The classification of all arithmetic groups Γ of a given real linear semisimple Lie group G, up to commensurability, relies on the classification of all algebraic absolutely simple groups defined over a number field k (see [22] for a reduction of this classification to the anisotropic case). For the groups G of classical type not D_4, this classification is due to Weil ([27]) and is equivalent to the classification of all central simple algebras D with antiinvolution $*$ over k (i.e $(a_1^*)^* = a_1$ and $(a_1a_2)^* = a_2^*a_1^*$, $\forall a_1, a_2 \in D$).

Since k is a number field, this is a classical topic in arithmetic which contains,
- the classification of central division algebras over k,
- the classification of bilinear symmetric or antisymmetric forms over k
- the classification of hermitian forms over a quadratic extension $k \supset k_0$.

The main tool in this classification is the local to global principle (see [18] or [28]).

According to a theorem of Borel ([4]), all linear real semisimple Lie groups contain at least one cocompact and one noncocompact lattice.

For an arithmetic group Γ of $G = \mathrm{SL}(d, \mathbb{R})$, Weil's classification implies that, up to commensurability, either,
- Γ is the group Γ_1 of unit of an order $\mathcal{O}_D$ in a central simple algebra D of rank d over $\mathbb{Q}$ which splits over $\mathbb{R}$, (this generalizes examples 1 and 4) or
- Γ is the group Γ_2 of $*$-invariant elements in the unit of an order $\mathcal{O}_D$ in a central simple algebra D of rank d over a real field k and $*$ is an antiinvolution of D nontrivial on k such that D is split over $\mathbb{R}$ and all the other embeddings of the fixed field k_0 of $*$ in k are real and extend to a complex embedding of k whose corresponding real unitary group is compact (this generalizes examples 3 and 7).

Moreover, Γ_1 is cocompact if and only if D is a division algebra and Γ_2 is cocompact if and only if either D is a division algebra or $k_0 \neq \mathbb{Q}$.

Conversely, Margulis arithmeticity theorem says: Let G be a real semisimple Lie group of rank at least 2, with no compact factor (a factor is a group G' which is a quotient of G). Then *all irreducible lattices Γ are arithmetic groups* (irreducible means that all the projections of Γ in any nontrivial factor of G are nondiscrete). This theorem is the main aim of Zimmer's book [29] and of Margulis's book [15].

Lecture 3: Representations

The aim of this lecture is to show how the properties of the unitary representations of a Lie group G have an influence on the algebraic structure of any lattice Γ of G.

We will deal here with one property due to Kazhdan. Namely, using the decreasing properties of the coefficients of unitary representations of G, when G is simple of rank at least 2, we will show that the abelianized of Γ is finite. We will also see that these properties imply mixing properties for some non relatively compact flows on G/Γ.

3.1 Coefficients decay

We will first prove a general decreasing property for coefficients of unitary representations of semisimple real Lie groups.

Definition 3.1 *A unitary representation π of a locally compact group G in a (separable) Hilbert space $\mathcal{H}_\pi$ is a morphism from G to the group $U(\mathcal{H}_\pi)$ of unitary transfomations of $\mathcal{H}_\pi$, such that $\forall v \in \mathcal{H}_\pi$, the map $G \to \mathcal{H}_\pi; g \mapsto \pi(g)v$ is continuous.*

For any $v, w \in \mathcal{H}_\pi$, the coefficient is the continuous function $c_{v,w} : G \to \mathbb{C}$ given by $c_{v,w}(g) = <\pi(g)v, w>$.

Examples - The trivial representation is the constant representation $\pi(g) = Id$. Its coefficients are constant maps.
- Suppose G acts continuously on a locally compact space X preserving a Radon measure ν. Then the formula $(\pi(g)\varphi)(x) = \varphi(g^{-1}x)$ defines a unitary representation π of G in $L^2(X, \nu)$. Its coefficients are the correlation coefficients $c_{\varphi,\psi} : g \to \int_G \varphi(x)\overline{\psi}(gx)d\nu(x)$.
- When G is compact, any unitary representation is a hilbertian orthogonal sum of irreducible unitary representations. By Peter-Weyl, these are finite dimensional.

Let us denote, for $H \subset G$, by

$$\mathcal{H}_\pi^H := \{v \in \mathcal{H}_\pi \ / \ \forall h \in H, \ \pi(h)v = v\}$$

the subspace of H-invariant vectors. Recall that a Lie group G is *semisimple* if its Lie algebra $\mathfrak{g}$ does not have any nonzero solvable ideals [or equivalently, if the group of automorphim of $\mathfrak{g}$ is a semisimple $\mathbb{R}$-group] and that G is *quasisimple* if $\mathfrak{g}$ is simple.

Theorem 3.2 (Howe, Moore) *Let G be a connected semisimple real Lie group with finite center and π be a unitary representation of G. Suppose that $\mathcal{H}_\pi^{G_i} = 0$, for any connected normal subgroup $G_i \neq 1$. Then, for all $v, w \in \mathcal{H}_\pi$, one has*

$$\lim_{g \to \infty} <\pi(g)v, w> \quad = \quad 0. \tag{3}$$

Remarks - The proof of this theorem will be postponed to section 3.4.
- The symbol $g \to \infty$ means that g goes out of any compact of G.
- When G is quasisimple, the hypothesis is $\mathcal{H}_\pi^G = 0$.

Corollary 3.3 *Let G be a connected semisimple real Lie group with finite center and π be a unitary representation of G without nonzero G-invariant vectors. Let H be a closed subgroup of G whose images in the factors $G/G_i \neq 1$ are noncompact. Then $\mathcal{H}_\pi^H = 0$.*

Remark - When $\mathfrak{g}$ is simple, the hypothesis is H *noncompact*.

Proof By induction, one can suppose that $\forall i$, $\mathcal{H}_\pi^{G_i} = 0$. Let v be an H-invariant vector. The coefficient $c_{v,v}$ is constant on H. By theorem 3.2, it has to be zero. Hence $v = 0$. $\diamondsuit$

3.2 Invariant vectors for SL(2)

Let us begin by a direct proof of corollary 3.3 for $\mathrm{SL}(2,\mathbb{R})$.

For $t > 0$ and $s \in \mathbb{R}$, let $a_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, $u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, $u_s^- = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$.

Proposition 3.4 *Let π be a unitary representation of $G = \mathrm{SL}(2,\mathbb{R})$, $t \neq 1$, $s \neq 0$ and $v \in \mathcal{H}_\pi$. If v is either a_t-invariant or u_s-invariant or u_s^--invariant then it is G-invariant.*

The proof uses the lemma

Lemma 3.5 (Mautner) *Let π be a unitary representation of a locally compact group G. For $v \in \mathcal{H}_\pi$, $\|v\| = 1$, let $S_v \subset G$ be its stabiliser $S_v = \{g \in G \;/\; \pi(g)v = v\}$. Then*
a) $S_v = \{g \in G \;/\; c_{v,v}(g) = 1\}$.
b) Let $g \in G$ such that there exist $g_n \in G$, $s_n \in S_v$, $s'_n \in S_v$ such that $\lim_{n\to\infty} g_n = g$, $\lim_{n\to\infty} s_n g_n s'_n = e$. Then g is in S_v.

Proof a) Use the equality $\|\pi(g)v - v\|^2 = 2\,\|v\|^2 - 2\,\mathrm{Re}(c_{v,v}(g))$.
b) Let n goes to ∞ in the equality $c_{v,v}(g_n) = c_{v,v}(s_n g_n s'_n)$ to get $c_{v,v}(g) = 1$. $\diamondsuit$

Proof of proposition 3.4 It is enough to prove that the invariance of v by one among a_t, u_s, u_s^- implies the invariance by the other two. Thanks to the symmetries, there are only two cases to deal with:

$\mathbf{a_t}$-invariant $\Longrightarrow$ $\mathbf{u_s}$-invariant. One may suppose $t > 1$. One uses lemma 3.5.b with $g_n = g = u_s$, $s_n = a_t^{-n}$ and $s'_n = a_t^n$. One easily check that $\lim_{n\to\infty} s_n g_n s'_n = \lim_{n\to\infty} u_{t^{-2n}s} = e$.

$\mathbf{u_s}$-invariant $\Longrightarrow$ $\mathbf{a_t}$-invariant. One may suppose that t is rational $t = \frac{p}{q}$. One uses lemma 3.5.b with $g = a_t$, $g_n = \begin{pmatrix} \frac{p}{q} & 0 \\ \frac{t-1}{snp} & \frac{q}{p} \end{pmatrix}$, $s_n = u_s^{-np}$ and $s'_n = u_s^{nq}$. One easily check that $\lim_{n\to\infty} s_n g_n s'_n = \lim_{n\to\infty} \begin{pmatrix} 1 & 0 \\ \frac{t-1}{snp} & 1 \end{pmatrix} = e$. $\diamondsuit$

3.3 Real semisimple Lie groups

To prove theorem 3.2, we recall without proof basic facts on the structure of semisimple Lie groups (see [12]). We use the language of root systems and parabolic subgroups which, since E.Cartan, is the only convenient one which allows to deal with all real semisimple Lie groups. At the end we will recall the meaning of these concepts for the important example $G = \mathrm{SL}(d, \mathbb{R})$.

Let G be a connected semisimple Lie group with finite center.

Maximal compact subgroups Then G contains a maximal compact subgroup K and all such subgroups are conjugate. Let $\mathfrak{k} \subset \mathfrak{g}$ be the corresponding Lie algebras. There exists an involution θ of $\mathfrak{g}$, called *Cartan involution* whose fixed point set is $\mathfrak{k}$. Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ where $\mathfrak{q}$ is the fixed point set of $-\theta$. The Killing form $K(X,Y) = tr(adX \circ adY)$ is positive definite on $\mathfrak{q}$ and negative definite on $\mathfrak{k}$.

Cartan subspaces An element X of $\mathfrak{g}$ is said to be *hyperbolic* if $\mathrm{ad}(X)$ is diagonalizable over $\mathbb{R}$. A *Cartan subspace* of $\mathfrak{g}$ is a commutative subalgebra whose elements are hyperbolic and which is maximal for these properties. All Cartan subspaces are conjugate and a maximal commutative algebra in $\mathfrak{q}$ is a Cartan subspaces. Let us choose one of them $\mathfrak{a} \subset \mathfrak{q}$ and denote by $A = \exp(\mathfrak{a})$. By definition the *real rank* of G is the dimension of $\mathfrak{a}$. The set of real characters of the Lie group A can be identified with the dual $\mathfrak{a}^\star$. Thanks to the Killing form, this space is Euclidean.

Restricted roots Let us diagonalize $\mathfrak{g}$ under the adjoint action of A. One denote by Δ the set of *restricted roots* i.e. the set of nontrivial weights for this action. It is a root system. One has a decomposition $\mathfrak{g} = \mathfrak{l} \oplus (\oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha)$ where $\mathfrak{g}_\alpha := \{Y \in \mathfrak{g} \;/\; \forall g \in A \;,\; \mathrm{Ad}g(Y) = \alpha(g)Y\}$ is the root space associated to α and $\mathfrak{l}$ is the centralizer of $\mathfrak{a}$.

Weyl chambers Let Δ^+ be a choice of positive roots, $\Delta^- = -\Delta^+$ and Π be the set of simple roots. Π is a basis of $\mathfrak{a}^\star$. Let $\mathfrak{u}^\pm := \oplus_{\alpha\in\Delta^\pm}\mathfrak{g}_\alpha$, $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}^+$ be the *minimal parabolic subalgebra* associated to Δ^+. Its normaliser $P := N_G(\mathfrak{p})$ is the *minimal parabolic subgroup* associated to Δ^+. Let $A^+ := \{a \in A \;/\; \forall\alpha \in \Delta^+ \;,\; \alpha(a) \geq 1\}$ be the corresponding *Weyl chamber* in A. One has the *Cartan decomposition* $G = KA^+K$. Let L be the centralizer of $\mathfrak{a}$ in G and $U^\pm$ be the connected groups with Lie algebra $\mathfrak{u}^\pm$. One has the equality $P = LU^+$.

Parabolic subgroups For every subset $\theta \subset \Pi$, one denotes by $<\theta>$ the vector space generated by θ, $\Delta_\theta = \Delta \cap <\theta>$, $\Delta_\theta^\pm = \Delta_\theta \cap \Delta^\pm$, $\mathfrak{l}_\theta = \mathfrak{l} \oplus \oplus_{\alpha\in\Delta_\theta}\mathfrak{g}_\alpha$, $\mathfrak{u}_\theta^\pm = \oplus_{\alpha\in\Delta^\pm-\Delta_\theta^\pm}\mathfrak{g}_\alpha$, $U_\theta^\pm$ the connected associated groups, $A_\theta = \{a \in A \;/\; \forall\alpha \in \theta \;,\; \alpha(a) = 1\}$, $A_\theta^+ = A^+ \cap A_\theta$, L_θ be the centralizer of A_θ in G. Let $\mathfrak{p}_\theta = \mathfrak{l}_\theta \oplus \mathfrak{u}_\theta^+$ and $P_\theta = L_\theta U_\theta^+$ be the *parabolic subalgebra and subgroup* associated to θ. One knows:
(1) *The quotient L/A is compact.*
(2) *Every group containing P is equal to some P_θ.*
(3) *P_θ is generated by the subgroups $P_{\{\alpha\}}$ for $\alpha \in \theta$.*

(4) *The multiplicitation $m : U^- \times P \to G$ is a diffeomorphism onto a open subset of full measure.*
(5) *If $\theta_1 \subset \theta_2$, then $\Delta_{\theta_1} \subset \Delta_{\theta_2}$, $P_{\theta_1} \subset P_{\theta_2}$ and $U^+_{\theta_1} \supset U^+_{\theta_2}$.*

Example $G = \mathrm{SL}(d, \mathbb{R})$. One can take
$K = \mathrm{SO}(d, \mathbb{R})$,
$A = \{a = \mathrm{diag}(a_1, \ldots, a_d) \,/\, a_i > 0\ ,\ a_1 \cdots a_d = 1\}$,
$A^+ = \{a \in A \,/\, a_1 \geq \cdots \geq a_d\}$,
$\Delta = \{\varepsilon_i - \varepsilon_j\ ,\ i \neq j\ ,\ 1 \leq i, j \leq d\}$,
$\Delta^+ = \{\varepsilon_i - \varepsilon_j\ ,\ 1 \leq i < j \leq d\}$,
$\Pi = \{\varepsilon_{i+1} - \varepsilon_i\ ,\ 1 \leq i < d\}$,
where $\varepsilon_i \in \mathfrak{a}^\star$ is the differential of the character of A denoted by the same symbol: $\varepsilon_i(a) = a_i$. The root spaces $\mathfrak{g}_{\varepsilon_i - \varepsilon_j}$ are 1-dimensional (with basis $E_{i,j} = e_j^\star \otimes e_i$) and one has $\mathfrak{l} = \mathfrak{a}$ [Note that these two properties are satisfied only for *split* semisimple Lie groups. They are not satisfied for $\mathrm{SO}(p, q)$ when $p \geq q + 2 \geq 3$]. One has then,

$$\mathfrak{u}^+ = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\} \quad , \quad \mathfrak{p} = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \quad , \quad \mathfrak{u}^- = \left\{ \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix} \right\} \quad ,$$

and, choosing θ^c with only 2 simple roots, one has in term of bloc matrices:

$$\mathfrak{u}_\theta^+ = \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\} \quad , \quad \mathfrak{p}_\theta = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \quad , \quad \mathfrak{u}_\theta^- = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix} \right\} \quad ,$$

$$\mathfrak{l}_\theta = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \quad , \quad A_\theta = \left\{ \begin{pmatrix} b_1 Id & 0 & 0 \\ 0 & b_2 Id & 0 \\ 0 & 0 & b_3 Id \end{pmatrix} \in A \right\}$$

and $A_\theta^+ = A_\theta \cap A^+$. Note that another value for θ would give different numbers and sizes of block matrices.

