FROM SALEM NUMBERS TO MAHLER MEASURE OF K3 SURFACES (LECTURE 1)

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1. Introduction

Looking for large primes, Pierce [Pi17] proposed the following construction. Consider $P \in \mathbb{Z}[x]$ monic, and write

$$P(x) = \prod_i (x - \alpha_i),$$

then, look at

$$\Delta_n = \prod_i (\alpha_i^n - 1).$$

The α_i are algebraic integers. By applying Galois theory, it is easy to see that $\Delta_n \in \mathbb{Z}$. Note that if $P = x - 2$, we get the Mersenne sequence $\Delta_n = 2^n - 1$. The idea is to look for primes among the factors of Δ_n. The prime divisors of such integers must satify some congruence conditions that are quite restrictive, hence they are easier to factorize than a randomly given number. Moreover, one can show that $\Delta_m | \Delta_n$ if $m|n$. Then we may look at the numbers

$$\frac{\Delta_p}{\Delta_1} \quad \text{for } p \text{ prime.}$$

Lehmer [Le33] noticed that the number of trial divisions would get minimized if the sequence Δ_n grows slowly. Thus, he studied $\frac{|\Delta_{n+1}|}{|\Delta_n|}$, observed that

$$\lim_{n\to\infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1, \\ 1 & \text{if } |\alpha| < 1, \end{cases}$$

and suggested the following definition.

Definition 1. *Given $P \in \mathbb{C}[x]$, such that*

$$P(x) = a \prod_i (x - \alpha_i)$$

define the (Mahler) [1] *measure of P*

$$\Omega(P) = |a| \prod_i \max\{1, |\alpha_i|\}.$$

The logarithmic (Mahler) measure is defined as

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[1] The name Mahler came later after the person who successfully extended this definition to the several-variable case.

$$\mathrm{m}(P) = \log \Omega(P) = \log |a| + \sum_i \log^+ |\alpha_i|.$$

Now I cite Lehmer [Le33].

1.1. **A problem in the theory of equations.** The following problem arises immediately.

If ϵ is a positive quantity, to find a polynomial of the form

$$f(x) = x^r + a_1 x^{r-1} + \ldots + a_r$$

where the a's are integers, such that the absolute value of the product of those roots of f which lie outside the unit circle, lies between 1 and $1 + \epsilon$.

This problem, of interest in itself, is especially important for our purposes. Whether or not the problem has a solution for $\epsilon < 0.176$ we do not know.

...

The best linear function f is of course $x - 2 = 0$.

The best quadratic is $x^2 - x - 1$, for which $\Omega = 1.61803399...$

The best cubic is $x^3 - x - 1$, for which $\Omega = 1.32471796..$

The best quartic is $x^4 - x - 1$, for which $\Omega = 1.380277569...$

The examination of all quintic equations would require much time. The above results would all suggest that good examples might be found among trinomial equations. We have accordingly examined all trinomials of degree 5,6, and 7 with the following results:

The best trinomial quintic is $x^5 - x^3 - 1$, $\Omega = 1.3625986...$

The best trinomial sextic is $x^6 - x - 1$, $\Omega = 1.3707...$

The best trinomial septic is $x^7 - x^3 - 1$, $\Omega = 1.3797...$

None of the equations given above have a root on the unit circle.

Another important class of polynomials are those which are symmetric. The best polynomials of this type have all their roots but two on the unit circle. The best polynomials of degree 2, 4, 6, and 8 are

$$\begin{array}{cc} x^2 - 3x + 1 & \Omega = 2.6180339885... \\ x^4 - x^3 - x^2 - x + 1 & \Omega = 1.722083806... \\ x^6 - x^4 - x^3 - x^2 + 1 & \Omega = 1.401268369... \\ x^8 - x^5 - x^4 - x^3 + 1 & \Omega = 1.280638157... \end{array}$$

We have not made an examination of all degree symmetric polynomials but a rather intensive search has failed to reveal a better polynomial than

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 \quad \Omega = 1.176280821...$$

In fact these measures Ω concern quite remarquable polynomials.

All these polynomials are monic, with integer coefficients and irreducible. The best linear, quadratic, cubic and quartic belong to the same class and the symmetric ones except the quadratic belong to another class. Use, for example PARI and observe polynomials of both classes have only one root outside the unit circle. In the first class the other roots lie inside the unit circle while in the second class there are also roots on the unit circle. So, ten years before Pisot and Salem numbers were discovered and studied, some of them and their measure Ω, which will be nothing else than the Mahler measure introduced in 1962 by Mahler [Ma62], play an important role in Lehmer's paper.

2. Pisot and Salem numbers

Definition 2. *A Pisot number is an algebraic integer θ, $\theta > 1$, with all its other conjugates of modulus less than* 1.

Definition 3. *A Salem number is an algebraic integer τ, $\tau > 1$, with all its other conjugates of modulus less or equal to* 1 *and with at least one conjugate of modulus* 1.

Theorem 4. *(Salem (1945))*[Sa45] *Every Pisot number is a limit point of Salem numbers on both sides*

PROOF. In order to give a self contained proof we follow the Salem's original proof. But you can also refer to [BDGPS92]

Denote θ (resp. $P(z)$) a Pisot number (resp. its minimal polynomial). Thus $P(z)$ is monic, irreducible with integer coefficients and degree p. Denote $Q(z) := z^p P(1/z)$ its reciprocal polynomial.

1) Suppose first that $Q(z)$ and $P(z)$ are not identical, that is θ is not a quadratic unit. Let m be a positive integer and define

$$R_m(z) := z^m P(z) + Q(z).$$

So $R_m(z)$ is a reciprocal polynomial whose zeros are algebraic integers. We have

$$\frac{P(z)}{Q(z)} = \frac{z-\theta}{1-\theta z} \prod_{i=1}^{p-1} \frac{z-\alpha_i}{1-\alpha_i^* z},$$

where the α_i are the conjugates of θ and $\alpha_i^* := \frac{|\alpha_i|^2}{\alpha_i}$. Choose $\epsilon > 0$ such that $Q(z)$ has no zeros for $1 \leq |z| \leq 1+\epsilon$. By properties of conformal representation, we get, for $|z| = 1+\epsilon$,

$$\left| \prod_{i=1}^{p-1} \frac{z-\alpha_i}{1-\alpha_i^* z} \right| > 1.$$

Moreover, for $|z| = 1+\epsilon$, supposing that $1+\epsilon < \theta$ we can prove the inequalities

$$\left| \frac{z-\theta}{1-\theta z} \right| > \frac{\theta-(1+\epsilon)}{\theta(1+\epsilon)-1} > 1 - \epsilon \frac{\theta+1}{\theta-1} > 0$$

for ϵ small enough.

Now, on $|z| = 1+\epsilon$, it follows

$$\left| z^m \frac{P(z)}{Q(z)} \right| > (1+m\epsilon)\left[1 - \epsilon\frac{\theta+1}{\theta-1}\right] = 1 + \left(m - \frac{\theta+1}{\theta-1}\right)\epsilon - m\frac{\theta+1}{\theta-1}\epsilon^2.$$

Supposing $\epsilon < \frac{1}{2}(\theta-1)/(\theta+1)$ and taking $m > 2(\theta+1)/(\theta-1)$, we get , for $|z| = 1+\epsilon$,

$$\left| z^m \frac{P}{Q} \right| > 1$$

and by Rouché's theorem, inside the circle $|z| < 1+\epsilon$, $R_m(z)$ has as many zeros as $z^m P$, i.e. $m+p-1$. Since ϵ is arbitrarily small, $R_m(z)$ has only one zero τ_m of modulus larger than 1, thus real.