3.4 Coefficients decay for SL(d)

In this section we give the proof of theorem 3.2

We will need the following lemma which is a special case of the corollary 3.3 we have not yet proven.

Lemma 3.6 *Let π be a unitary representation of a connected quasisimple real Lie group with finite center G, $a \neq 1$ be a hyperbolic element of G and $u \neq 1$ be a unipotent element of G. If v is either a-invariant or u-invariant then it is G-invariant.*

Proof First case: v is a-invariant. One can suppose $a \in A^+$. Let $\theta = \{\alpha \in \Pi \,/\, \alpha(a) = 1\}$. The same argument as in proposition 3.4, shows that v is invariant by U_θ and U_θ^-. One conclude that v is G-invariant thanks to the following fact than the reader can easily check for $G = \mathrm{SL}(d, \mathbb{R})$: *the two groups U_θ and U_θ^- generate G.*

Second case: v is u-invariant. According to Jacobson-Morozov, there exists a Lie subgroup S of G containing u with Lie algebra $\mathfrak{s} \simeq \mathfrak{sl}(2, \mathbb{R})$. By proposition 3.4, v is S-invariant. Since S contains hyperbolic elements, we are back in the first case. ♢

Proof of theorem 3.2 If the coefficient $<\pi(g)v, w>$ does not decrease to 0, one can find sequences $g_n = k_{1,n} a_n k_{2,n} \in G = KA^+K$ such that

$$\lim_n <\pi(g_n)v, w> = \ell \neq 0 \ , \ \lim_n k_{1,n} = k_1 \ , \ \lim_n k_{2,n} = k_2$$

and for some $\alpha \in \Pi$, $\lim_n \alpha(a_n) = \infty$. One can suppose that $k_1 = k_2 = e$.

Using the weak compactness of the unit ball of $\mathcal{H}_\pi$ one can suppose that the sequence $\pi(a_n)v$ has a weak limit $v_0 \in \mathcal{H}_\pi$. This vector v_0 is nonzero since

$$<v_0, w> = \lim_n <\pi(a_n)\pi(k_{2,n})v, \pi(k_{1,n}^{-1})w> = \lim_n <\pi(g_n)v, w> \neq 0$$

Moreover, this vector is u-invariant for all $u \in U_{\{\alpha\}^c}$. Because, since $a_n^{-1} u a_n \to e$,

$$\|\pi(u)v_0 - v_0\| \leq \overline{\lim_n} \|\pi(a_n)(\pi(a_n^{-1} u a_n)v_0 - v_0)\| = 0 \ .$$

This contradicts lemma 3.6. ♢

3.5 Uniform coefficients decay

In this section, we prove that, for higher rank semisimple Lie group G and K-finite vectors, the coefficients decay is uniform.

For v in a unitary representation $\mathcal{H}_\pi$ of G, let

$$\delta(v) = \delta_K(v) = (\dim <Kv>)^{1/2} \in \mathbb{N} \cup \{\infty\}.$$

Theorem 3.7 (Howe) *Let G be a connected semisimple real Lie group with finite center, such that, for all normal subgroup $G_i \neq 1$ of G, one has $\mathrm{rank}_{\mathbb{R}}(G_i) \geq 2$. Then there exists a K-biinvariant function $\eta_G \in \mathrm{C}(G)$ satisfying $\lim_{g \to \infty} \eta_G(g) = 0$ and such that, for all unitary representation π of G with $\mathcal{H}_\pi^{G_i} = 0$, $\forall\, i$, for any $v, w \in \mathcal{H}_\pi$, with $\|v\| = \|w\| = 1$, one has, for $g \in G$,*

$$|<\pi(g)v, w>| \leq \eta_G(g)\delta(v)\delta(w).$$

Remark The (most often) best function η_G has been computed by H.Oh ([17]), thanks to Harish-Chandra's function

$$\xi(t) = (2\pi)^{-1} \int_0^{2\pi} (t \cos^2 s + t^{-1} \sin^2 s)^{-1/2} ds \qquad (4)$$

$$\asymp t^{-1/2} \log t \ \text{ for } \ t \gg 1 \ . \qquad (5)$$

For instance, for $G = \mathrm{SL}(d, \mathbb{R})$, $d \geq 3$, one can take, with $a = \mathrm{diag}(t_1, \ldots, t_d)$,

$$\eta_G(a) = \prod_{1 \leq i \leq [n/2]} \xi\Big(\frac{t_i}{t_{n+1-i}}\Big) \ . \tag{6}$$

The proof is based on the following two propositions

Definition 3.8 *Let σ, τ be unitary representations of G. One says that σ is weakly contained in τ, and one writes $\sigma \prec \tau$, if,*

$$\forall \varepsilon > 0,\ \forall C \text{ compact in } G,\ \forall v_1, \ldots, v_n \in \mathcal{H}_\sigma\ ,\ \exists w_1, \ldots, w_n \in \mathcal{H}_\tau\ /$$
$$|<\pi(g)v_i, v_j> - <\tau(g)w_i, w_j>| \leq \varepsilon\ \ ,\ \ \forall g \in C\ ,\ \forall i, j \leq n\ .$$

For $g = kan \in G$ let us denote $H(g) = a$ and let us introduce *Harish-Chandra spherical function* ξ_G given by

$$\xi_G(g) = \int_K \rho(H(gk))^{-1/2} dk \quad \text{with} \quad \rho(a) = \det_{\mathfrak{n}}(\mathrm{Ad}(a))\ .$$

The following proposition will be applied not directly to G but to a subgroup of G isomorphic to $\mathrm{SL}(2, \mathbb{R})^e$.

Proposition 3.9 *Let G be a connected real semisimple Lie group with finite center and π be a unitary representation of G which is weakly contained in the left regular representation λ_G. Then for $v, w \in \mathcal{H}_\pi$, with $\|v\| = \|w\| = 1$, $g \in G$, one has*

$$|<\pi(g)v, w>| \leq \xi_G(g)\, \delta_K(v)\, \delta_K(w) \tag{7}$$

Proof Let us first prove these inequalities for the left regular representation λ_G.

First note that,
For all v in $L^2(G)$, left K-finite, with $\|v\| = 1$, there exists a positive left K-invariant function $\varphi \in L^2(G)$ with $\|\varphi\| = 1$ such that, for all $x \in G$, one has $|v(x)| \leq \delta(v)\varphi(x)$.
One can take $\varphi(x) := \delta(v)^{-1}(\sum_i |v_i(x)|^2)^{1/2}$ where v_i is an orthogonal basis of $<Kv>$.

If ψ is the positive K-invariant function associated in the same way to w, one gets

$$\begin{aligned} |<\pi(g)v, w>| &\leq \int_G |v(g^{-1}x)w(x)| d\mu(x) \\ &\leq \delta(v)\delta(w) \int_G \varphi(g^{-1}x)\psi(x) d\mu(x) \leq \delta(v)\, \delta(w)\ <\pi(g)\varphi, \psi>\ . \end{aligned}$$

These functions $\varphi, \psi \in L^2(G)$ are left K-invariant, positive and of norm 1. We want to prove the majoration

$$|<\pi(g)\varphi, \psi>| \leq \xi_G(g)\ . \tag{8}$$

Using the formula for the Haar measure as in lemma 2.3, one computes,

$$
\begin{aligned}
|<\pi(g)\varphi,\psi>| &= \int_K \Big(\int_{AN} \varphi(an)\psi(gkan)\rho(a)dadn\Big)\,dk \\
&\le \|\varphi\|_{L^2} \int_K \Big(\int_{AN} \psi(H(gk)an)^2\rho(a)dadn\Big)^{1/2} dk \\
&= \|\varphi\|_{L^2}\|\psi\|_{L^2} \int_K \rho(H(gk))^{-1/2}dk.
\end{aligned}
$$

thanks to Cauchy-Schwarz inequality in $L^2(AN)$ and the K-invariance of φ and ψ.

Let us now deduce these inequalities for π. The main point is to show that, starting from a finite family v_i of vectors in $\mathcal{H}_\pi$ such that, $\forall k \in K$, $\forall i$, one has $\pi(k)v_i = \sum_j u_{i,j}(k)v_j$, then one can find vectors $w'_i \in L^2(G)$ as in definition 3.8 satisfying moreover $\lambda_G(k)w'_i = \sum_j u_{i,j}(k)w'_j$. For that, just replace the family w_i given in definition 3.8 by $w'_i = \int_K \sum_j u_{i,j}(k^{-1})\lambda_G(k)w_j$. ♢

Let us compute this function ξ_G for $G = \mathrm{SL}(2,\mathbb{R})^e$.
Let us show that for $a_t = (\mathrm{diag}(t_1^{1/2}, t_1^{-1/2}), \ldots, \mathrm{diag}(t_e^{1/2}, t_e^{-1/2})) \in A^+$, one has

$$\xi_G(a_t) = \xi(t_1)\cdots\xi(t_e). \tag{9}$$

For that, one can suppose $e = 1$, i.e. $G = \mathrm{SL}(2,\mathbb{R})$. Then for

$$k = \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \text{ and } a_t = \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix} \text{ one has}$$

$$\rho(H(a_t k)) = \|a_t k\, e_1\|^2 = t\cos^2 s + t^{-1}\sin^2 s\ .$$

And the formula (9) is a consequence of the definition (4). ♢

Proposition 3.10 *Let V be a finite dimensional representation of $G = \mathrm{SL}(2,\mathbb{R})$, without nonzero invariant vectors. Let π be an irreducible unitary representation of the semidirect product $V \rtimes G$ such that $\mathcal{H}_\pi^V = 0$. Then the restriction of π to G is weakly contained in the regular representation λ_G.*

Sketch of proof of proposition 3.10 By Mackey's theorem (see [13]), such a representation π of $V \rtimes G$ is induced from an irreducible representation σ of a proper subgroup H of $V \rtimes G$ containing V. Such a subgroup is solvable hence amenable, hence σ is weakly contained in the regular representation λ_H (see definition 4.1). Therefore π is weakly contained in $\lambda_{V\rtimes G}$. ♢

Proof of theorem 3.7 We will prove this theorem only for $SL(d,\mathbb{R})$, but with the bound given by (6). The proof of the general case is similar.

Let $e = [\frac{d}{2}]$ and

$$S = S_1 \times \cdots \times S_e \subset G = \mathrm{SL}(d,\mathbb{R})$$

where $S_i \simeq \mathrm{SL}(2,\mathbb{R})$ is the subgroup whose Lie algebra has basis $E_{i,j}, E_{j,i}, E_{i,i} - E_{j,j}$ with $j = n+1-i$. These subgroups commute. Let

$$B = A \cap S = \{a = \mathrm{diag}(t_1, \ldots, t_d) \in A \;/\; t_i t_{n+1-i} = 1 \;\; \forall\, i \leq e\} \;\text{ and}$$

$$C = \{a = \mathrm{diag}(t_1, \ldots, t_d) \in A \;/\; t_i = t_{n+1-i} \;\; \forall\, i \leq e\}\;.$$

For $g = k_1 a k_2 \in G = KA^+K$, write $a = bc$ with $b \in B$, $c \in C$.

Note that G contains some subgroups $G_i = V_i \rtimes S_i$ where $V_i \simeq \mathbb{R}^2$ has a nontrivial S_i-action. According to lemma 3.6, $\mathcal{H}_\pi$ does not contains any V_i-invariants vectors. Hence, by proposition 3.10, the restriction $\pi|_{S_i}$ is weakly contained in the regular representation $\infty\lambda_{S_i}$, hence, $\pi|_S$ is also weakly contained in $\infty\lambda_S$. Therefore, one can apply proposition 3.9 and formulas (6) and (9) to get the following upper bound, where $K_S = K \cap S$:

$$\begin{aligned} |<\pi(g)v, w>| &= |<\pi(b)\pi(ck_2)v, \pi(k_1^{-1})w>| \\ &\leq \xi_S(b)\, \delta_{K_S}(\pi(ck_2)v)\, \delta_{K_S}(\pi(k_1^{-1})w) \\ &= \eta_G(a)\, \delta_{K_S}(\pi(k_2)v)\, \delta_{K_S}(\pi(k_1^{-1})w) \\ &\leq \eta_G(a)\, \delta_K(v)\, \delta_K(w) \end{aligned}$$

because S and C commute. $\diamondsuit$

3.6 Property T

Definition 3.11 *One says that a continuous representation of a locally compact group G in a Banach space B almost has invariant vectors if, one has*

$$\forall \varepsilon > 0,\ \forall C \text{ compact in } G,\ \exists v \in B \;/\; \|v\| = 1 \text{ and } \forall g \in C,\ \|g\,v - v\| \leq \varepsilon\;.$$

Such a v is called (ε, C)-invariant.

One says that G has Kazhdan's property T if any unitary representation of G which almost has invariant vectors does have nonzero invariant vectors.

The main motivation for this definition are the following 3 propositions which are essentially due to Kazhdan.

Proposition 3.12 (Kazhdan) *Let G be a connected quasisimple real Lie group with finite center. Suppose that* $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$. *Then G has property T.*

Proposition 3.13 *Let Γ be a lattice in a locally compact group G.*
If G has property T, then Γ has property T too.

Proposition 3.14 *Let Γ be a discrete group with property T. Then*
a) the group Γ is finitely generated.
b) The quotient $\Gamma/[\Gamma,\Gamma]$ is finite.

As a consequence of these 3 propositions, one gets the main result in this section.

Corollary 3.15 *Let Γ be lattice in a connected quasisimple real Lie group with finite center. Suppose that* $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$. *Then Γ is finitely generated and $\Gamma/[\Gamma, \Gamma]$ is finite.*

Proof of proposition 3.12 Let π be a unitary representation which almost has G-invariant vectors. This means that for any $\varepsilon > 0$ and C compact of G, one can find a $(\varepsilon/2, C)$-invariant vector v in $\mathcal{H}_\pi$. One may suppose $\varepsilon < 1$ and C K-biinvariant, containing K and containing an element g with $\eta_G(g) \leq 1/2$.

The average $w = \int_K \pi(k) v \, dk$ is a K-invariant vector of norm $\|w\| \geq 1/2$ such that, for all g in C, $\|\pi(g)w - w\| \leq \varepsilon/2$. Hence the vector $v' := w/\|w\|$ is K-invariant and (ε, C)-invariant.

If $\mathcal{H}_\pi$ does not have G-invariant vectors, this contradicts the bound $c_{v',v'}(g) \leq \xi(g) \leq 1/2$ given by theorem 3.7. ♢

Proof of proposition 3.13 Let π be a unitary representation of Γ which almost has Γ-invariant vectors. We want to prove that it actually has a Γ-invariant vector. For that we will construct a representation σ of G, the *induced representation*, and show that it almost has G-invariant vectors.

Let μ be the right Haar measure on G such that $\mu(G/\Gamma) = 1$ and note that by conservation of the volume, this measure is also left invariant. One can choose a Borel subset $F \subset G$ such that the map $m : F \times \Gamma \to G; (x, \gamma) \mapsto x\gamma$ is a bijection. It is easy to choose F such that, for any compact $C \subset G$, $m^{-1}(C)$ is relatively compact in $G \times \Gamma$. Let us write, for $g \in G$ and $x \in G$,

$$m^{-1}(gx) = (x_g, g_x) \; .$$

The representation σ is given by

$$\mathcal{H}_\sigma = \{ f : G \to \mathcal{H}_\pi \text{ measurable} \quad / \quad \forall g \in G \; , \; \forall \gamma \in \Gamma \; , \; f(g) = \pi(\gamma) f(g\gamma) \text{ and } \int_F \|f(x)\|^2_{\mathcal{H}_\pi} d\mu < \infty \},$$

$$(\sigma(g^{-1})f)(x) = f(gx) = \pi(g_x) f(x_g).$$

Let C be a compact of G and $\varepsilon > 0$. Let $F_1 \subset F$ be a relatively compact subset of G such that $\mu(F - F_1) < \varepsilon^2/8$ and Γ_1 be the finite set $\Gamma_1 := \{g_x \; / \; g \in C \; , \; x \in F_1\}$. Let $v \in \mathcal{H}_\pi$ be an $(\varepsilon/2, \Gamma_1)$-invariant vector and define $f \in \mathcal{H}_\sigma$ by $f|_F = v$. This vector is (ε, C)-invariant, since one has the majoration, for all g in C,

$$\begin{aligned} \|\sigma(g^{-1})f - f\|^2_{\mathcal{H}_\sigma} &\leq \int_{F - F_1} 4 \, d\mu(x) + \int_{F_1} \|\pi(g_x) f(x_g) - f(x)\|^2_{\mathcal{H}_\pi} d\mu(x) \\ &\leq \varepsilon^2/2 + (\varepsilon/2)^2 \leq \varepsilon^2 \; . \end{aligned}$$

Since G has property T, $\mathcal{H}_\sigma$ contains a nonzero G-invariant vector f_0. This function is almost always equal to a nonzero vector $v_0 \in \mathcal{H}_\pi$ which is then Γ-invariant. ♢

Proof of proposition 3.14 a) Consider the unitary representation of Γ in the Hilbert direct sum $\oplus_\Delta \ell^2(\Gamma/\Delta)$ where Δ describe all finitely generated subgroups of Γ. The vectors $\delta_\Delta \in \ell^2(\Gamma/\Delta) \subset \mathcal{H}$ are $(0, \Delta)$-invariant. Hence, this representation almost has invariant vectors. Property T implies that $\mathcal{H}^\Gamma \neq 0$, hence, for some Δ, $\ell^2(\Gamma/\Delta)^\Gamma \neq 0$, Γ/Δ is finite and Γ is finitely generated.

b) Since Γ is finitely generated, if the quotient $\Gamma/[\Gamma, \Gamma]$ were infinite, there would exist a surjective morphism $\Gamma \to \mathbb{Z}$ and $\mathbb{Z}$ would have property T. However, the regular representation of $\mathbb{Z}$ in $\ell^2(\mathbb{Z})$ almost has invariant vectors, $v_n := n^{-1/2} \sum_{0 \leq k < n} \delta_k$, but has no invariant vector. Contradiction. $\diamondsuit$

3.7 Ergodicity

One of the main application of the coefficients decay is the ergodicity of some flows on the finite volume quotients G/Γ.

Proposition 3.16 *Let G be a connected semisimple real Lie group with finite center and H be a closed subgroup of G whose images in the factors $G/G_i \neq 1$ are noncompact. Let Γ be a lattice in G and ν be the G-invariant probability measure on $X := G/\Gamma$.*

Then the action of H on X is ergodic and mixing.

Remarks - *ergodicity* means that any H-invariant measurable subset A of X satisfies $\nu(A) = 0$ or 1.
- *mixing* is a stronger property. It means that $\forall A, B \subset X$, $\lim_{h \to \infty} \nu(A \cap hB) = \nu(A)\nu(B)$.
- Note that the ergodicity of the geodesic flow or of the horocycle foliation for the associated locally symmetric space $K \backslash G/\Gamma$ is a special case of this corollary.

Proof Let $A \subset X$ be an H-invariant measurable subset. Let π be the unitary representation of G in $L^2_0(X) := \{f \in L^2(X, \nu) \;/\; \int_X f d\nu = 0\}$. One has $L^2_0(X)^G = 0$. The vector $v_A = \mathbf{1}_A - \nu(A)$ is an H-invariant vector in $L^2_0(X)$. By the corollary 3.3, one has $v_A = 0$. Hence $\nu(A) = 0$ or 1. This proves ergodicity. Mixing uses the same proof with theorem 3.2 and the equality

$$\nu(hA \cap B) - \nu(A)\nu(B) = <\pi(h)v_A, v_B>. \qquad \diamondsuit$$

Corollary 3.17 *Let Γ be a lattice in a quasisimple real Lie group G and $a \in A^+$, $a \neq 1$. Then, for almost all $x \in G/\Gamma$, the semi-orbit $\{a^n x \;,\; n \geq 0\}$ is dense in G/Γ.*

Proof The density of almost all quasi-orbits is a classical consequence of ergodicity: since G/Γ is metrisable separable, it is enough to show that, for almost all open set O, the union $\cup_{n \geq 0} a^{-n} O$ is of full measure. This follows from its a-invariance and from ergodicity. $\diamondsuit$

Remark The mixing speed can be estimated thanks to the uniform decay of coefficients given in theorem 3.7.

Lecture 4: Boundaries

The aim of this lecture is to show how measurable Γ-equivariant maps between "boundaries" can be used to prove some algebraic properties for a lattice Γ in a higher rank simple Lie group G.

We will prove a theorem of Kazhdan and Margulis which says that Γ is almost simple, i.e. any normal subgroup of Γ is either finite or of finite index.

Note that the same tool is at the heart of the proof of Margulis superrigidity theorem, that we will not discuss here.

4.1 Amenability

Let us recall a few different equivalent definitions of amenability.

In this section G is a locally compact (metrisable and separable) space and μ is a left Haar measure on G. Let

$$\mathrm{UCB}(G) = \{f \in L^\infty(G) \ / \ \lim_{y\to e} \sup_{x\in G} |f(yx) - f(x)| = 0\}$$

be the set of bounded functions $f : G \to \mathbb{C}$ which are left uniformly continuous.

A *mean* on $L^\infty(G)$ or on $\mathrm{UCB}(G)$ is a linear form m such that

$$m(1) = 1 \text{ and } (f \geq 0 \Rightarrow m(f) \geq 0).$$

Such a m is real, i.e. $m(\mathrm{Re}(f)) = \mathrm{Re}(m(f))$ and continuous: $|m(f)| \leq \|f\|_\infty$.

Let 1_G be the trivial representation and λ_G be the regular representation of G and $\infty\lambda_G$ the Hilbert direct sum of infinitely many copies of λ_G.

Definition 4.1 (Godement, Hulanicki) *A locally compact goup G is amenable if it satisfies one of the following equivalent properties.*
(1) *Every continuous action of G on a compact space X, has an invariant probability measure on X.*
(1′) *Same with X metrisable.*
(2) *Every continuous affine action of G on a compact convex subset A of a Hausdorff locally convex topological vector space E has a fixed point in A.*
(2′) *Same with E metrisable.*
(3) $\mathrm{UCB}(G)$ *has a left-invariant mean.*
(4) $L^\infty(G)$ *has a left-invariant mean.*
(5) $L^1(G)$ *almost has left-invariant vectors.*
(6) $L^2(G)$ *almost has left-invariant vectors i.e.* $1_G \prec \lambda_G$ *(see definition 3.8).*
(7) *For every irreducible unitary representation π of G, one has $\pi_G \prec \infty\lambda_G$.*
(8) $1_G \prec \infty\lambda_G$.

At first glance, these eight properties look quite different. After a careful rereading, one realizes that all of them have to do with fixed points or almost fixed point of some G-actions.

Using these definitions, one gets easily:

Examples - The groups $\mathbb{Z}$ and $\mathbb{R}$ are amenable. Note that these two groups do not have invariant probability measures. Examples of invariant means on $L^\infty(\mathbb{Z})$ are given by taking limits with respect to some ultrafilter of $\mathbb{Z}$. The invariant means are not unique. Property (2) for $\mathbb{Z}$ is an old result of Kakutani, whose proof is the following sentence: one can get a fixed point as a cluster point of the sequence of barycenters of the n first points of an orbit of $\mathbb{Z}$ in A. Property (6) for $\mathbb{Z}$ has already been proven in the proof of proposition 3.14.
- Any compact group is amenable: use property (6).
- Any extension of two amenable groups is amenable: use property (2).
- Any solvable Lie group is amenable.
- A noncompact semisimple real Lie group is not amenable: property (1) is not satisfied since the action of G on G/P does not have any invariant probability measure.

The main tools in the proof of these equivalences are those provided by the classical functional analysis.

<u>Let us prove (1) $\Leftrightarrow$ (1′) $\Leftrightarrow$ (2) $\Leftrightarrow$ (2′).</u>
(1′) $\Rightarrow$ (2′) The barycenter of a G-invariant probability measure on A is a fixed point.
(2′) $\Rightarrow$ (1) Let $F = \mathrm{C}(X) = \{$continous functions on $X\}$, $E = F^\star = \mathcal{M}(X) = \{$bounded measures on $X\}$ and $A = \mathcal{P}(X) = \{$probabilities on $X\} \subset E$. The action of G on A is continuous and affine. One does not know wether E is metrisable or not but, since G is separable, one can write $F = \cup F_\alpha$ where the F_α are the separable closed G-invariant vector subspaces of F. Let $p_\alpha : E \to F_\alpha^\star$ the restriction to F_α. Since $F_\alpha^\star$ is metrisable, G has a fixed point in $p_\alpha(A)$. The intersection of the family of nonempty compact sets $A_\alpha := \{a \in A \ / \ p_\alpha(a)$ is G-invariant$\}$ is also nonempty. It is the set of fixed points of G in A.
(1) $\Rightarrow$ (2) Same as (1′) $\Rightarrow$ (2′) without the metrisability hypothesis.
(2) $\Rightarrow$ (1′) Same as (2′) $\Rightarrow$ (1) but easier.

<u>Let us prove the equivalences of (3) and (4) with the previous ones.</u>
(2) $\Rightarrow$ (3) The action of G on UCB(G) by left-translations: $\pi(g)f(x) = f(g^{-1}x)$, $\forall f \in$ UCB(G), $\forall g, x \in G$ is continuous. Hence the action on the set A of means on UCB(G) is also continuous. This set is closed, convex and bounded hence weakly compact. Any fixed point of this action is an invariant mean.
(3) $\Rightarrow$ (4) First recall a few basic definitions and properties of convolution.
For $\nu \in \mathcal{P}(G)$, $F \in L^\infty(G)$, $f \in$ UCB(G), $\alpha \in \mathrm{C}_c(G)$ and $x \in G$, one has

$$\nu \star F\,(x) = \textstyle\int_G F(y^{-1}x)d\nu(y)\ ,$$
$$\alpha \star f\,(x) = \textstyle\int_G \alpha(y)f(y^{-1}x)d\mu(y).$$

Recall that one has $\nu\star F \in L^\infty(G)$ and $\alpha\star F \in$ UCB(G). Moreover, for any approximation of identity $\beta_n \in \mathrm{C}_c(G)$, the sequence $\beta_n\star f$ converges uniformly to f. One has also similar statements with $F \star \alpha$ and $f \star \beta_n$.