The polynomial $R_m(z)$ being reciprocal, $1/\tau_m$ is the only zero inside the unit circle, all the other zeros being of modulus 1. Now dividing $R_m(z)$ by its cyclotomic

factors, after proving there are still zeros of modulus 1, the remaining polynomial is a Salem polynomial and τ_m a Salem number. By definition of a Pisot number, $P(\theta) = 0$ and $P'(\theta) > 0$, hence $P'(z) \geq \mu > 0$ for $z \in [\theta-\sigma, \theta+\sigma]$, $\sigma > 0$ being small enough. Thus taking $|\delta| < \sigma$,$P(\theta + \delta)$ is never 0 and has the sign of δ. Moreover if $|\delta| = \frac{1}{\sqrt{m}}$, for m large enough, $R_m(\theta + \delta)$ has the sign of $P(\theta + \delta)$ hence the sign of δ. Since $R_m(\theta) = Q(\theta)$, we get, if $Q(\theta) > 0$

$$R_m(\theta + \delta)R_m(\theta) < 0 \text{ if } \delta < 0$$

and, if $Q(\theta) < 0$

$$R_m(\theta + \delta)R_m(\theta) < 0 \text{ if } \delta > 0$$

It follows $\theta < \tau_m < \theta + \frac{1}{\sqrt{m}}$ if $Q(\theta) < 0$ and $\theta - \frac{1}{\sqrt{m}} < \tau_m < \theta$ if $Q(\theta) > 0$.

Hence we obtain by the previous process an infinite sequence of τ_m, all different and tending to the Pisot number θ as m tends to infinity. We then deduce two things. First these τ_m have arbitrarily large degree (if not since their minimal polynomials have bounded coefficients there would be a contradiction). Secondly, for m large enough, they are not quadratic units hence have effectively conjugates on the unit circle thus are Salem numbers.

Considering now, instead of the polynomial $z^m P + Q$, the reciprocal polynomial $\frac{z^m P - Q}{z-1}$, we construct, in the same way, a sequence of Salem numbers tending to the Pisot number θ on the other side.

2) If θ is a quadratic unit, then $\theta + 1/\theta$ is an integer $r \geq 3$. Let $T_m(x)$ be the first kind Tchebycheff polynomial of degree m, i.e. $T_m(x) = \cos m\nu$ for $x = 2\cos\nu$. The polynomial $T_m(x)$ has m distinct real roots between -2 and $+2$. We can prove that the equation

$$2T_m(x)(x - r) - 1 = 0$$

possess m real roots between -2 and $+2$ and one real root between r and $r + \epsilon_m$, ϵ_m being positive and tending to 0 as m tends to infinity. Putting $x = y + 1/y$ we get a Salem polynomial defining a Salem number τ_m tending on one side to the Pisot quadratic unit θ. Doing the same trick in the equation

$$2T_m(x)(x - r) + 1 = 0$$

we get another sequence of Salem numbers approaching the quadratic Pisot unit the other side.

□

And Salem adds

I have not been able to solve the problem of the existence of other limit points than the P. V. numbers for the numbers of the class (T).

Remark 5. *The above problem is still unsolved.*

Remark 6. $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, $T_4 = 8x^4 - 8x^2 + 1$, $T_5 = 16x^5 - 20x^3 + 5x$, $T_6 = 32x^6 - 48x^4 + 18x^2 - 1$.

2.1. **Reciprocal to Salem's construction.** In a paper at Duke, Boyd [Bo77] proved in 1977 that every Salem number can be obtained from a Pisot number using Salem's construction.

Theorem 7. *Let Q be a monic polynomial with integer coefficients, of degree q, satisfying $z^q Q(z^{-1}) = \epsilon Q(z)$ with ϵ either 1 or -1, having $q - 2$ distinct roots on*

the unit circle and one root $\sigma > 1$ outside the unit circle. If $\epsilon = 1$ and q is even we also require the middle coefficient of Q to be even. Then there is a Pisot polynomial P_0 of degree k and a positive integer n such that

$$Q(z) = z^n P_0(z) + \epsilon z^k P_0(z^{-1}). \tag{1}$$

PROOF.