We will show the following assertion

Lemma 4.2 *If* UCB(G) *has a left invariant mean* m*, then there exists an invariant mean* $\widetilde{m}$ *on* $L^\infty(G)$ *such that* $\forall\alpha \in \mathrm{C}_c(G)$ *with* $\int_G \alpha d\mu = 1$*,* $\forall F \in L^\infty(G)$*, one has* $\widetilde{m}(\alpha \star F) = \widetilde{m}(F)$*.*

Proof Notice first that, since $\alpha \star f$ is a uniform limit of averages of translates of f one has $m(\alpha \star f) = m(f)$ for any $f \in$ UCB(G). Then fix an element $\alpha_0 \in \mathrm{C}_c(G)$ such that $\int_G \alpha_0 d\mu = 1$ and define a mean $\widetilde{m}$ on $L^\infty(G)$ by $\tilde{m}(F) = m(\alpha_0 \star F)$. One computes

$$\begin{aligned}\widetilde{m}(\alpha \star F) &= \lim_n m(\alpha_0 \star \alpha \star F \star \beta_n) = \lim_n m(F \star \beta_n) = \lim_n m(\alpha_0 \star F \star \beta_n)\\ &= m(\alpha_0 \star F) = \widetilde{m}(F).\end{aligned}$$

Now we will infer the invariance of $\widetilde{m}$. Define as above $\pi(g)F : x \to F(g^{-1}x)$, note that $\alpha \star \pi(g)F = \alpha_g \star F$ where $\alpha_g(x) = \Delta(g)^{-1}\alpha(xg^{-1})$ with $\Delta =$ the modulus function, and compute

$$\widetilde{m}(\pi(g)F) = \widetilde{m}(\alpha \star \pi(g)F) = \widetilde{m}(\alpha_g \star F) = \widetilde{m}(F). \qquad \diamondsuit$$

(4) $\Rightarrow$ (1) Let m be an invariant mean on $L^\infty(G)$. Fix a point $x_0 \in X$ and associate to any function $h \in \mathrm{C}(X)$ a function $\widetilde{h} \in L^\infty(G)$ by $\widetilde{h}(g) = f(g\,x_0)$. The formula $\mu(h) = m(\widetilde{h})$ defines then a G-invariant probability measure on X.

Let us prove the equivalences of (5) with the previous ones.

(3) $\Rightarrow$ (5) Recall that $L^\infty(G) \simeq L^1(G)^\star$ and denote by $\varphi \to m_\varphi$ the natural injection of $L^1(G)$ in the dual of $L^\infty(G)$ given by $m_\varphi(F) = \int_G \varphi F d\mu$ and let

$$P(G) := \{\varphi \in L^1(G) \;/\; \varphi \geq 0 \;\text{ and }\; \int_G \varphi d\mu = 1\}.$$

Let us first show that $P(G)$ *is weakly dense in the set of means on* $L^\infty(G)$*.* For that notice that, if we endow $L^\infty(G)^\star$ with the weak topology, its dual is $L^\infty(G)$. If a mean m were not in the weak closure of the convex set $P(G)$, Hahn-Banach theorem would give an element $F \in L^\infty(G)$ such that $m(F) > \sup_{\varphi\in P(G)} \int_G \varphi F d\mu = \|F\|_{L^\infty}$. Contradiction.

Choose a mean $\widetilde{m}$ on $L^\infty(G)$ as in lemma 4.2. Let $\varphi_j \in P(G)$ be a filter such that m_{φ_j} converges to $\widetilde{m}$. Note that, since $L^\infty(G)^*$ is not metrisable, one has to use filters instead of sequences. From the equalities $m_{\alpha\star\varphi_j}(F) = m_{\varphi_j}(\alpha' \star F)$ with $\alpha'(g) = \Delta(g)^{-1}\alpha(g^{-1})$, one deduces that

$\alpha \star \varphi_j - \varphi_j$ *weakly converges to* 0*, for all* $\alpha \in \mathrm{C}_c(G)$ *with* $\int_G \alpha d\mu = 1$.

Let us show that *one can choose* φ_j *such that this convergence is strong.* Let $\prod_\alpha L^1(G)$ be the product of infinitely many copies of $L^1(G)$ indexed by all test functions α of integral equal to 1. Its dual is the direct sum $\oplus_\alpha L^\infty(G)$. Let $T : L^1(G) \longrightarrow \prod_\alpha L^1(G)$ be the linear map defined by $T(\varphi)_\alpha = \alpha \star \varphi - \varphi$.

We have shown that 0 is in the weak closure of the convex set $T(P(G))$. Hahn-Banach theorem implies that the weak closure of a convex set is equal to its strong closure. Hence, there is a filter still denoted $\varphi_j \in P(G)$ such that,

$$\|\alpha \star \varphi_j - \varphi_j\|_{L^1} \text{ converges to 0, for all } \alpha \in \mathrm{C}_c(G) \text{ with } \int_G \alpha d\mu = 1. \qquad (10)$$

Let $\varepsilon > 0$ and C be a compact of G. Recall that we want to find an element $\varphi \in P(G)$ such that, for all $g \in K$, one has $\|\pi(g)\varphi - \varphi\|_{L^1} \leq \varepsilon$. Let us fix some $\beta \in P(G)$, we will show that some $\varphi = \beta \star \varphi_j$ works.

By continuity of the left translation in $L^1(G)$, there exists an open neighborhood $\mathcal{U}$ of e in G such that, for all $y \in \mathcal{U}$, one has $\|\pi(y)\beta - \beta\|_{L^1(G)} \leq \varepsilon/3$.

Let us choose a covering of C by finitely many translates $x_1\mathcal{U}, \ldots, x_n\mathcal{U}$. Thanks to (10), one can find j such that, $\|(\pi(x_i)\beta) \star \varphi_j - \varphi_j\|_{L^1} \leq \varepsilon/3$, $\forall\, i = 1, \ldots, n$. Then writing $g = x_i y$ with $i \leq n$ and $y \in E$, one gets
$\|\pi(g)\varphi - \varphi\|_{L^1}$
$\leq \|(\pi(x_i y)\beta) \star \varphi_j - (\pi(x_i)\beta) \star \varphi_j\|_{L^1} + \|(\pi(x_i)\beta) \star \varphi_j - \varphi_j\|_{L^1} + \|\varphi_j - \beta \star \varphi_j\|_{L^1}$
$\leq \|(\pi(y)\beta) \star \varphi_j - \beta \star \varphi_j\|_{L^1} + 2\varepsilon/3 \leq \|\pi(y)\beta - \beta\|_{L^1}\, \|\varphi_j\|_{L^1} + 2\varepsilon/3 \leq \varepsilon.$

(5) $\Rightarrow$ (4) By hypothesis, there exists a sequence $\varphi_j \in L^1(G)$ such that $\|\varphi_j\|_{L^1} = 1$ and, $\forall g \in G$, $\lim_{j\to\infty} \|\pi(g)\varphi_j - \varphi_j\|_{L^1} = 0$. Replacing φ_j by $|\varphi_j|$, one may suppose that $\varphi_j \in P(G)$. Since the set of means on $L^\infty(G)$ is closed and bounded, it is weakly compact. Any cluster value m of the sequence φ_j is an invariant mean since, $\forall F \in L^\infty(G)$, one has

$$\begin{aligned} |m(\pi(g)F - F)| &= \lim \left| m_{\varphi_j}(\pi(g)F - F) \right| = \lim \left| \int_G (\pi(g^{-1})\varphi_j - \varphi_j) F d\mu \right| \\ &\leq \overline{\lim}\, \|F\|_{L^\infty} \|\pi(g^{-1})\varphi_j - \varphi_j\|_{L^1} = 0. \end{aligned}$$

Let us prove the equivalences of (6), (7) and (8) with the previous ones.

(5) $\Rightarrow$ (6) If φ is a norm 1 vector of $L^1(G)$ such that $\|\pi(g)\varphi - \varphi\|_{L^1} \leq \varepsilon$, then $\psi := |\varphi|^{\frac{1}{2}}$ is a norm 1 vector of $L^2(G)$ such that $\|\pi(g)\psi - \psi\|_{L^2} \leq \varepsilon$.

(6) $\Rightarrow$ (5) If ψ is a norm 1 vector of $L^2(G)$ such that $\|\pi(g)\psi - \psi\|_{L^2} \leq \varepsilon$, then $\varphi := |\psi|^2$ is a norm 1 vector of $L^1(G)$ such that, by Cauchy-Schwarz:
$\|\pi(g)\varphi - \varphi\|_{L^1} \leq \|\pi(g)|\psi| + |\psi|\|_{L^2}\, \|\pi(g)|\psi| - |\psi|\|_{L^2} \leq 2\varepsilon.$

(6) $\Rightarrow$ (7) The operator $U : L^2(G) \otimes \mathcal{H}_\pi \to L^2(G, \mathcal{H}_\pi)$ given by $U(\psi \otimes v)(x) = \psi(x)\pi(x^{-1})v$ defines a unitary equivalence between $\lambda_G \otimes \pi$ and the representation of G in $L^2(G, \mathcal{H}_\pi)$ given by $(g\, F)(x) = F(g^{-1}x)$ which is equivalent to $\dim(\mathcal{H}_\pi)\, \lambda_G$. Therefore, if 1_G is weakly contained in λ_G, then $\pi = 1 \otimes \pi$ is weakly contained in $\lambda_G \otimes \pi$ hence in $\infty\, \lambda_G$.

(7) $\Rightarrow$ (8) Clear.

(8) $\Rightarrow$ (4) Let us first show that *the function* $1 \in L^\infty(G)$ *is in the weak closure of the set* $\mathcal{C}$ *of coefficients* $c_{\psi,\psi}$ *of functions* $\psi \in L^2(G)$*, with* $\|\psi\|_{L^2} = 1$. Our hypothesis means that, for all $\varepsilon > 0$ and all compact C of G, there exists a sequence $\psi_i \in L^2(G)$ of elements of norm 1 and a sequence $a_i \in \mathbb{C}$ such that $\sum_i |a_i|^2 = 1$ and, for all $g \in C$, $|1 - \sum_i |a_i|^2 c_{\psi_i,\psi_i}(g)| \leq \varepsilon$. Hence the function 1 is in the weak closure of the closed convex hull $co(\mathcal{C})$ of $\mathcal{C}$. But the funcion 1 is an extremal point in the unit ball of $L^\infty(G)$. Hence it is also an extremal point of $co(\mathcal{C})$. Such a point is in the weak closure of $\mathcal{C}$ (this is almost $1_G \prec \lambda_G$ but not quite).

Otherwise stated, we have found a sequence $\psi_j \in L^2(G)$ with $\|\psi_j\|_{L^2} = 1$ such that the following sequence of elements of $L^\infty(G)$: $g \mapsto \|\pi(g)\psi_j - \psi_j\|_{L^2}$ weakly converges to 0. Let $\varphi_j = |\psi_j|^2 \in L^1(G)$. The same argument than (6) $\Rightarrow$ (5) shows that the sequence of elements of $L^\infty(G)$: $g \mapsto \|\pi(g)\varphi_j - \varphi_j\|_{L^1}$ weakly converges to 0. This hypothesis is just enough to follow the arguments of the implication (5) $\Rightarrow$ (4). Therefore, every mean m in the closure of the sequence m_{φ_j} is G-invariant. $\diamondsuit$

4.2 Normal subgroup theorem

The following theorem is the aim of this lecture.

Theorem 4.3 (Kazhdan, Margulis) *Let G be a real linear quasisimple Lie group. Suppose that* $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$. *Then any lattice Γ in G is quasisimple, i.e. every normal subgroup N of Γ is either finite or of finite index.*

Remark This theorem is still true for G semisimple if the lattice Γ is irreducible. But the following proof has to be modified when G has a factor of real rank 1.

Sketch of proof of theorem 4.3 The main idea hidden behind the proof of this theorem is to consider the σ-algebra $\mathfrak{M}(Y)^N$ of N-invariant Borel subsets of $Y = G/P$, up to negligeable ones. Since N is normal in Γ, this σ-algebra is Γ-invariant. One first prove that any Γ-invariant σ-algebra of G/P is the inverse image of the σ-algebra of all Borel subsets on G/P' for a bigger group $P' \supset P$ (theorem 4.5 and lemma 4.6).

When $\mathfrak{M}(Y)^N$ is non trivial, i.e. when $P' \neq G$, then N acts trivially on G/P' and N is in the center of G.

When $\mathfrak{M}(Y)^N$ is trivial, one shows that Γ/N is amenable. For that, one constructs, for every continuous action of Γ/N on a compact metrisable space X, a boundary map, i.e. a measurable Γ-equivariant map

$$\Phi : G/P \longrightarrow \mathcal{P}(X).$$

Since $\mathfrak{M}(Y)^N$ is trivial, such a boundary map must be constant and the image is a Γ/N-invariant probability measure on X which proves the amenability of Γ/N. Thanks to property T, one deduces that Γ/N is finite.

The detailed proof will last up to the end of this lecture.

4.3 The boundary map

Starting from any action of Γ on a compact space X, one constructs a *boundary map.*

Proposition 4.4 (Furstenberg) *Let Γ be a lattice in a semisimple Lie group G acting continuously on a compact metrisable space X and P be a minimal parabolic subgroup of G.*

Then there exist a measurable Γ-equivariant map $\Phi : G/P \to \mathcal{P}(X)$

- Recall that $\mathrm{C}(X) = \{$continuous functions on $X\}$, $\mathcal{M}(X) = \{$bounded measures on $X\}$ and $\mathcal{P}(X) = \{$probabilities on$X\}$.