The proof needs first some remarks concerning the algebraic curve defined by

$$q(z,t) = z^n P_0(z) + \epsilon t z^k P_0(z^{-1}).$$

From Hille [Hi62], chapter 12 "Algebraic functions" in Analytic function theory, Vol. II, the equation $q(z,t) = 0$ defines an algebraic curve $z = Z(t)$ with $n+k$ branches and a finite number of singularities which are multiple points. Moreover this curve is an analytic function Z(t) that means that all the branches are branches of the same analytic function, so that one can obtain all these functions by analytic continuation of a single one of them along suitably chosen closed paths.

We consider each branch of the algebraic curve for $0 \leq t < 1$ oriented with t increasing.

If $0 \leq t < 1$, then for $|z| = 1$,

$$|z^n P(z)| > |\epsilon t z^k P(z^{-1})|$$

since P has real coefficients and $\bar{z} = z^{-1}$ on the unit circle.

If $t > 1$ and $|z| = 1$, then

$$|z^n P(z)| < |\epsilon t z^k P(z^{-1})|.$$

It follows:

(A) The equality $z = z_i(t)$ is impossible for $|z| = 1$ and $t \neq 1$.

Thus, if P has s zeros in $|z| > 1$ and $k - s$ zeros in $|z| < 1$, the algebraic curve $Z(t)$ is made of s branches outside the unit circle and $n + k - s$ branches inside the unit circle.

From the relation $q(z^{-1}, t^{-1}) = \epsilon t^{-1} z^{-n-k} q(z,t)$ we deduce

(B) $\frac{1}{Z(t^{-1})} = Z(t)$ i.e. the branches are exchanged under reflection in the x-axis, with preservation of orientation and under inversion in the unit circle with reversal of orientation.

If α is a simple root of modulus 1 of the equation $Q(z) = 0$, there exists a branch of the algebraic curve for example $z = z_1(t)$ satisfying $z_1(1) = \alpha$. From (A) we deduce, either $|z_1(0)| < 1$ and α is an *exit* of the branch beginning inside the unit circle or $|z_1(0)| > 1$ and α is an *entrance* of the branch beginning outside the unit circle.

If α is a root of order p, $p > 1$, of the equation $Q(z) = 0$, a certain number p_1 (resp. p_2) of branches enter (resp. leave) at α, with $p_1 + p_2 = p$.

We now give a criterium for α to be an entrance or an exit.

Lemma 8. *Denote by Q a polynomial with real coefficients satisfying*

$$Q(z) = z^n P(z) + \epsilon z^k P(z^{-1}), \tag{2}$$

with $\deg(P) = k$*, without zero on the unit circle and α a simple root of modulus* 1 *of Q.*

Then α is an exit (resp. entrance) if and only if

$$\epsilon\bar{\alpha}^{k-1}Q'(\alpha)P(\alpha) < 0 \quad (resp. > 0), \tag{3}$$

or equivalently

$$n - k + 2\Re(\alpha P'(\alpha)/P(\alpha)) > 0 \quad (resp. < 0). \tag{4}$$

If α is a root of order p, then at least $[p/2]$ branches enter the unit circle at α and at least $[p/2]$ leave at α.

PROOF. Denote $z(t)$ the branch of $Z(t)$ satisfying $z(1) = \alpha$. From (B) we deduce that this branch is orthogonal at $t = 1$ to the unit circle hence $z'(1)\bar{z}(1)$ is real. From theorem 12.2.1 of Hille, $z(t)$ is an analytic function of t so has an expression of the form near $t = 1$

$$z(t) = \alpha + (t-1)z'(1) + ... \tag{5}$$

Now, the relation $q(z(t), t) = 0$ can be written as

$$q(z(t), t) = (1-t)(z(t))^n P(z(t)) + tQ(z(t)) = 0 \tag{6}$$

Then using (5), since $Q(z) = (z-\alpha)Q'(\alpha) + \epsilon'(z-\alpha)$, the expression (6) has a factor $1-t$. Simplifying by this factor and letting $t \to 1$, we deduce

$$\alpha^n P(\alpha) - z'(1)Q'(\alpha) = 0,$$

which, in turn, using (2), becomes

$$\epsilon\alpha^k P(\alpha^{-1}) + z'(1)Q'(\alpha) = 0.$$

Notice that $z'(1) \neq 0$ since P has no zero on the unit circle.