- We endowed implicitly G/P with a G-quasiinvariant measure, for instance a K-invariant measure.
- *Measurable* means: for every Borel subset E of the compact metrisable space $\mathcal{P}(X)$, the pullback $\Phi^{-1}(E)$ is measurable in G/P i.e. is equal to a Borel subset of G/P up to some negligeable set.
- Γ-*equivariant* means: for all γ in Γ and almost all x in G/P, one has $\Phi(\gamma x) = \gamma\Phi(x)$.
- The map Φ is called *boundary map*, since G/P may be thought as a boundary of the symmetric space G/K.

Proof Let $F = L^1_\Gamma(G, \mathrm{C}(X))$ be the space of Γ-equivariant measurable maps $f : G \to \mathrm{C}(X)$ such that $\|f\| := \int_{\Gamma\backslash G} \|f(g)\|_\infty dg < \infty$. Let $E = L^\infty_\Gamma(G, \mathcal{M}(X))$ be the space of bounded, Γ-equivariant measurable maps $m : G \to \mathcal{M}(X)$. The duality

$$<m, f>= \int_{\Gamma\backslash G} <m(g), f(g)> dg$$

gives an identification of E with the continuous dual of F, because if Y is a fundamental domain of Γ in G, one has $F \simeq L^1(Y, \mathrm{C}(X))$ and $E \simeq L^\infty(Y, \mathrm{C}(X)^\star) \simeq F^\star$. The subset $A = L^\infty_\Gamma(G, \mathcal{P}(X)) \subset E$ is convex, closed and bounded, hence is weakly compact. The right translation on G induces continuous actions of G on F, E and A.

Since P is a compact extension of a solvable group, it is amenable and hence has a fixed point Φ in A. This Φ is the required measurable map, since a P-invariant element of E is almost surely equal to a measurable function which is constant on the orbits of P. $\diamondsuit$

4.4 Quotients of G/P

For every measured space (Z, μ) where μ is a σ-finite measure, one denotes $\mathfrak{M}(Z) = \mathfrak{M}(Z, \mu)$ the σ-algebra of measurable subsets of Z modulo those of zero measure. Otherwise stated,

$$\mathfrak{M}(Z) \simeq \{f \in L^\infty(Z) \ / \ f^2 = f\}.$$

The theorem 4.3 will be a consequence of the following theorem that we will show in the next sections.

Theorem 4.5 (Margulis) *Let G be a quasisimple Lie group of real rank at least 2, Γ be a lattice in G, P be a minimal parabolic subgroup and $\mathfrak{M} \subset \mathfrak{M}(G/P)$ be a Γ-invariant sub-σ-algebra. Then $\mathfrak{M}$ is G-invariant.*

Remarks - This theorem is still true for G semisimple if the lattice Γ is irreducible.
- Note that, by proposition 3.16, the action of Γ on G/P is ergodic, i.e. any Γ-invariant Borel subset of G/P is of zero or full measure. Theorem 4.5 is a far-reaching extension of this assertion.
- When $\mathrm{rank}_{\mathbb{R}}(G) = 1$, any cocompact lattice Γ in G contains an infinite normal subgroup N such that Γ/N is non amenable. Then the σ-algebra $\mathcal{M}^N$ is Γ-invariant but not G-invariant. The younger reader will check the existence of N when Γ is the π_1 of a compact

surface, noting that such a group has a nonabelian free quotient. The more advanced reader will notice that this property is true for any Gromov hyperbolic group.

The following lemma, which is true for any separable locally compact group G, emphasizes the conclusion of theorem 4.5.

For a closed subgroup H of G, the σ-algebra $\mathfrak{M}(G/H)$ can be identified with the σ-algebra $\mathfrak{M}(G, H)$ of Borel right H-invariant subset of G.

Lemma 4.6 *For all left G-invariant σ-algebra of $\mathfrak{M}(G)$, there exists a closed subgrooup H of G such that $\mathfrak{M} = \mathfrak{M}(G, H)$.*

Remark For all closed subgroups $H_1, H_2 \subset G$, one has the equivalence

$$H_1 \subset H_2 \Longleftrightarrow \mathfrak{M}(G, H_1) \supset \mathfrak{M}(G, H_2).$$

Therefore, the σ-algebra generated by $\mathfrak{M}(G, H_1)$ and $\mathfrak{M}(G, H_2)$ is $\mathfrak{M}(G, H_1 \cap H_2)$.

Proof of lemma 4.6 Let $\Omega := L^\infty(G, \mathfrak{M}) \subset L^\infty(G)$ be the subspace of $\mathfrak{M}$-measurable bounded functions. Recall the *topology of convergence in measure* on $L^\infty(G)$, i.e. the topology of uniform convergence outside a suitable subset of arbitrarily small measure. The sets

$$O_{C,\varepsilon}(f) = \{g \in L^\infty(G) \;/\; \mu(\{x \in C \;/\; |g(x) - f(x)| > \varepsilon) < \varepsilon\},$$

for $C \subset G$ of finite measure and $\varepsilon > 0$ constitute a basis of neighborhood of an element $f \in L^\infty(G)$. Let $\Omega_0 := \Omega \cap \mathrm{C}(G)$. One checks successively

- Ω is closed in $L^\infty(G)$ for the convergence in measure.
- $\forall \varphi \in \mathrm{C}_c(G)$, $\forall f \in \Omega$, $\varphi \star f \in \Omega_0$.
- Ω_0 is dense in Ω for the convergence in measure.

For x in G, the closed subgroup $H_x = \{h \in G \;/\; f(xh) = f(x) \; \forall f \in \Omega_0\}$ does not depend on x because $\mathfrak{M}$ is G-invariant. Let us write $H = H_x$. By definition, Ω_0 is a subalgebra of $\mathrm{C}(G/H)$ which is closed for the topology of uniform convergence on all compacts and which separates points. The Stone-Weierstrass theorem shows then that $\Omega_0 \simeq \mathrm{C}(G/H)$. Hence, one has $\Omega = \mathfrak{M}(G, H)$. $\diamondsuit$

Proof of (theorem 4.5 $\Rightarrow$ theorem 4.3)

First case: Γ/N is amenable. Since $rank_{\mathbb{R}} G \geq 2$, G has property T. Then, by proposition 3.13, Γ and its quotient Γ/N have also property T. But an amenable group with property T is compact, because, since the regular representation must have an invariant vector, its Haar measure is finite. In our case Γ/N is finite.

Second case: Γ/N is not amenable. There exists then a continuous action of Γ/N on a compact metrisable space X with no invariant probability measure. According to proposition 4.4, there exists a measurable Γ-equivariant map $\Phi : G/P \longrightarrow \mathcal{P}(X)$ which is not essentially constant.

Let $\mathfrak{M} := \{\Phi^{-1}(A) \;/\; A$ Borel subset of $\mathcal{P}(X)\}$ modulo the negligeable subsets. This σ-algebra $\mathfrak{M}$ on G/P is Γ-invariant and all $M \in \mathfrak{M}$ are N-invariant. According to theorem

4.5 and lemma 4.6, there exists a subgroup $P' \neq G$ such that $\mathfrak{M} = \mathfrak{M}(G, P')$. But then all the Borel subset of G/P' are N-invariant, hence, the action of N on G/P' is trivial and N is included in $Z(G) := \bigcap_{g \in G} gP'g^{-1}$, the center of G since G is quasisimple. ♢

4.5 Contracting automorphisms

The proof of theorem 4.5 relies on the following proposition which will be proved in the next section.

Proposition 4.7 *Let H be a separable locally compact group, $\varphi : H \to H$ be a contracting automorphism and $E \subset H$ be a measurable Borel subset.*

Then, for almost all h in H, one has the following convergence in measure:

$$\lim_{n\to\infty} \varphi^{-n}(hE) = \begin{cases} H & \text{if } \ hE \ni e\ , \\ \emptyset & \text{if } \ hE \not\ni e\ . \end{cases}$$

Remarks - *Contracting* means that any compact of H can be sent on any neighborhood of e by φ^n if n is sufficiently large.
- *Convergence in measure* means convergence in measure of the characteristic functions.
- To get a little feeling of what happens think of the extreme cases when hE contains or avoid a neighborhood of e.

Let us denote $L_\theta^- = U^- \cap L_\theta$, so that one has $U^- = U_\theta^- L_\theta^-$. For every subset $E \subset U^-$, let us denote

$$\psi_\theta(E) = U_\theta^-\,(E \cap L_\theta^-).$$

Corollary 4.8 *Let $a \in A^+$, $\theta := \{\alpha \in \Pi \ / \ \alpha(a) = 1\}$ and $M \subset U^-$ be a Borel subset. Then, for almost all u in U^-, one has the following convergence in measure on U^-:*

$$\lim_{n\to\infty} a^{-n}uMa^n = \psi_\theta(uM).$$

Proof Since we can replace M by ℓM with ℓ in L_θ^-, it is enough to prove this assertion, for almost all u in U_θ^-. Let $M_\ell := M \cap U_\theta^- \ell$ be the fibers of the projection of M on L_θ^-.

By definition, the conjugation by a is a contracting automorphism of U_θ^-. According to proposition 4.7, for all ℓ in L_θ^- and almost all u in U_θ^-, one has

$$\lim_{n\to\infty} (a^{-n}uMa^n)_\ell = (\psi_\theta(uM))_\ell$$

for the convergence in measure on U_θ^-. Hence, for almost all u in U_θ^-, this assertion is true for almost all ℓ in L_θ^-. For such a u, Fubini's theorem and Lebesgue's dominated convergence theorem allows us to conclude that $\lim\limits_{n\to\infty} a^{-n}uMa^n = \psi_\theta(uM)$ for the convergence in measure on U_θ^-. ♢

Corollary 4.9 *Let $\mathfrak{M} \subset \mathfrak{M}(G/P)$ be a Γ-invariant sub-σ-algebra, $M \in \mathfrak{M}$ and $\theta \subset \Pi$, with $\theta \neq \Pi$. Then, for almost all u in U^-, one has, for all g in G, $g\psi_\theta(uM) \in \mathfrak{M}$.*

The corollary 4.9 is an important step toward theorem 4.5. It allows us to construct in any Γ-invariant σ-algebra $\mathfrak{M} \subset \mathfrak{M}(G/P)$, a G-invariant sub-σ-algebra $\mathfrak{M}_0 \subset \mathfrak{M}$. To have a chance that this σ-algebra $\mathfrak{M}_0$ is non trivial, we will need to find $\theta \subset \Pi$, with $\emptyset \neq \theta \neq \Pi$, because, for every Borel subset $E \subset G/P$, one has $\psi_\emptyset(E) = \emptyset$ or G/P.
This explains the higher-rank hypothesis in theorem 4.5

To prove this corollary, will need the following lemma

Lemma 4.10 *Let Γ be a lattice in a quasisimple real Lie group G and $a \in A^+$, $a \neq 1$. Then, for almost all $u \in U^-$, the semi-orbit $\{\Gamma u^{-1} a^n \ , \ n \geq 0\}$ is dense in $\Gamma\backslash G$.*

Proof of lemma 4.10 This lemma is a consequence of corollary 3.17, after exchanging right and left. However, one needs one more argument because the almost all statement is relative to the Lebesgue measure of U^-. For that, just notice that U^-P is of full measure in G, and that for any $p \in P$, since the limit $\ell := \lim\limits_{n\to\infty} a^{-n}pa^n$ exists, one has the equivalence: ($\{\Gamma u a^n \ , \ n \geq 0\}$ is dense) $\iff$ ($\{\Gamma u p a^n \ , \ n \geq 0\}$ is dense). $\diamondsuit$

Proof of corollary 4.9 Let $a \in A^+$ such that one has $\theta := \{\alpha \in \Pi \ / \ \alpha(a) = 1\}$. Since $\theta \neq \Pi$, one has $a \neq 1$. According to the point (4) in section 3.3, one has an identification $\mathfrak{M}(U^-) \simeq \mathfrak{M}(G/P)$. We will prove that the assertion is true for all u satisfying the conclusion of corollary 4.8 and lemma 4.10.
Let g in G, one can write $g = \lim\limits_{i\to\infty} g_i$ where $g_i = \gamma_{n_i} u^{-1} a^{n_i}$ with $\gamma_{n_i} \in \Gamma$ and $n_i \to \infty$. According to corollary 4.8, the Borel subsets $a^{-n_i}uM = a^{-n_i}uMa^{n_i}$ converges in measure to $\psi_\theta(uM)$. Hence $\gamma_{n_i}M = g_i a^{-n_i}uM$ converges in measure to $g\psi_\theta(uM)$. Since $\gamma_{n_i}M$ is in $\mathfrak{M}$, $g\psi_\theta(uM)$ too. $\diamondsuit$

Proof of (corollary 4.9 $\Rightarrow$ theorem 4.5) Let θ_1 be a minimal subset of Π such that $\mathfrak{M} \supset \mathfrak{M}(G/P_{\theta_1})$. Suppose that one does not have equality. Since P_{θ_1} is generated by the $P_{\{\alpha\}}$ with $\alpha \in \theta_1$, there exists $\alpha \in \theta_1$ and a subset $M \in \mathfrak{M}$ which is not right $L^-_{\{\alpha\}}$-invariant. This means that the set

$$W = \{u \in U^- \ / \ uM \cap L^-_{\{\alpha\}} \ \text{ and } \ (uM)^c \cap L^-_{\{\alpha\}} \text{ are not negligeable in } L^-_{\{\alpha\}}\}$$

is of nonzero measure in U^-. Since $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$, one has $\{\alpha\} \neq \Pi$ and one can find such a u which also satisfies the conclusion of corollary 4.9. Hence the σ-algebra $\mathfrak{M}_2$ generated by the subsets $g\psi_{\{\alpha\}}(uM)$ is a sub-σ-algebra of $\mathfrak{M}$.
By construction $\mathfrak{M}_2$ is G-invariant. According to lemma 4.6 and the point (2) of section 3.3, there exists a subset $\theta_2 \subset \Pi$ such that $\mathfrak{M}_2 = \mathfrak{M}(G/P_{\theta_2})$. Since $\mathfrak{M}_2$ contains non $L^-_{\{\alpha\}}$-invariant subset, α is not in θ_2. According to lemma 4.6, the σ-algebra generated by $\mathfrak{M}(G/P_{\theta_1})$ and $\mathfrak{M}(G/P_{\theta_2})$ is $\mathfrak{M}(G/P_{\theta_3})$ with $\theta_3 = \theta_1 \cap \theta_2$. This contradicts the minimality of θ_1. $\diamondsuit$

4.6 Lebesgue density theorem

The proof of proposition 4.7 relies on a generalization of Lebesgue's density points theorem. Instead of working with small Euclidean balls associated to distances one works with small balls associated to b-distances.

Definition 4.11 *A b-distance on a space X is a map $d : X \times X \to [0, \infty[$ such that $\forall\, x, y, z \in X$, $d(x, y) = 0 \Leftrightarrow x = y$, $d(x, y) = d(y, x)$ and $d(x, z) \le b\,(d(x, y)+d(y, z))$.*

- One says that X is a b-metric space. Then, there exists a topology on X for which the balls $B(x, \varepsilon) := \{y \in X \ / \ d(x, y) \le \varepsilon\}$, with $\varepsilon > 0$, form a basis of neighborhood of the points x.
- One has the inclusion $\overline{B(x, \varepsilon)} \subset B(x, b\,\varepsilon)$.
- For each subset $Y \subset X$, let $B(Y, \varepsilon) := \cup_{y\in Y} B(y, \varepsilon)$ be the ε-neighborhood of Y and $\delta(Y) = \sup_{x,y\in Y} d(x, y)$ be the diameter of Y.

Example Let φ be a contracting automorphism of a locally compact group H. It is easy to construct a compact neighborhood C of e such that, $C = C^{-1}$ and $\varphi(C) \subset C$. Choose $N \ge 1$ such that $\varphi^N(C^2) \subset C$ and define, for $x, y \in H$,

$$d_C(x, y) = 2^{-n_C(x,y)} \text{ where } n_C(x, y) = \sup\{n \in \mathbb{Z} \ / \ x^{-1}y \in \varphi^n(C)\} \ .$$

This d_C is a left H-invariant 2^N-distance on H.

Let (X, d) be a locally compact b-metric space and μ be a Radon measure on X i.e. a Borel measure which is finite on compact sets. One says that X is of *finite μ-dimension* if, for all $x \in X$ and $\varepsilon > 0$ one has $\mu(B(x, \varepsilon)) > 0$ and, for all $c > 1$

$$\sup_{x\in X} \overline{\lim_{\varepsilon\to 0}}\, \mu\left(\overline{B(x, c\varepsilon)}\right) / \mu\left(\overline{B(x, \varepsilon)}\right) < \infty \ .$$

Note that if one checks this property for one $c > 1$, it is true for all $c > 1$.
Let $E \subset X$ be a measurable subset. A point $x \in E$ is a *density point* if

$$\lim_{\varepsilon\to 0} \mu\left(\overline{B(x, \varepsilon)} \cap E\right) / \mu\left(\overline{B(x, \varepsilon)}\right) = 1 \ .$$

Theorem 4.12 (Lebesgue) *Let (X, d) be a locally compact b-metric space which is of finite μ-dimension for a Radon measure μ on X and E be a measurable subset of X.*
Then μ-almost every point of E is a density point.

Proof of (theorem 4.12 ⇒ proposition 4.7) Let μ be a left Haar measure on H. The b-metric space (H, d_C) of the above example is of finite μ-dimension because, for all $h \in H$, one has $B(h, 2^{-n}) = h\varphi^n(C)$. Theorem 4.12 with $\varepsilon = 2^{-n}$ implies that, for almost every h in E^{-1},

$$\lim_{n\to\infty} \mu(h^{-1}\varphi^n(C) \cap E)/\mu(h^{-1}\varphi^n(C)) = 1 \ .$$

But the automorphism φ sends the Haar measure on one of its multiples. Hence

$$\lim_{n\to\infty} \mu(C \cap \varphi^{-n}(hE)))/\mu(C) = 1 \ .$$

This is true for an exhausting family of compact sets C. Otherwise stated, $\varphi^{-n}(hE)$ converges in measure to H.

The same discussion with E^c shows that, for almost all h in $(E^c)^{-1}$, $\varphi^{-n}(hE)$ converges in measure to $\emptyset$. $\diamondsuit$

To prove theorem 4.12, we will need the following two lemmas

Lemma 4.13 *Let (X, d) be a compact b-metric space, $Y \subset X$ and $\mathcal{F}$ be a family of closed subsets of X, such that, for all $x \in X$, there exists a closed set $F \in \mathcal{F}$ containing x whose diameter is nonzero but arbitrarily small.*

Then, either Y is included in a finite disjoint union of elements of $\mathcal{F}$ or there exists a sequence $(F_n)_{n>0}$ of disjoint elements of $\mathcal{F}$ such that, for all $n \geq 1$,

$$Y \ \subset \ F_1 \cup \cdots \cup F_n \cup (\cup_{k>n} B(F_k, 3b\,\delta(F_k))) \ .$$

Proof By induction, if $F_1, \ldots, F_k$ have been chosen and do not cover Y, the set

$$\mathcal{F}_k = \{F \in \mathcal{F} \ / \ \forall\, i \leq k \ , \ F_i \cap B(F, \delta(F)) = \emptyset\}$$

is nonempty. Let $\varepsilon_k = \sup_{F \in \mathcal{F}_k} \delta(F)$ and choose $F_{k+1} \in \mathcal{F}_k$ such that $\delta(F_{k+1}) \geq 2\varepsilon_k/3$.

It is clear that one has $\lim_k \varepsilon_k = 0$. If not, a sequence of points $p_k \in F_k$ such that $d(p_i, p_j) \geq 2\varepsilon_j/3$, $\forall\, i < j$ would contradict the compacity of X.

Let us show, by contradiction, that this sequence satisfies the required properties. Let y be a point of Y which is not in $F_1 \cup \cdots \cup F_n \cup (\cup_{k>n} B(F_k, 3b\,\delta(F_k)))$. There exists $F \in \mathcal{F}_n$ with nonzero diameter and containing y. Let us show by induction on $k \geq n$ that F is in $\mathcal{F}_k$. In fact, one has

$$B(F, \delta(F) \subset B(y, 2b\,\delta(F)) \subset B(y, 2b\,\varepsilon_k) \subset B(y, 3b\,\delta(F_{k+1})) \ .$$

According to y's choice, this last ball does not meet F_{k+1}. Hence $B(F, \delta(F)) \cap F_{k+1} = \emptyset$ and $F \in \mathcal{F}_{k+1}$. Therefore, one finds $\varepsilon_k \geq \delta(F) > 0$, $\forall\, k \geq n$. Contradiction. $\diamondsuit$

Let us suppose again now that (X, d) is a locally compact b-metric space and that μ is a Radon measure on X.

A *Vitali covering* $\mathcal{F}$ of a subset $Y \subset X$ is a covering of Y by closed subsets of X of non zero measure such that, $\exists\, \lambda > 1$, $\forall\, y \in Y$, $\exists F \in \mathcal{F}$ such that

$$y \in F \ \ , \ \ \delta(F) \ \text{ is arbitrarily small and } \ \mu(B(F, 3b\,\delta(F)))/\mu(F) \leq \lambda \ .$$

Lemma 4.14 (Vitali) *With these notations, for any Vitali covering $\mathcal{F}$ of a subset Y of X, there exists a sequence $(F_n)_{n>0}$ of disjoint elements of $\mathcal{F}$ such that*

$$\mu(Y - \bigcup_{n>0} F_n) = 0 \ .$$

Proof First suppose X compact. Let $(F_n)_{n>0}$ be the sequence of elements of $\mathcal{F}$ given by lemma 4.13. One has,

$$\mu(Y - \bigcup_{k\leq n} F_k) \leq \mu(\bigcup_{k>n} B(F_k, 3b\,\delta(F_k))) \leq \lambda \sum_{k>n} \mu(F_k) \to 0 \ ,$$

for $n \to \infty$, because $\sum_k \mu(F_k) \leq \mu(X) < \infty$. Therefore, one has $\mu(Y - \bigcup_{n>0} F_n) = 0$.

When X is only locally compact, one constructs thanks to a proper continuous function $f : X \to [0, \infty)$, a sequence of disjoint open relatively compact sets X_i such that $\mu(X - \bigcup_{i>0} X_i) = 0$. One then applies the previous argument to each subset $Y \cap X_i$ of the compact $\overline{X}_i$ and the covering $\mathcal{F}_i := \{F \in \mathcal{F} \ / \ F \subset X_i\}$. ♢

Proof of theorem 4.12 Let

$$B_i = \{x \in E/\underline{\lim}\ \mu(\overline{B(x,\varepsilon)} \cap E\,)/\mu(\overline{B(x,\varepsilon)}\,) < \tfrac{i-1}{i}\}.$$

It is enough to prove that, $\forall\, i$, $\mu(B_i) = 0$. For that, let us choose a sequence $(U_j)_{j>0}$ of open subsets of X containing E and such that $\lim\limits_{j\to\infty} \mu(U_j - E) = 0$ and let

$$\mathcal{F}^{ij} = \{\overline{B(x,\varepsilon)} \ / \ x \in E,\ \varepsilon > 0,\ \overline{B(x,\varepsilon)} \subset U_j \text{ and } \mu(\overline{B(x,\varepsilon)} \cap E\,)/\mu(\overline{B(x,\varepsilon)}\,) < \tfrac{i-1}{i}\}.$$

Since X is of finite μ-dimension, the family $\mathcal{F}^{ij}$ is a Vitali covering of B_i. Hence, there exists a sequence $(F_n^{ij})_{n>0}$ of disjoint elements of $\mathcal{F}^{ij}$ such that $\mu(B_i - \bigcup_{n>0} F_n^{ij}) = 0$. But then,

$$\mu(B_i) \leq \sum_{n>0} \mu(F_n^{ij}) \leq i \sum_{n>0} \mu(F_n^{ij} - E) \leq i\, \mu(U_j - E) \ ,$$

for all j. Hence $\mu(B_i) = 0$. ♢

Lecture 5: Local fields

Local fields is an important tool for discrete groups. For instance they are a decisive ingredient in the proof of Tits alternative or of Margulis arithmeticity theorem. We will not discuss these points here. Instead, we will show how they allow us to understand a larger class of groups than arithmetic groups, the so called S-arithmetic groups.

These groups happen to be lattices in locally compact groups G which are products of real and p-adic Lie groups. Moreover, many theorems for lattices in real Lie groups can be extended to lattices in such a group G with a very similar proof. In fact, the main property of $\mathbb{R}$ used in these proofs was "locally compact field" and not "archimedean field".

Hence, this lecture will be a rereading of the proofs in the previous chapters.

As a by-product of this point of view, we will construct cocompact lattices Γ in $\mathrm{SL}(d,L)$, for a p-adic field L and we will see that when $d \geq 3$, such Γ have property T and are quasisimple.

5.1 Examples

Here, as in section 2.1, we give a few explicit examples of lattices.

Let p, p_1, p_2 be prime numbers and $d \geq 2$, $m \geq 1$ be integers such that m is prime to p and $-m$ is a square in $\mathbb{Q}_p$. Let I_d be the $d \times d$ identity matrix.

In the following examples, the embedding of Γ is the diagonal embedding.

Example 1 *The group* $\Gamma := \mathrm{SL}(d,\mathbb{Z}[\frac{1}{p}])$ *is a noncocompact lattice in* $\mathrm{SL}(d,\mathbb{R})\times\mathrm{SL}(d,\mathbb{Q}_p)$.

Example 2 *The group* $\Gamma := \{g \in \mathrm{SL}(d,\mathbb{Z}[\frac{1}{p_1p_2}]) \ / \ g\,{}^tg = I_d\}$ *is a cocompact lattice in* $\mathrm{SO}(d,\mathbb{Q}_{p_1}) \times \mathrm{SO}(d,\mathbb{Q}_{p_2})$, *when* $d \geq 3$.

Example 3 *The group* $\Gamma := \{g \in \mathrm{SL}(d,\mathbb{Z}[\frac{\sqrt{-m}}{p}]) \ / \ g\,{}^t\overline{g} = I_d\}$ *is a cocompact lattice in* $\mathrm{SL}(d,\mathbb{Q}_p)$.

Example 4 Let L be a finite extension of $\mathbb{Q}_p$. One can then choose a totally real algebraic integer α over $\mathbb{Z}$ of degree $[L:\mathbb{Q}_p]$ such that $L = \mathbb{Q}_p[\alpha]$. *The group* $\Gamma := \{g \in \mathrm{SL}(d,\mathbb{Z}[\alpha,\frac{\sqrt{-m}}{p}]) \ / \ g\,{}^t\overline{g} = I_d\}$ *is a cocompact lattice in* $\mathrm{SL}(d,L)$.

Example 5 Using the two square roots of $-m$ in $\mathbb{Q}_p$, *the group* $\Gamma := \mathrm{SL}(d,\mathbb{Z}[\frac{\sqrt{-m}}{p}])$ *is a noncocompact lattice in* $\mathrm{SL}(d,\mathbb{C}) \times \mathrm{SL}(d,\mathbb{Q}_p) \times \mathrm{SL}(d,\mathbb{Q}_p)$.

Example 6 Let $F_p((t))$ be the field of Laurent series over $\mathbb{F}_p$. *The group* $\Gamma := \mathrm{SL}(d,\mathbb{F}_p[t^{-1}])$ *is a noncocompact lattice in* $\mathrm{SL}(d,\mathbb{F}_p((t)))$.

We will give a proof for the examples 1 to 4. The proof for the last ones is similar.

5.2 S-completions

Let us recall a few definitions related to the completions of $\mathbb{Q}$.

S-completions For p prime, let $\mathbb{Q}_p$ be the p-adic completion of $\mathbb{Q}$ for the absolute value $|.|_p$ such that $|p|_p = p^{-1}$. Let $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \ / \ |x|_p \leq 1\}$ and let μ_p be the Haar measure on $\mathbb{Q}_p$ such that $\mu_p(\mathbb{Z}_p) = 1$.