After multiplication by $Q'(\alpha)$ of the conjugate expression, it follows

$$\epsilon\bar{\alpha}^k P(\alpha)Q'(\alpha) + \bar{z'}(1)Q'(\bar{\alpha})Q'(\alpha) = 0. \tag{7}$$

If α is an exit, we have $|z(0)| < 1$ and $|z(t)| < 1$ for $0 \leq t < 1$ so $\bar{z}(1)z'(1) = \frac{z'(1)}{\alpha} > 0$. Thus, from (7)

$$\epsilon\bar{\alpha}^{k-1}P(\alpha)Q'(\alpha) < 0$$

which is (3).

A similar argument can be developped if α is an "entrance" leading to a reverse inequality.

Differentiating (1) and using the relation $\alpha^n P(\alpha) + \epsilon\alpha^k P(\bar{\alpha}) = 0$ since α is a root of Q, we can express

$$\epsilon\bar{\alpha}^{k-1}P(\alpha)Q'(\alpha) = -(n-k)|P(\alpha)|^2 - \alpha P'(\alpha)P(\bar{\alpha}) - \bar{\alpha}P'(\bar{\alpha})P(\alpha).$$

Since $|P(\alpha)|^2 > 0$, it follows the equivalence of (3) and (4). □

Corollary 9. *Suppose the polynomial P of degree k has exactly s roots of modulus strictly greater than 1, the other roots being in $|z| < 1$. Suppose that Q has $s-1$ distinct roots on $|z| = 1$ each of which satisfies $n-k+2\Re(\alpha P'(\alpha)/P(\alpha)) \leq 0$. Then Q has at most one root outside the unit circle.*

PROOF. Since $q(z, 0) = z^n P(z)$ there are s branches of $Z(t)$ outside the unit circle for $0 \leq t < 1$. The $s-1$ roots of $q(z, 1)$ on the unit circle are either simple roots satisfying (2) or else multiple roots. Each of these supplies an entrance for at least one branch. Thus at most one branch can end outside the unit circle. □

Proof of the theorem We look for a polynomial P of degree $q-1$ with $q-2$ roots of modulus less than 1 and a root of modulus greater than 1 satisfying

$$Q(z) = zP(z) + \epsilon z^{q-1}P(z^{-1}). \tag{8}$$

Hence $P(z) = z^{n-1}P_0(z)$, P_0 being a Pisot polynomial and $n \geq 1$.

We consider only the case $\epsilon = -1$ and $q = 2m+1$, the proof being similar in the other case.

The following abbreviation will be used in writing the polynomial

$$a_0 z^k + a_1 z^{k-1} + \ldots + a_k \equiv a_0 \;\; a_1 \;\; \ldots a_k.$$

Thus we suppose the polynomial Q to be of the form

$$1 \;\; d_1 \;\; d_2 \;\; \ldots \;\; d_m \;\; -d_m \;\; \ldots \;\; -d_1 \;\; -1.$$

Hence P will be

$$1 \;\; (c_1+d_1) \;\; (c_2+d_2) \;\; \ldots \;\; (c_m+d_m) \;\; c_m \;\; c_{m-1} \;\; \ldots \;\; c_2 \;\; c_1.$$

with $c_i \in \mathbb{Z}$. We seek integers $c_1, ..., c_m$ such that the $2m-1$ distinct roots of modulus 1 of Q, denoted by z_1,...,z_{m-1},$\bar{z_1}$,...,$\bar{z}_{m-1}$,1, are "exits".