For $p = \infty$, let $\mathbb{Q}_\infty = \mathbb{R}$ be the completion of $\mathbb{Q}$ for the usual absolute value $|.|_\infty$ such that $|x|_\infty = x$ when $x > 0$ and μ_∞ be the Haar measure on $\mathbb{Q}_\infty$ such that $\mu_\infty([0,1]) = 1$.

The $\mathbb{Q}_p$, for $p \in \mathcal{V} := \{p \in \mathbb{N}, \text{prime}\} \cup \{\infty\}$, are all the completions of $\mathbb{Q}$.

Moreover, for x in $\mathbb{Q}^\times$, one has the product formula

$$\prod_p |x|_p = 1.$$

For $S \subset \mathcal{V}$, let

$$\mathbb{Z}_S \ := \ \mathbb{Z}[(\tfrac{1}{p})_{p \in S - \infty}], \quad \mathbb{Q}_S := \prod_{p \in S} \mathbb{Q}_p \ \text{ and } \ \mathbb{Q}_S^\circ := \{x \in \mathbb{Q}_S^\times \ / \ \prod_{p \in S} |x_p|_p = 1\}.$$

The ring $\mathbb{Z}_S$ is a subring of the field $\mathbb{Q}$. The field $\mathbb{Q}$ will be seen as a subring of the ring $\mathbb{Q}_S$ thanks to the diagonal embedding. When S is finite, the ring $\mathbb{Q}_S$ is locally compact.

Lemma 5.1 *Let S be a finite subset of $\mathcal{V}$ containing ∞. Then*
a) $\mathbb{Z}_S$ is a discrete cocompact subgroup of $\mathbb{Q}_S$.
b) $\mathbb{Z}_S^\times$ is a cocompact lattice in $\mathbb{Q}_S^\circ$.
c) For any $S' \subset S$ with $S' \neq S$, $\mathbb{Z}_S$ is dense in $\mathbb{Q}_{S'}$.

Example For p prime, the group $\mathbb{Z}[\frac{1}{p}]$ is discrete cocompact in $\mathbb{R} \times \mathbb{Q}_p$ and dense in both $\mathbb{R}$ and $\mathbb{Q}_p$.

Proof a) Let $O_S := \mathbb{R} \times \prod_{p \in S - \infty} \mathbb{Z}_p$. One has $O_S + \mathbb{Z}_S = \mathbb{Q}_S$ and $O_S \cap \mathbb{Z}_S = \mathbb{Z}$.

b) Let $U_S := \mathbb{R}^\times \times \prod_{p \in S - \infty} \mathbb{Z}_p^\times$. One has $U_S \mathbb{Z}_S^\times = \mathbb{Q}_S^\circ$ and $U_S \cap \mathbb{Z}_S^\times = \{\pm 1\}$.

c) Exercise. ♢

Adèles and idèles The language of adèles is a more concise and efficient way to deal with all completions of a number field than the S-arithmetic one. For instance it will allow us to say in a simple way that the cocompactness and density in lemma 5.1 are uniform in S. For our purposes, the concept of adèles can be avoided and the reader may forget the following paragraph.

Let $\mathbb{A} = \mathbb{A}_\mathbb{Q}$ be the ring of adèles of $\mathbb{Q}$. It is the *restricted product* of all $\mathbb{Q}_p$. More precisely, for a finite subset $S \subset \mathcal{V}$,

$$\mathbb{A}(S) = \{x \in \prod_{p \in \mathcal{V}} \ / \ \forall p \notin S \ , \ |x_p|_p \leq 1\} \ \text{ and } \ \mathbb{A} = \bigcup_{S\, finite} \mathbb{A}(S) \ .$$

The rings $\mathbb{A}(S)$ are endowed with the product topology and $\mathbb{A}$ is endowed with the inductive limit topology. The ring $\mathbb{A}$ of adèles is locally compact. We denote by μ its Haar measure whose restriction to the $\mathbb{A}(S)$ is the product measure of μ_p.

The subring $\mathbb{A}(\infty)$ is the ring of *integer adèles*. The field $\mathbb{Q}$ embedded diagonally in $\mathbb{A}$ is the subring of *principal adèles*.

The group of idèles is the multiplicative group $\mathbb{I} = \mathbb{A}^\times$ endowed with the induced topology for the embedding $\mathbb{I} \to \mathbb{A} \times \mathbb{A}$; $x \mapsto (x, x^{-1})$. The group $\mathbb{I}$ is locally compact. It contains the subgroup of *integral idèles* $I(\infty) := \mathbb{A}(\infty)^\times$, the subgroup of *principal idèles* $\mathbb{Q}^\times$ and

$$\mathbb{I}^\circ := \{x \in \mathbb{I} \; / \; |x| = 1\} \quad \text{where} \quad |x| = \prod_{p \in \mathcal{V}} |x_p|_p \; .$$

The following lemma is a stronger version of lemma 5.1.

Lemma 5.2 *a)* $\mathbb{Q}$ *is a discrete cocompact subgroup of* $\mathbb{A}$.
b) $\mathbb{Q}^\times$ *is a cocompact lattice in* $\mathbb{I}^\circ$.
c) For any $p \in \mathcal{V}$, $\mathbb{Q}$ *is dense in* $\mathbb{A}_p := \mathbb{A}/\mathbb{Q}_p$.

Proof a) One has $\mathbb{A}(\infty) + \mathbb{Q} = \mathbb{A}$ and $\mathbb{A}(\infty) \cap \mathbb{Q} = \mathbb{Z}$.
b) One has $\mathbb{I}(\infty)\, \mathbb{Q}^\times = \mathbb{I}$ and $\mathbb{I}(\infty) \cap \mathbb{Q}^\times = \{\pm 1\}$.
c) For $p = \infty$, $\mathbb{Z}$ is dense in $\prod_{\ell \neq \infty} \mathbb{Z}_\ell$. For $p < \infty$, $\mathbb{Z}[\frac{1}{p}]$ is dense in $\mathbb{R} \times \prod_{\ell \neq p, \infty} \mathbb{Z}_\ell$. $\diamondsuit$

Remark One has a similar construction for any number field k using all the absolute values v of k: $\mathbb{A}_k$ is the restricted product of all the completions k_v. We will not develop this important point of view here, since thanks to Weil's restriction of scalar we will mostly work with the field $k = \mathbb{Q}$.

5.3 The space of lattices in $\mathbb{Q}_S^d$

In this section and the next one, we extend to the S-arithmetic setting the results of lecture 2. The proofs are almost the same and will be only sketched.

Here is the extension of proposition 2.1

Proposition 5.3 *a) The group* $\mathrm{SL}(d, \mathbb{Z}_S)$ *is a lattice in* $\mathrm{SL}(d, \mathbb{Q}_S)$.
b) The group $\mathrm{SL}(d, \mathbb{Q})$ *is a lattice in* $\mathrm{SL}(d, \mathbb{A})$.

Thanks to the local Iwasawa decomposition for $G_p = \mathrm{SL}(d, \mathbb{Q}_p)$, $G_p = K_p A_p N_p$ as in section 2.2 for $p = \infty$ and with $K_p := \mathrm{SL}(d, \mathbb{Z}_p)$, $A_p := \{g = \mathrm{diag}(p^{n_1}, \ldots, p^{n_d}) \in G_p\}$, $N_p := \{g \in G_p \; / \; g - 1$ is strictly upper triangular$\}$ for $p < \infty$, one gets an Iwasawa decomposition for $G_{\mathbb{Q}_S} = \mathrm{GL}(d, \mathbb{Q}_S)$ $G_{\mathbb{Q}_S} = K^{\mathbb{Q}_S} A^{\mathbb{Q}_S} N^{\mathbb{Q}_S}$. We introduce again
$A_s^{\mathbb{Q}_S} := \{a \in A^{\mathbb{Q}_S} \; / \; |a_{i,i}| \leq s\, |a_{i+1,i+1}|$, for $i = 1, \ldots, d-1$ $\}$, with $s \geq 1$,
$N_t^{\mathbb{Q}_S} := \{n \in N^{\mathbb{Q}_S} \; / \; |n_{i,j}| \leq t$, for $1 \leq i < j \leq d$ $\}$, with $t \geq 0$. Let $S_{s,t}^{\mathbb{Q}_S}$ be the *Siegel domain* $S_{s,t}^{\mathbb{Q}_S} := K^{\mathbb{Q}_S} A_s^{\mathbb{Q}_S} N_t^{\mathbb{Q}_S}$ and $G_{\mathbb{Z}_S} = \mathrm{GL}(d, \mathbb{Z}_S)$.

Similarly, the group $G_{\mathbb{A}} = \mathrm{GL}(d, \mathbb{A})$ has an Iwasawa decomposition $G_{\mathbb{A}} = K^{\mathbb{A}} A^{\mathbb{A}} N^{\mathbb{A}}$ and one defines in the same way the Siegel domain $S_{s,t}^{\mathbb{A}} := K^{\mathbb{A}} A_s^{\mathbb{A}} N_t^{\mathbb{A}}$ and put $G_{\mathbb{Q}} = \mathrm{GL}(d, \mathbb{Q})$.

Lemma 5.4 *For $s \geq \frac{2}{\sqrt{3}}$, $t \geq \frac{1}{2}$, one has $G_{\mathbb{Q}_S} = S^{\mathbb{Q}_S}_{s,t} G_{\mathbb{Z}_S}$ and $G_{\mathbb{A}} = S^{\mathbb{A}}_{s,t} G_{\mathbb{Q}}$.*

Proof Same as lemma 2.2. Just replace the norm on $\mathbb{R}^d$ by the canonical norm on $\mathbb{Q}_S^d$ or $\mathbb{A}^d$ which is the product $\|v\| = \prod_p \|v_p\|_p$ of the canonical local norms $\|w\|_p = \sup_i |w_i|_p$ for $p < \infty$ and $\|w\|_\infty = (\sum_i w_i^2)^{\frac{1}{2}}$. And note that, for all $x \in \mathbb{Q}_S$ or $\mathbb{A}$, there exists $y \in \mathbb{Q}$ such that, for all $p < \infty$, $|x_p - y|_p \leq 1$ and $|x_\infty - y|_\infty \leq \frac{1}{2}$. $\diamondsuit$

The proposition 5.3 is now a consequence of the following lemma whose proof is the same as lemma 2.3. Let $R^{\mathbb{Q}_S}_{s,t} := S^{\mathbb{Q}_S}_{s,t} \cap \mathrm{SL}(d, \mathbb{Q}_S)$ and $R^{\mathbb{A}}_{s,t} := S^{\mathbb{A}}_{s,t} \cap \mathrm{SL}(d, \mathbb{A})$.

Lemma 5.5 *a) The volume of $R^{\mathbb{Q}_S}_{s,t}$ in $\mathrm{SL}(d, \mathbb{Q}_S)$ for the Haar measure is finite.*
a) The volume of $R^{\mathbb{A}}_{s,t}$ in $\mathrm{SL}(d, \mathbb{A})$ for the Haar measure is finite.

The set $X^{\mathbb{Q}_S}$ of lattices (i.e. discrete cocompact subgroups) in $\mathbb{Q}_S^d$ is the quotient space $X^{\mathbb{Q}_S} := G_{\mathbb{Q}_S}/G_{\mathbb{Z}_S}$. For any lattice Λ of $\mathbb{Q}_S^d$, one denotes by $d(\Lambda)$ the volume of $\mathbb{Q}_S^d/\Lambda$. It is given by the formula $d(\Lambda) = |\det(f_1, \ldots, f_d)|$ for any $\mathbb{Z}_S$-basis $(f_1, \ldots, f_d)$ of Λ.

Similarly $X^{\mathbb{A}} := G_{\mathbb{A}}/G_{\mathbb{Q}}$ is the set of lattices Λ in $\mathbb{A}^d$ and the covolume $d(\Lambda)$ is given by the same formula.

We still have Hermite's bound and Mahler's criterion with the same proofs.

Lemma 5.6 *Any lattice Λ in $\mathbb{Q}_S^d$ or $\mathbb{A}^d$ contains a vector v with $0 < \|v\| \leq (\frac{4}{3})^{\frac{d-1}{4}} d(\Lambda)^{\frac{1}{d}}$.*

Proposition 5.7 *A subset $Y \subset X^{\mathbb{Q}_S}$ or $X^{\mathbb{A}}$ is relatively compact if and only if there exist constants $\alpha, \beta > 0$ such that for all $\Lambda \in Y$, one has $d(\Lambda) \leq \beta$ and $\inf_{v \in \Lambda - 0} \|v\| \geq \alpha$.*

5.4 Cocompact lattices

Let $G \subset \mathrm{GL}(d, \mathbb{C})$ be a $\mathbb{Q}$-group. Note that, the groups $G_{\mathbb{Q}_S}$ and $G_{\mathbb{A}}$ are well defined and are locally compact groups in which $G_{\mathbb{Z}_S}$ and $G_{\mathbb{Q}}$, respectively, are discrete. We want to know when these groups are cocompact.

Godement's criterion remains the same:

Theorem 5.8 (Borel, Harish-Chandra) *Let G be a $\mathbb{Q}$-group. Then*
$G_{\mathbb{Q}_S}/G_{\mathbb{Z}_S}$ compact $\Leftrightarrow$ $G_{\mathbb{A}}/G_{\mathbb{Q}}$ compact $\Leftrightarrow$ $G_{\mathbb{R}}/G_{\mathbb{Z}}$ compact $\Leftrightarrow$ G is $\mathbb{Q}$-anisotropic.

When G is semisimple, the proof is the same as in lecture 2, we just have to replace the intermediate lemmas and propositions by the following.

Lemma 5.9 *Let $G \subset H$ be an injective morphism of $\mathbb{Q}$-groups. Suppose that G has no nontrivial $\mathbb{Q}$-characters. Then the injections $G_{\mathbb{Q}_S}/G_{\mathbb{Z}_S} \hookrightarrow H_{\mathbb{Q}_S}/H_{\mathbb{Z}_S}$ and $G_{\mathbb{A}}/G_{\mathbb{Q}} \hookrightarrow H_{\mathbb{A}}/H_{\mathbb{Q}}$ are homeomorphisms onto closed subsets.*

Lemma 5.10 *Let $V_{\mathbb{Q}}$ be a $\mathbb{Q}$-vector space, $V = V_{\mathbb{Q}} \otimes \mathbb{C}$, $G \subset \mathrm{GL}(V)$ be a $\mathbb{Q}$-subgroup without nontrivial $\mathbb{Q}$-character. Suppose there exists a G-invariant polynomial $P \in \mathbb{Q}[V]$ such that*

$$\forall v \in V_{\mathbb{Q}} \ , \ P(v) = 0 \Longleftrightarrow v = 0 \ .$$

Then the quotients $G_{\mathbb{Q}_S}/G_{\mathbb{Z}_S}$ and $G_{\mathbb{A}}/G_{\mathbb{Q}}$ are compact.

Lemma 5.11 *Let $\varphi : G \to H$ be a $\mathbb{Q}$-isogeny between two semisimple $\mathbb{Q}$-group. Then*
a) the groups $\varphi(G_{\mathbb{Z}_S})$ and $H_{\mathbb{Z}_S}$ are commensurable.
b) The induced map $G_{\mathbb{A}}/G_{\mathbb{Q}} \to H_{\mathbb{A}}/H_{\mathbb{Q}}$ is proper.

Corollary 5.12 *a) In example* 5.1.2*, Γ is cocompact in* $\mathrm{SO}(d,\mathbb{Q}_{p_1}) \times \mathrm{SO}(d,\mathbb{Q}_{p_2})$.
b) In example 5.1.3*, Γ is cocompact in* $\mathrm{SL}(d,\mathbb{Q}_p)$.
c) In example 5.1.4*, Γ is cocompact in* $\mathrm{SL}(d,\mathbb{L})$.

Proof of corollary 5.12 a) Apply lemma 5.10, to the orthogonal $\mathbb{Q}$-group $G = SO(d,\mathbb{C}) = \{g \in \mathrm{SL}(d,\mathbb{C}) \ / \ g\,{}^t g = I_d\}$, the set $S = \{p_1, p_2, \infty\}$, the natural representation in $V_{\mathbb{Q}} = \mathbb{Q}^d$, the polynomial $P(v) = \sum_i v_i^2$ and notice that $G_{\mathbb{R}}$ is compact.

b) Apply lemma 5.10, to the unitary $\mathbb{Q}$-group, with $\mu = \sqrt{-m}$,

$$G = \left\{ \begin{pmatrix} a & -m\,b \\ b & a \end{pmatrix} \in \mathrm{GL}(2d,\mathbb{C}) \ / \ (a + \mu\, b)\,({}^t a - \mu\, {}^t b) = I_d \ , \ \det(a + \mu\, b) = 1 \right\},$$

the set $S = \{p, \infty\}$, the natural representation in $V_Q = \mathbb{Q}^d \times \mathbb{Q}^d$, the polynomial $P(v,w) = \sum_i (v_i^2 + m\, w_i^2)$ and notice that $G_{\mathbb{R}} \simeq \mathrm{SU}(d,\mathbb{R})$ is compact.

The map $(a,b) \to a + \sqrt{-m}\, b$ gives isomorphisms $G_{\mathbb{Z}_S} \simeq \Gamma$ and $G_{\mathbb{Q}_p} \simeq \mathrm{SL}(d,\mathbb{Q}_p)$.

c) Let $S = \{p, \infty\}$ and G be the group as as in b), but consider it as a k_0-group with $k_0 = \mathbb{Q}[\alpha]$ and restrict it as a $\mathbb{Q}$-group H so that $H_{\mathbb{Z}_S} \simeq G_{\mathbb{Z}[\frac{\alpha}{p}]}$. The d real completions of k_0 give compact real unitary groups hence $H_{\mathbb{R}}$ is compact and H is $\mathbb{Q}$-anisotropic. The field L is the only p-adic completion of k_0, hence $H_{\mathbb{Q}_p} = G_L \simeq \mathrm{SL}(d,L)$. One has just to apply theorem 5.8. ♢

Remark To convince the reader that the examples c) do exist for any finite extension $L/\mathbb{Q}_p$ of degree n, we will give a construction of

a totally real algebraic integer α of degree n such that $L = \mathbb{Q}_p[\alpha]$.

For that, let $\beta \in L$ be such that $L = \mathbb{Q}_p[\beta]$, $Q \in \mathbb{Q}_p[X]$ be the minimal polynomial of β over $\mathbb{Q}_p$ and $R \in \mathbb{R}[X]$ be a polynomial of degree n whose roots are real and distinct. Thanks to the density of $\mathbb{Q}$ in $\mathbb{R} \times \mathbb{Q}_p$, one can find $P \in \mathbb{Q}[X]$ sufficiently near both Q and R. Take for α a suitable multiple $\alpha = N\alpha_0$ of a root α_0 of P, and note that, thanks to Hensel lemma, one has $L = \mathbb{Q}_p[\alpha]$. ♢

A general overview Let H be a $\mathbb{Q}$-group. Suppose that H is semisimple or more generally that H does not have any $\mathbb{Q}$-character $\chi : H \to GL(1,\mathbb{C})$, then, by a theorem of Borel and Harish-Chandra,

$H_{\mathbb{Z}_S}$ is a lattice in $H_{\mathbb{Q}_S}$ and $H_{\mathbb{Q}}$ is a lattice in $H_{\mathbb{A}}$.

Note that, according to the weak and strong approximation theorem of Kneser and Platonov: for any semisimple simply connected $\mathbb{Q}$-group H and $p \in \mathcal{V}$ with $H_{\mathbb{Q}_p}$ noncompact, and $S \ni p$, then

$$H_{\mathbb{Z}_S} \textit{ is dense in } H_{\mathbb{Q}_{S-p}} \textit{ and } H_{\mathbb{Q}} \textit{ is dense in } H_{\mathbb{A}}/H_{\mathbb{Q}_p}.$$

The examples above are the main motivation for the following definition.

Let G be a finite product with no compact factors $G = \prod_i G_i$ where the G_i are the $\mathbb{Q}_{p_i}$ points of some quasisimple $\mathbb{Q}_{p_i}$-groups, with $p_i \in \mathcal{V}$.

An irreducible subgroup Γ of G is said to be arithmetic, if there exists an algebraic group H defined over $\mathbb{Q}$, a finite set $S \subset \mathcal{V}$ and a group morphism $\pi : H_{\mathbb{Q}_S} \to G$ with compact kernel and cocompact image such that the groups Γ and $\pi(H_{\mathbb{Z}_S})$ are commensurable

Note that π is automatically "algebraic".

The classification of all arithmetic groups Γ, up to commensurability, relies again on the classification of all algebraic absolutely simple groups defined over a number field k. According to a theorem of Borel and Harder ([7]),

for any semisimple $\mathbb{Q}_p$-group H, the group $H_{\mathbb{Q}_p}$ contains at least one lattice.

Note that, since $G = H_{\mathbb{Q}_p}$ contains a compact open subgroup U without torsion,

any lattice Γ in $H_{\mathbb{Q}_p}$ is cocompact

(to prove this basic fact, just check that U acts freely on G/Γ and hence that all U-orbits in G/Γ have same volume).

Margulis arithmeticity theorem also applies to this case:

if G has no compact factors and the total rank of G is at least 2,
then all irreducible lattices Γ of G are arithmetic groups.

See [15]. Here, the total rank of G is the sum of all the $\mathbb{Q}_{p_i}$-rank of the G_i.

5.5 Coefficients decay

The coefficients decay and uniform coefficients decay are still true for p-adic semisimple Lie groups.

Theorem 5.13 *Let $G = \prod_i G_i$ be a product of groups G_i which are $\mathbb{Q}_{p_i}$-points of quasisimple $\mathbb{Q}_{p_i}$-groups for some $p_i \in \mathcal{V}$. Let π be a unitary representation of G such that $\mathcal{H}_\pi^{G_i} = 0$, $\forall i$. Then, for all $v, w \in \mathcal{H}_\pi$, one has* $\lim_{g\to\infty} <\pi(g)v, w> = 0$.

Corollary 5.14 *Let G be quasisimple $\mathbb{Q}_p$-group, π be a unitary representation of $G_{\mathbb{Q}_p}$ without nonzero $G_{\mathbb{Q}_p}$-invariant vectors and H be a non relatively compact subgroup of $G_{\mathbb{Q}_p}$. Then $\mathcal{H}_\pi^H = 0$.*

A group G as in theorem 5.13 still has maximal compact subgroups K, Cartan subspaces A, restricted roots Δ, Weyl chambers A^+ and parabolic subgroups P_θ as in section 3.3. And those share almost all the same properties as in the real case... well... the maximal compact subgoups are not all conjugate, the Cartan subspaces have to be replaced by the product of $\mathbb{Q}_{p_i}$-points of maximally $\mathbb{Q}_{p_i}$-split torus, A^+ is not always a subsemigroup of A... these are technical details I do not want to enter. The recipe is: for what we want it works the same. Let us just give an example:

For $G = \mathrm{SL}(d, \mathbb{Q}_p)$, one can take $K = \mathrm{SL}(d, \mathbb{Z}_p)$, $A = \{a = \mathrm{diag}(p^{-n_1}, \dots, p^{-n_d}) \in G\}$, $A^+ = \{a \in A \,/\, n_1 \geq \dots \geq n_d\}$. Then Δ, Δ^+, Π, $\mathfrak{u}^+$, $\mathfrak{p}$, $\mathfrak{u}^-$, $\mathfrak{u}_\theta^+$, $\mathfrak{p}_\theta$, $\mathfrak{u}_\theta^-$, $\mathfrak{l}_\theta$, A_θ and A_θ^+ are given by the same formula as in section 3.3 and one has $G = KA^+K$.

For v in a unitary representation $\mathcal{H}_\pi$ of G, one let $\delta(v) = \delta_K(v) = (\dim <Kv>)^{1/2}$.

Theorem 5.15 *Let $G = \prod_i G_i$ be as in theorem 5.13 and suppose that* $\mathrm{rank}_{\mathbb{Q}_{p_i}}(G_i) \geq 2$ $\forall i$. *Then there exists a K-biinvariant function $\eta_G \in \mathrm{C}(G)$ satisfying $\lim_{g\to\infty} \eta_G(g) = 0$ and such that, for all unitary representations π of G with $\mathcal{H}_\pi^{G_i} = 0$, $\forall i$, for any $v, w \in \mathcal{H}_\pi$, with $\|v\| = \|w\| = 1$, one has, for $g \in G$, $|<\pi(g)v, w>| \leq \eta_G(g)\delta(v)\delta(w)$.*

Remark The function η_G has also been computed by H.Oh in the p-adic case, thanks to Harish-Chandra's function

$$\xi(|p^{-n}|) \;=\; \xi(p^n) \;=\; \frac{(p-1)n + (p+1)}{(p+1)p^{n/2}}$$

For instance, for $G = \mathrm{SL}(d, \mathbb{Q}_p)$, $d \geq 3$, $a = \mathrm{diag}(t_1, \dots, t_d)$ with $|t_1| \geq \dots \geq |t_d|$,

$$\eta_G(a) = \prod_{1\leq i\leq [n/2]} \xi(|\tfrac{t_i}{t_{n+1-i}}|) \;.$$

The proofs of these theorems are the same as in lecture 3, we just have to replace the intermediate lemmas and propositions by the following lemmas with same proofs.

Lemma 5.16 *Let π be a unitary representation of $G = \mathrm{SL}(2, \mathbb{Q}_p)$, and $v \in \mathcal{H}_\pi$. If v is either A-invariant or U^+-invariant or U^--invariant then it is G-invariant.*

For $g = kan \in G$ let us denote $H(g) = a$ and let us reintroduce *Harish-Chandra spherical function* ξ_G given by $\xi_G(g) = \int_K \rho(H(gk))^{-1/2} dk$ with $\rho(a) = \det_{\mathfrak{n}}(\mathrm{Ad} a)$.

Lemma 5.17 *Let $G = \prod_i G_i$ be as in theorem 5.13 and π be a unitary representation of G which is weakly contained in λ_G. Then for $v, w \in \mathcal{H}_\pi$, with $\|v\| = \|w\| = 1$, $g \in G$, one has $|<\pi(g)v, w>| \leq \xi_G(g)\delta_K(v)\delta_K(w)$.*

Lemma 5.18 *Let V be a $\mathbb{Q}$-representation of the $\mathbb{Q}$-group $G = \mathrm{SL}(2)$, without nonzero invariant vectors. Let π be an irreducible unitary representation of the semidirect product $V_{\mathbb{Q}_p} \rtimes G_{\mathbb{Q}_p}$ without $V_{\mathbb{Q}_p}$-invariant vectors. Then the restriction of π to $G_{\mathbb{Q}_p}$ is weakly contained in the regular representation.*

5.6 Property T and normal subgroups

The previous control on the coefficients of unitary representations of G leads to similar algebraic properties for its lattices

Proposition 5.19 *Let* $G = \prod_i G_i$ *be as in theorem 5.13 with* $\mathrm{rank}_{\mathbb{Q}_{p_i}}(G_i) \geq 2$*,* $\forall\, i$*. Then*
a) G *has property* T*.*
b) Any lattice Γ *in* G *is finitely generated and has a finite abelianization* $\Gamma/[\Gamma, \Gamma]$.

Remark For all nontrivial $\mathbb{Q}$-group G, the groups $G_{\mathbb{Q}}$ and $G_{\mathbb{A}}$ do not have property T, because $G_{\mathbb{Q}}$ is not finitely generated and $G_{\mathbb{A}}$ is not compactly generated.

If one adapts the arguments of lecture 4 to G, one gets:

Theorem 5.20 *Let* $G = \prod_i G_i$ *be as in theorem 5.13, and* Γ *be a lattice in* G*. If* $\mathrm{rank}_{\mathbb{Q}_{p_i}}(G_i) \geq 2$*,* $\forall\, i$*, then* Γ *is quasisimple.*

Remark One can replace the rank assumption by: total $\mathrm{rank}(G) \geq 2$.

If the reader wants to know more on one of these five lectures, he should read [25] for lecture 1 or, respectively, [5], [13], [15] and [18] for lectures 2, 3, 4 and 5.

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