Thus the following m inequalities must be satisfied

$$\bar{z_j}^{2m-1} Q'(z_j) P(z_j) > 0, \quad 1 \leq j \leq m-1, \quad Q'(1)P(1) > 0. \tag{9}$$

More precisely, if we write $Q(z) = (z-1)R(z)$ and $R(z) = z^m A(z+z^{-1})$, the polynomial A has degree m and $m-1$ roots in $]-2,2[$ denoted $\rho_j = 2\cos\phi_j$ ($z_j = \exp i\phi_j$) and one root $\rho = \sigma + \sigma^{-1} > 2$. Hence we get

$$Q'(z_j) = (z_j - 1)R'(z_j) = (z_j - 1)[z_j^{m-1}(z_j - \bar{z_j})]A'(\rho_j)$$

and

$$\bar{z_j}^{2m-1} Q'(z_j) P(z_j) = (z_j + \bar{z_j} - 2)A'(\rho_j)(z_j^{1/2} + \bar{z_j}^{1/2})$$
$$[c_1(z_j^{m-1/2} + \bar{z_j}^{m-1/2}) + c_2(z_j^{m-1/2-1} + \bar{z_j}^{m-1/2-1}) + \ldots + c_m(z_j^{1/2} + \bar{z_j}^{1/2}) +$$
$$z_j^{1/2}(z_j^m + d_1 z_j^{m-1} + \ldots + d_m)] > 0$$

for $1 \leq j \leq m-1$

and

$Q'(1)P(1) = R(1)P(1) < 0$.

Since $z_j + \bar{z_j} - 2 < 0$ and $R(1) < 0$ the inequalities (9) are of the form

$$b_{1j}c_1 + b_{2j}c_2 + \ldots + b_{mj}c_m > b_j, \quad 1 \leq j \leq m \tag{10}$$

b_{ij} and b_j being real numbers. Moreover $\det((b_{ij})) = \det(B) \neq 0$. Otherwise there would be a vector $x = (x_1, ..., x_m) \neq (0, 0, ..., 0)$ satisfying $Bx = 0$ that is

$$x_1 z_j^{m-1/2} + x_2 z_j^{m-1/2-1} + \ldots + x_m z_j^{1/2} + x_m z_j^{-1/2} + \ldots + x_1 z_j^{-m+1/2} = 0;$$

thus there would be a polynomial

$$x_1 X^{2m-1} + x_2 X^{2m-2} + \ldots + x_m X^m + x_m X^{m-1} + \ldots + x_1$$

with the $2m-1$ roots z_1, ...,z_{m-1}, $\bar{z_1}$, ..., $\bar{z}_{m-1}$, 1. Since it possesses also the root -1, this polynomial of degree $2m-1$ would have $2m$ roots, a contradiction.

Since the matrix B is non singular, it follows that the region in $\mathbb{R}^m$ determined by (10) is a polyhedral cone affinely equivalent to an orthant of $\mathbb{R}^m$ and hence containing infinitely many lattice points (for example, each sphere of radius $m^{1/2}/2$

contained in the cone contains a lattice point). Each of these points determines a polynomial P satisfying (8) and the above inequalities. Now P can have no roots on the unit circle since, by (8), they would be roots of Q thus among the z_j and 1, a contradiction by (9). From the lemma we deduce that the $2m-1$ roots of modulus 1 of Q are "exit" of branches beginning at roots of $zP(z)$. Thus P has at least $2m-2$ roots of modulus strictly less than 1. The root $\sigma > 1$ of Q is the end of a branch beginning outside the unit circle at a root of P. Since there is no other branch beginning outside the unit circle and coming inside, P has exactly one root of modulus greater than 1.

□

2.2. **A list of "small" Salem numbers.** [Bo77] By "small" Salem numbers, we mean Salem numbers less than 1.3. Known Pisot numbers give by the previous theorem only a few "small" Salem numbers. For example, the smallest Pisot θ_0, $\theta_0 = 1.3247179572...$, with minimal polynomial $P_0(x) = x^3 - x - 1$ gives only σ_1, σ_7, σ_{19}, σ_{22}, σ_{38}, where σ_k denotes the kth smallest Salem [Bo77]. As for the second smallest Pisot number θ_1, $\theta_1 = 1.3802775691...$, with minimal polynomial $P_1(x) = x^4 - x^3 - 1$, it gives only σ_5 and σ_{22}. The other Pisot numbers less than $\frac{1+\sqrt{5}}{2}$ give no other "small" Salem.

Thus Boyd used the corollary, applied for example to $P_0(z^2)$, for $s = 2$ and $\alpha = -1$, giving σ_2, σ_3, σ_5, σ_{12}, σ_{22}. Also $P_1(z^2)$, for $s = 2$ and $\alpha = -1$ give σ_6, σ_{10} and σ_{19}.

Other polynomials, one due to Cantor, $x^6 - x^5 - 1$, another due to Lehmer, $x^5 - x^3 - 1$ give respectively σ_{12}, σ_{13}, σ_{19}, σ_{21} to σ_{36}, σ_{38}, σ_{39} and σ_1, σ_4, σ_5, σ_7 to σ_{12}, σ_{14}, σ_{19}, σ_{22}.

The remaining "small" Salem numbers were detected using a method based on Schur's algorithm [Bo78].

2.3. **The list from Mossinghoff's website.** [Mo]

There are 47 known Salem numbers less than 1.3. The most recently discovered Salem numbers are marked with an asterisk. This list is known to be complete for degree at most 44. There is only one known small Salem number with larger degree.

Note: Salem numbers are truncated, not rounded.

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	D	Salem Number
1.)	10	1.176280818259917506544070338474
2.)	18	1.188368147508223588142960958629
3.)	14	1.200026523987391518902962100414
4.)	14	1.202616743688604261118295415948
5.)	10	1.216391661138265091626806311199
6.)	18	1.219720859040311844169606760414
7.)	10	1.230391434407224702790177938975
8.)	20	1.232613548593121003962731694807
9.)	22	1.235664580389747308105169351531
10.)	16	1.236317931803230489899094869802
11.)	26	1.237504821217490608171021829989
12.)	12	1.240726423652541392056148161575
13.)	18	1.252775937410113900864582824053
14.)	20	1.253330650201489757028162788986
15.)	14	1.255093516763722879173091003232
16.)	18	1.256221154391670233067434043309
17.)	24	1.260103540354990920321649852331
18.)	22	1.260284236896492963739228435283
19.)	10	1.261230961137138851946671503074
20.)	26	1.263038139930169261222022798085
21.)	14	1.267296442523068692734077407604
22.)	22	1.276779674019016861136497157605
23.)	8	1.280638156267757596701902532710
24.)	26	1.281691371528106310055107748672
25.)	20	1.282495560639960169561207128806
26.)	18	1.284616550925536736743131441485
27.)	26	1.284746821544843035729838140650
28.)	30	1.285099363651876557117420034512
29.)	30	1.285121520153207532780681369174
30.)	30	1.285185670752909791356387310393
31.)	26	1.285196726769853432068127270005
32.)	44	1.285199179205612167312192383918
33.)	30	1.285235436228923770828415884707
34.)	34	1.285409064765363764030309848277
35.)	18	1.286395966836277224044411092745
36.)	26	1.286730182048201274368747282841
37.)	24	1.291741425714500483635599066106
38.)	20	1.292039106017929461943480560567
*39.)	40	1.292418657582426546281031229140
*40.)	46	1.292900721780102794630870596243
41.)	10	1.293485953125454106519909883794
42.)	18	1.295675371944048235295741653561
*43.)	34	1.296210659593309216851783179125
44.)	22	1.296421365194547218873224266498
45.)	28	1.296821373714950077456125855369
*46.)	36	1.298429835475111538327805425740
47.)	26	1.299744869472170731620096386139

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